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BANACH SPACES WHICH SATISFY LINEAR IDENTITIES

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In 1935, Jordan and von Neumann proved that any Banach space which satisfies the parallelogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \quad \text{for all elements } x \text{ and } y \tag{1}$$

must be a Hilbert space.

Subsequent authors have found norm conditions weaker than (1) which require a Banach space to be a Hilbert space. Notable examples include the results of Day, Lorch, Senechalle and Carlsson.

In this paper, we study nontrivial linear identities such as

$$\sum_{k=0}^{m} a_k\|c_k(0)x_0+\cdots+c_k(n)x_n\|^p=0 \quad \text{for all elements } x_i \tag{2}$$

on a Banach space X.

A necessary condition for (2) to hold in X is that $\|x+ty\|^p$ must be a polynomial in t for all choices of elements x and y. A sufficient condition for (2) to hold in X is that (2) must hold in the field of scalars. Specific identities are presented including a generalized parallelopiped law first observed by Koehler, and some isometric results are stated.

2. **The parallelogram law revisited.** In 1909 [4], Fréchet proved the following result.

LEMMA 1 (Fréchet). *If g is continuous function on $\boldsymbol{R}$ and, for all real r and s, equation* (3) *holds, then g is a polynomial with degree less than N.*

$$\sum_{k=0}^{N}(-1)^{N-k}\binom{N}{k}g(r+ks)=0\,. \tag{3}$$

Proof. It is well-known that any sequence $\{a_n\}$ satisfying $\Sigma(-1)^{N-k}\binom{N}{k}a_{k+M}=0$ for all M is generated by a polynomial; that is, there is a polynomial P with degree less than N for which $a_n=P(n)$.

In (3), put $g(n)=a_n$, $s=1$ and let r range over the integers. Then there is a polynomial P with $P(n)=a_n=g(n)$. Now put $g(n/2)=b_n$, $s=1/2$ and let r range over the half-integers. There is

a polynomial Q with $Q(n) = b_n = g(n/2)$. Thus $Q(2n) = P(n)$ for all n and $Q(x) \equiv P(x/2)$, so $g(n/2) = P(n/2)$. A repetition of this argument demonstrates that $P(n2^{-m}) = g(n2^{-m})$ for all integers m and n. By the continuity of g, $P(x) = g(x)$ for all x and the lemma is proved.

The parallelogram law has a second difference nature: $\|x + y\|^2 - 2\|x\|^2 + \|x - y\|^2 = 2\|y\|^2$, (see Johnson [6]). Putting successively $x = u + 2v$, $y = v$ and $x = u + v$, $y = v$ and subtracting, we get (4). Fix w and z, elements of any space in which (1) holds.

$$\sum_{k=0}^{3} (-1)^k \binom{3}{k} \|u + kv\|^2 = 0 . \tag{4}$$

Let $u = w + rz$, $v = sz$ and substitute in (4). Setting $\|w + tz\|^2 = g(t)$, we obtain (5). By the triangle inequality, $|g^{1/2}(t_0) - g^{1/2}(t_1)| \leqq |t_1 - t_0| \cdot \|z\|$, hence g is continuous.

$$\sum_{k=0}^{3} (-1)^k \binom{3}{k} g(r + ks) = 0 . \tag{5}$$

Applying Lemma 1 to (5) we see that $g(t)$ is quadratic in t. Indeed, if $\|w + tz\|^2 = A(w, z) + 2B(w, z)t + C(w, z)t^2$, then clearly $A(w, z) = \|w\|^2$ and $C(w, z) = \|z\|^2$. It is not hard to verify that $B(w, z)$ satisfies the definition of a real inner-product and $B(w, z) + iB(w, iz)$ that of a complex inner-product. This provides an alternative proof to the Jordan-von Neumann theorem.

We shall return to the parallelogram law in §6 as an embarkation point for a series of linear identities which hold in more spaces than Hilbert space. As an appetizer, consider (6), a generalization of (1) to a three-dimensional parallelopiped.

$$\begin{aligned} &\|x + y + z\|^k + \|x + y - z\|^k + \|x - y + z\|^k + \|x - y - z\|^k \\ &\quad -2(\|x + y\|^k + \|x - y\|^k + \|x + z\|^k + \|x - z\|^k \\ &\quad +\|y + z\|^k + \|y - z\|^k) \\ &\quad +4(\|x\|^k + \|y\|^k + \|z\|^k) = 0 . \end{aligned} \tag{6}$$

Observe that (6) holds for $k = 2$ in Hilbert space and for $k = 4$ in Hilbert space and in $L_4(x, \mu)$ for any (x, μ). Indeed, it may be verified that (6) holds in any Banach space in which $\|rx + sy + tz\|^4$ is a homogeneous polynomial in r, s, and t for fixed elements x, y, and z. This condition turns out to be necessary as well, and the situation will prove to be typical.

3. Spaces which satisfy linear identities. The main result of this section will be Theorem 4, which follows from Theorem 2 by an intervening lemma.

THEOREM 2. *Suppose X is a Banach space in which (7) holds for all elements x and y, where $a_k \neq 0$, $p > 0$, $b_0 = 0$ and the b_k's are distinct.*

$$\sum_{k=0}^{m} a_k \|x + b_k y\|^p = 0 . \tag{7}$$

Then for every x and y in X, $\|x + ty\|^p$ is a polynomial in t. In particular, p is an even integer.

Proof. Set $A(u, v) = a_0\|u\|^p + \sum_{k=1}^{m} \|u + b_k v\|^p$. By the hypotheses, $A(u, v) = 0$ for all elements u and v and $b_k \neq 0$ for $k \geqq 1$. Fix elements x and y; let z be arbitrary. Then $0 = A(x + z, y - b_1^{-1}z) - A(x, y)$. Writing this out, we obtain (8). Notice that the term for $k = 1$ in the sum in (8) vanishes.

$$\begin{aligned} 0 &= a_0(\|x + z\|^p - \|x\|^p) \\ &\quad + \sum_{k=1}^{m} a_k(\|x + b_k y + (1 - b_k b_1^{-1})z\|^p - \|x + b_k y\|^p) . \end{aligned} \tag{8}$$

We repeat this procedure (due to Wilson [14]) and obtain (9), where the inner sum is taken over all choices of $1 \leqq i_1 < \cdots < i_j \leqq m$.

$$0 = \sum_{j=0}^{m} (-1)^j (\Sigma A(x + jz, y - (b_{i_1}^{-1} + \cdots + b_{i_j}^{-1})z)) . \tag{9}$$

Equation (9) may be expanded using the definition of $A(u, v)$ as (10), where the inner sum is as before.

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|x + jz\|^p \\ &\quad + \sum_{k=1}^{m} a_k \sum_{j=0}^{m} (-1)^j (\Sigma \|x + b_k y + (j - b_k(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}))z\|^p) . \end{aligned} \tag{10}$$

For any fixed k, the 2^m subsets of $\{1, \cdots, m\}$ divide into two corresponding classes: if $k \notin I = \{i_1, \cdots, i_j\}$, then I and $I \cup \{k\}$ are associated. Using this pairing, the terms

$$(-1)^j \|x + b_k y + (j - b_k(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}))z\|^p$$

and $(-1)^{j+1}\|x + b_k y + (j + 1 - b_k(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1} + b_k^{-1}))z\|^p$ cancel out in the triple sum. Hence, (10) reduces to (11).

$$0 = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|x + jz\|^p . \tag{11}$$

As in §2, we choose u and v arbitrary nonzero elements in X, let r and s be arbitrary reals and set $x = u + rv$, $y = sv$ and $g(t) =$

$||u+tv||^p$ in (11). We obtain (12), and by Lemma 1, $g(t)=||u+tv||^p$ must be a polynomial.

$$0=\sum_{j=0}^{m}(-1)^j\binom{m}{j}g(r+js)\,. \tag{12}$$

As $t^{-p}g(t)\to||v||^p\neq 0$, g has degree p and as $g(t)\geqq 0$, p is an even integer.

LEMMA 3. *Suppose X is a Banach space in which* (13) *holds for all elements x and y, where $a_k\neq 0$, $p>0$ and the (b_k, c_k)'s are pairwise linearly independent.*

$$\sum_{k=0}^{m}a_k||b_kx+c_ky||^p=0\,. \tag{13}$$

Then $||x+ty||^p$ is a polynomial in t for all elements x and y.

Proof. We shall reduce (13) to (7). Permute the k's so that $|b_0|\leqq\cdots\leqq|b_m|$ and let $d_k=b_k^{-1}c_k$. If $b_0\neq 0$ then $b_k\neq 0$; if $b_0=0$ then $c_0\neq 0$ and $b_k\neq 0$ for $k\geqq 1$ and the d_k's are distinct by the linear independence. In the first case, rewrite (13) as (14) and then put into it $x=u-d_0v$, $y=v$ where u and v are arbitrary.

$$\sum_{k=0}^{m}a_k|b_k|^p||x+d_ky||^p=0\,. \tag{14}$$

We obtain (15) which is in the form of (7).

$$\sum_{k=0}^{m}a_k|b_k|^p||u+(d_k-d_0)v||^p=0\,. \tag{15}$$

In the second case, let s be a number for which $b_k+sc_k\neq 0$ and put $x=u$, $y=su+v$ into (13) where u and v are again arbitrary. (If (b_j, c_j) and (b_k, c_k) are independent then so are (b_j+sc_j, c_j) and (b_k+sc_k, c_k).) We obtain (16) which now falls under the first case.

$$\sum_{k=0}^{m}a_k||(b_k+sc_k)u+c_kv||^p=0\,. \tag{16}$$

Carlsson [1] proved Lemma 3 for $p=2$ and a_k, b_k, c_k real.

THEOREM 4. *Suppose X is a Banach space in which* (17) *holds for all elements x_i, where $a_k\neq 0$, $p>0$, the $(c_k(0), \cdots, c_k(n))$'s are pairwise linearly independent $(n+1)$-tuples, and for every i there is at least one k with $c_k(i)\neq 0$.*

$$\sum_{k=0}^{m}a_k||c_k(0)x_0+\cdots+c_k(n)x_n||^p=0\,. \tag{17}$$

Then, for all x and y, $\|x + ty\|^p$ is a polynomial in t.

Proof. Permute the k's so that $|c_0(0)| \geqq |c_1(0)| \geqq \cdots \geqq |c_m(0)|$, then $c_0(0) \neq 0$. If $c_k(0) = 0$ for $k \geqq 1$, put $x_1 = \cdots = x_n = 0$ in (17), then $a_0\|c_0(0)x_0\|^p = 0$ for all x_0, that is, X is trivial. Otherwise, suppose $c_k(0) \neq 0$ for $k \leqq j$, $j \geqq 1$ and (17) may be rewritten as (18), where $d_k(i) = c_k(i)c_k(0)^{-1}$.

$$\sum_{k=0}^{j} a_k |c_k(0)|^p \|x_0 + d_k(1)x_1 + \cdots + d_k(n)x_n\|^p + \sum_{k=j+1}^{m} a_k \|c_k(1)x_1 + \cdots + c_k(n)x_n\|^p = 0 . \tag{18}$$

From the linear independence, it follows that the n-tuples $\{(d_k(1), \cdots, d_k(n))\}$ are distinct. Define the sets

$$A_{ik} = \left\{(s_1, \cdots, s_n) \in \boldsymbol{C}^n \mid \sum_{q=1}^{n} s_q d_i(q) = \sum_{q=1}^{n} s_q d_k(q)\right\} .$$

These sets do not exhaust $\boldsymbol{C}_n^n$ and so we can find r_q for which the b_k, defined by $b_k = \sum_{q=1}^{n} r_q d_k(q)$, are distinct.

For arbitrary x and y, let $x_0 = x$, $x_k = r_k y$ in (18). We obtain (19) which is in the form of (13).

$$\sum_{k=0}^{j} a_k |c_k(0)|^p \|x + b_k y\|^p + \left(\sum_{k=j+1}^{m} \sum_{i=1}^{n} |c_k(i)r_i|^p\right)\|y\|^p = 0 . \tag{19}$$

This completes the proof.

To obtain the natural corollary we need a lemma.

LEMMA 5. *Suppose f is a nonnegative function and q and r are positive integers such that f^q and f^r are polynomials. Then f^s is a polynomial, where $s = (q, r)$, the greatest common divisor.*

Proof. We have $f^q(x) = A^q \prod_{i=1}^{k} (x - x_i)^{\lambda_i}$, $x_i \neq x_j$, $A \geqq 0$. Thus, $f^r(x) = A^r \prod_{i=1}^{k} (x - x_i)^{\lambda_i r/q}$, and so $q \mid r\lambda_i$. Write $q = s\bar{q}$, $r = s\bar{r}$, $(\bar{q}, \bar{r}) = 1$. As $s\bar{q} \mid s\bar{r}\lambda_i$, $\bar{q} \mid \lambda_i$ and so $f^s = A^s \prod_{i=1}^{k} (x - x_i)^{\lambda_i/\bar{q}}$ is a polynomial.

COROLLARY 6. *Suppose X is a Banach space which satisfies* (20) *for all elements x_i, where $p_k > 0$ and the same restrictions on constants apply as in Theorem* 4.

$$\sum_{k=0}^{m} a_k \|c_k(0)x_0 + \cdots + c_k(n)x_n\|^{p_k} = 0 . \tag{20}$$

Then p_k is an even integer for each k and $\|x + ty\|^p$ is a polynomial

in t for all x and y, where $p = (p_1, \cdots, p_k)$, the greatest common divisor.

Proof. Fix $y_0, \cdots, y_n$ and let $b_k = a_k \| c_k(0)y_0 + \cdots + c_k(n)y_n \|^{p_k}$. Let $x_k = \lambda y_k$ for $\lambda > 0$, then by (20), $\sum_{k=0}^{m} b_k \lambda^{p_k} = 0$. Upon collecting equal p_k's as q_j's we find that $\sum_{j=1}^{s} (\sum_{p_k = q_j} b_k)\lambda^{q_j} = 0$. Hence $\sum_{p_k = q_j} b_k = 0$. Thus X satisfies an equation of form (17) for each p_k and $\|x + ty\|^{p_k}$ is a polynomial in t. We now apply Lemma 5 to this situation and conclude that f^p is a polynomial.

4. **The Class $\mathscr{P}_{2n}$.** The necessary condition of Theorem 4 suggests the following definition: a Banach space X is *polynomial of degree* $2n$ if, for all elements x and y, $\|x + ty\|^{2n}$ is a polynomial of degree $2n$ in real t. The class $\mathscr{P}_{2n}$ consists of all Banach spaces which are polynomial of degree $2n$.

THEOREM 7. (i) *If X is in $\mathscr{P}_2$, then X is a Hilbert space.*

(ii) *If m divides n, then $\mathscr{P}_{2m}$ is contained in $\mathscr{P}_{2n}$.*

(iii) *If $r = (m, n)$, then $\mathscr{P}_{2m} \cap \mathscr{P}_{2n} = \mathscr{P}_{2r}$.*

(iv) *If k is an integer dividing n, then for all measure spaces (X, μ), $L_{2k}(X, \mu)$ is in $\mathscr{P}_{2n}$.*

(v) *If p is not an even integer and (X, μ) is not trivial, then $L_p(X, \mu)$ is not in $\mathscr{P}_{2n}$ for any n.*

Proof. (i) This is the Jordan-von Neumann theorem; see also §2.

(ii) If $\|x + ty\|^{2m}$ is a polynomial and $2n/2m$ is an integer, then $\|x + ty\|^{2n}$ is a polynomial.

(iii) Combine (ii) and Lemma 5.

(iv) It suffices to show that $L_{2n}(X, \mu)$ is in $\mathscr{P}_{2n}$. Pick elements f and g in $L_{2n}(X, \mu)$; then

$$\|f + tg\|^{2n} = \int |f + tg|^{2n} d\mu = \int (|f|^2 + t(f\bar{g} + \bar{f}g) + t^2)^n d\mu$$
$$= \int \Sigma c_i(f, g) t^i d\mu ,$$

where $c_i(f, g)$ is a sum of terms of the form $f^a \bar{f}^b g^c \bar{g}^d$ with $c + d = i$, $a + b = 2n - i$. As $\int |f|^{2n} d\mu < \infty$ and $\int |g|^{2n} d\mu < \infty$, each $c_i(f, g)$ is integrable by Hölder's inequality. Thus $\|f + tg\|^{2n} = \Sigma \int c_i(f, g) d\mu \cdot t^i$ and $L_{2n}(X, \mu)$ is in $\mathscr{P}_{2n}$.

(v) If (X, μ) has two disjoint sets of positive measure, then one may easily construct elements x and y in $L_p(X, \mu)$ with $\|x + ty\|^p = 1 + |t|^p$. If p is not an even integer then $(1 + |t|^p)^{2n/p}$ is not a

polynomial as it is not in $C^{[p]+1}$. (See also Sundaresan [13].) The embedding properties of $\mathscr{P}_{2n}$ will be described in a subsequent paper [11]. We state without proof the following theorem.

THEOREM 8. (i) *If X is a real two-dimensional space in $\mathscr{P}_4$, then it is isometrically isomorphic to a subspace of $L_4[0, 1]$.*

(ii) *There exists a three-dimensional space in $\mathscr{P}_4$ which is not isometrically isomorphic to any subspace of $L_4(X, \mu)$ for any (X, μ).*

(iii) *There exist two-dimensional spaces in $\mathscr{P}_{2n}(n \geqq 3)$ which are not isometrically isomorphic to any subspace of $L_{2n}(X, \mu)$ for any (X, μ).*

5. **The sufficient condition.** In this section, we prove that the classes $\mathscr{P}_{2n}$ form the finest possible gradation of Banach spaces according to the linear identities they satisfy: if an identity of type (16) holds with $p = 2n$ for one space in $\mathscr{P}_{2n}$ then it holds for all spaces in $\mathscr{P}_{2n}$. We begin with a few preliminaries.

LEMMA 9. *Suppose a function $g(u_1, \cdots, u_n)$ is a polynomial in each of its variables separately, that is,* (21) *holds for each r, $1 \leqq r \leqq n$, where the $g_{k,r}$'s are continuous and a carat over a variable signifies its omission.*

$$g(u_1, \cdots, u_n) = \sum_{k=1}^{s_r} g_{k,r}(u_1, \cdots, \hat{u}_r, \cdots, u_n)u_r^k \,. \tag{21}$$

Then g is in fact a polynomial in the variables together.

Proof. The proof will be by induction on n. The theorem is certainly true for $n = 1$. Suppose it is true if $n = m$. For $n = m + 1$ we have by hypothesis a representation of $g(u_1, \cdots, u_{m+1})$ in the form (22). Let $M = \max s_r + 1$.

$$g(u_1, \cdots, u_{m+1}) = \sum_{k=1}^{m-1} g_{k,m+1}(u_1, \cdots, u_m)u_{m+1}^k \,. \tag{22}$$

Define the Mth difference $\Delta^M h(r, v)(u_1, \cdots, u_s)$ by (23).

$$\Delta^M h(r, v)(u_1, \cdots, u_s) = \sum_{i=0}^{M} (-1)^{M-i}\binom{M}{i} h(u_1, \cdots, u_r + iv, \cdots, u_s) \,. \tag{23}$$

Since Δ^M is certainly linear, we can compute for $1 \leqq t \leqq m + 1$ $\Delta^M g(t, v)(u_1, \cdots, u_{m+1})$ obtaining (24).

$$\Delta^M g(t, v)(u_1, \cdots, u_{m+1}) = \sum_{i=0}^{M} \Delta^M g_{k,m+1}(t, v)(u_1, \cdots, u_m)u_{m+1}^k \,. \tag{24}$$

Since a representation of form (21) holds for t and the Mth difference annihilates all polynomials with degree less than M, $\Delta^M g(t, v)(u_1, \cdots, u_{m+1}) = 0$. Thus, $\Delta^M g_{k,m+1}(t, v)(u_1, \cdots, u_m) = 0$ for each t. By Lemma 5, $g_{k,m+1}(u_1, \cdots, u_m)$ is a polynomial in each u_i separately. The induction hypothesis for $n = m$ ensures that $g_{k,m+1}(u_1, \cdots, u_m)$ is a polynomial in $u_1, \cdots, u_m$ together. By (22), $g(u_1, \cdots, u_{m+1})$ is therefore a polynomial in $u_1, \cdots, u_{m+1}$. This establishes the induction step and completes the proof.

LEMMA 10. *A space X is in $\mathscr{P}_{2n}$ if and only if, for all m and elements x_i, $||t_0x_0 + \cdots + t_mx_m||^{2n}$ is a polynomial in $t_0, \cdots, t_m$.*

Proof. It is easy to see that $||t_0x_0 + \cdots + t_mx_m||^{2n}$ must be a homogeneous polynomial if it is a polynomial at all, and that this condition is equivalent to $||x_0 + t_1x_1 + \cdots + t_mx_m||^{2n}$ being a polynomial in $t_1, \cdots, t_m$.

If $||x_0 + t_1x_1 + \cdots + t_mx_m||^{2n}$ is a polynomial, set $x_0 = x$, $x_1 = y$, $x_2 = \cdots = x_m = 0$, arbitrary x and y. We find that X is in $\mathscr{P}_{2n}$. To prove the converse, define $f(t_1, \cdots, t_m) = ||x_0 + t_1x_1 + \cdots + t_mx_m||^{2n}$. Fix all variables save t_r, then $f(t_1, \cdots, t_m)$ is a polynomial in t_r with coefficients depending on $t_1, \cdots, \hat{t}_r, \cdots, t_m$. For any polynomial $p(t) = \sum a_kt^k$ of degree $2n$ we have the formulae $a_k = \sum_{j=0}^{2n} c_{j,k}p(j)$, where $c_{j,k}$ can be found by solving the linear system: $\Sigma a_kj^k = p(j)$, $0 \leqq j \leqq 2n$. Now $f(t_1, \cdots, t_m)$ can be put into form (25) for each r and by the above, we can write $a_{k,r}(t_1, \cdots, \hat{t}_r, \cdots, t_m) = \sum_{j=0}^{2n} c_{j,k}f(t_1, \cdots, j, \cdots, t_m)$.

$$f(t_1, \cdots, t_m) = \sum_{k=0}^{2n} a_{k,r}(t_1, \cdots, \hat{t}_r, \cdots, t_m)t_r^k . \tag{25}$$

Hence $a_{k,r}(t_1, \cdots, \hat{t}_r, \cdots, t_m)$ is a sum of terms $||x_0 + t_1x_1 + \cdots + jx_r + t_mx_m||^{2n}$ each of which is continuous in $t_1, \cdots, \hat{t}_r, \cdots, t_m$. We can now apply Lemma 9 and conclude that $f(t_1, \cdots, t_m)$ is a polynomial in $t_1, \cdots, t_m$ jointly. As $|f(t_1, \cdots, t_m)| \leqq (||x_0|| + |t_1|\,||x_1|| + \cdots + |t_m|\,||x_m||)^{2n}$, f has degree at most $2n$.

THEOREM 11. *If an identity of the form* (17) *(with restrictions on constants as in Theorem* 4*) holds for one space in $\mathscr{P}_{2n}$, then it holds for all spaces in $\mathscr{P}_{2n}$. In particular, an identity holds in $\mathscr{P}_{2n}$ if and only if it holds in Hilbert space.*

Proof. An ordered p-tuple $(i_0, \cdots, i_{p-1})$ of nonnegative integers is a p-partition of d if $i_0 + \cdots + i_{p-1} = d$. For each p and d there are a finite number of such partitions. Suppose (17) holds for X in $\mathscr{P}_{2n}$. The ith $(m + 1)$-partition of t, $(\pi(t, i, 0), \cdots, \pi(t, i, m))$ will be

called $\pi(t, i)$ for short. We write $\pi(n, i) = \pi(i)$. In (17), we restrict x_i to a one-dimensional subspace of X generated by x, $\|x\| = 1$: $x_i = z_i x$, $\|x_i\| = |z_i|$, complex z_i. We obtain (26), which can be rewritten as (27) using $|z|^{2n} = (|z|^2)^n$.

$$\sum_{k=0}^{r} a_k |c_k(0)z_0 + \cdots + c_k(m)z_m|^{2n} = 0 . \tag{26}$$

$$\sum_{k=0}^{r} a_k \left(\sum_{i=0}^{m} \sum_{j=0}^{m} c_k(i)\bar{c}_k(j) z_i \bar{z}_j \right)^n = 0 . \tag{27}$$

Index the $(m+1)$-partitions of n from 1 to s, then (27) can be written as (28); where $c_k^{\pi_p}$ denotes $c_k(0)^{\pi_p(0)} \cdots c_k(m)^{\pi_p(m)}$, z^{π_p} denotes $z_0^{\pi_p(0)} \cdots z_m^{\pi_p(m)}$, $\bar{c}_k^{\pi_q}$ and $\bar{z}^{\pi_p}$ are defined analogously, where the double sum ranges independently over all pairs of $(m+1)$-partitions of n and where $d_{p,q}$ is the positive multinomial coefficient depending on π_p and π_q.

$$\sum_{k=0}^{r} a_k \sum_{p=0}^{s} \sum_{q=0}^{s} d_{p,q} c_k^{\pi_p} \bar{c}_k^{\pi_q} z^{\pi_p} \bar{z}^{\pi_q} = 0 . \tag{28}$$

We now rewrite (28) as (29).

$$\sum_{p=0}^{s} \sum_{q=0}^{s} d_{p,q} \left(\sum_{k=0}^{r} a_k c_k^{\pi_p} \bar{c}_k^{\pi_q} \right) z^{\pi_p} \bar{z}^{\pi_q} = 0 . \tag{29}$$

A polynomial in $z_0, \bar{z}_0, \cdots, z_m, \bar{z}_m$ which vanishes identically must have vanishing coefficients. As $d_{p,q} > 0$ we deduce (30) for all partitions π_p and π_q.

$$\sum_{k=0}^{r} a_k c_k^{\pi_p} \bar{c}_k^{\pi_q} = 0 . \tag{30}$$

Now let Y be a space in $\mathscr{P}_{2n}$ and fix elements $x_0, \cdots, x_m$ in Y. Write $f(z_0, \cdots, z_m) = \|z_0 x_0 + \cdots + z_m x_m\|^{2n}$ for complex $z_k = v_k + iw_k$, (v_k, w_k real). Then $f(z_0, \cdots, z_m) = \|v_0 x_0 + \cdots + v_m x_m + w_0(ix_0) + \cdots + w_m(ix_m)\|^{2n}$ is a polynomial of degree $2n$ in $v_0, \cdots, v_m, w_0, \cdots, w_m$ by Lemma 10. From $2v_k = z_k + \bar{z}_k$, $2iw_k = z_k - \bar{z}_k$, it follows that $f(z_0, \cdots, z_m)$ is a polynomial in z_k and $\bar{z}_k$ of degree $2n$. Rewrite $f(z_0, \cdots, z_m)$ in form (31), where the sum is taken over all $(m+1)$-partitions of t and $2n - t$, indexed by j and k respectively, where the $b_{t,j,k}$'s are the coefficients and the condensed $z^{\pi(t,j)}$ is as before.

$$f(z_0, \cdots, z_m) = \sum_{t=0}^{2n} \sum_{\pi(t,j)} \sum_{\pi(2n-t,k)} b_{t,j,k} z^{\pi(t,j)} \bar{z}^{\pi(2n-t,k)} . \tag{31}$$

Because $f(z_0, \cdots, z_m) = f(e^{i\theta}z_0, \cdots, e^{i\theta}z_m)$, (31) does not depend on θ. Viewing (32) as a polynomial in $e^{i\theta}$ having as coefficients polynomials in z and $\bar{z}$, it follows that $b_{t,j,k} = 0$ unless $t = n$.

$$\sum_{t=0}^{n} e^{(2t-2n)i\theta} \sum_{\pi(t,j)} \sum_{\pi(2n-t,j)} b_{t,j,k} z^{\pi(t,j)} \bar{z}^{\pi(2n-t,k)} . \tag{32}$$

We can thus rewrite (31) as (33), and insert this into (17).

$$f(z_0, \cdots, z_m) = \sum_{p=0}^{s} \sum_{q=0}^{s} b_{p,q} z^{\pi p} \bar{z}^{\pi q} . \tag{33}$$

We obtain (34).

$$\begin{aligned} &\sum_{k=0}^{r} a_k \| c_k(0)x_0 + \cdots + c_k(m)x_m \|^{2n} \\ &\quad = \sum_{k=0}^{r} a_k \Big(\sum_{p=0}^{s} \sum_{q=0}^{s} b_{p,q} c_k^{\pi p} \bar{c}_k^{\pi q} \Big) = \sum_{p=0}^{s} \sum_{q=0}^{s} b_{p,q} \sum_{k=0}^{r} a_k c_k^{\pi p} \bar{c}_k^{\pi q} = 0 . \end{aligned} \tag{34}$$

This last equality is a consequence of (30).

6. The parallelopiped law. Frechét [5] proved that any Banach space which satisfies (35) for all elements x, y, and z is a Hilbert space.

$$\begin{aligned} &\| x + y + z \|^2 - \| x + y \|^2 - \| x + z \|^2 - \| y + z \|^2 \\ &\qquad + \| x \|^2 + \| y \|^2 + \| z \|^2 = 0 . \end{aligned} \tag{35}$$

Jordan and von Neumann reduced (35) to (1) by putting $z = -y$ and proving that the new condition is as strong as the old. The same sort of reduction applied to Theorem 12 will lead to a generalized parallelopiped law.

THEOREM 12. *If X is in $\mathscr{P}_{2n}$ and $m > 2n$, then for all elements $x_1, \cdots, x_m$, equation* (36) *holds, where the inner sum is taken over all k-tuples $1 \leqq i_1 < \cdots < i_k \leqq m$.*

$$\sum_{i=1}^{m} (-1)^k \sum \| x_{i_1} + \cdots + x_{i_k} \|^{2n} = 0 . \tag{36}$$

Proof. In light of Theorem 11 it is sufficient to prove the theorem for elements in Hilbert space, indeed, we need only prove that (36) holds for all complex numbers z_j. The sum on the left-hand side is, in any case, a polynomial in the z_j's and $\bar{z}_j$'s.

$$\sum_{i=1}^{m} (-1)^k \Sigma | z_{i_1} + \cdots + z_{i_k} |^{2n} = 0 . \tag{37}$$

A monomial $z_{j_1}^{r_1} \cdots z_{j_s}^{r_s} \cdot \bar{z}_{j_1}^{t_s} \cdots \bar{z}_{j_s}^{t_s}$ with $\Sigma r_k = \Sigma t_k = n$, $r_k + t_k > 0$ will occur in $| z_{i_1} + \cdots + z_{i_k} |^{2n}$ either with multinomial coefficient

$$(n!)^2/(r_1!) \cdots (r_s!)(t_1!) \cdots (t_s!)$$

or not at all depending on whether the j_l's are contained in the i_k's or not. Because $r_k + t_k > 0$, $s \leqq 2n < m$, hence each monomial occurs

in (37). Indeed, for $k \geqq s$, the j_l's are contained in precisely $\binom{m-s}{k-s}$ k-tuples and so, altogether, a monomial $z_{j_1}^{r_1} \cdots z_{j_s}^{r_s} \bar{z}_{z_1}^{t_1} \cdots \bar{z}_{j_s}^{t_s}$ will occur in (37) with total coefficient (38). As $\sum_{j=0}^{m-s} (-1)^j \binom{m-s}{j} = 0$, the sum in (38) vanishes and so so (37) is proved.

$$\sum_{k=s}^{m} (-1)^k \binom{m-s}{k-s} \frac{(n!)^2}{r_1! \cdots r_s! t_1! \cdots t_s!} . \tag{38}$$

The identity holds in Hilbert space and hence in all $\mathscr{P}_{2n}$.

THEOREM 13 (The parallelopiped law). *If X is in $\mathscr{P}_{2n}$ and $r>n$, then for all elements $x_1, \cdots, x_r$, the identity* (39) *holds in X, where the inner sum is taken over all choices of sign $\varepsilon_i = \pm 1$ and all ordered k-tuples with $1 \leqq i_1 < \cdots < i_k \leqq r$.*

$$\sum_{k=1}^{r} (-1)^k 2^{r-k} \Sigma \| \varepsilon_1 x_{i_1} + \cdots + \varepsilon_k x_{i_k} \|^{2n} = 0 . \tag{39}$$

Proof. In Theorem 12, let $m = 2r$, $x_{2k-1} = x_k'$ and $x_{2k} = -x_k'$ for $1 \leqq k \leqq r$. Each term $\varepsilon_1 x_{i_1}' + \cdots + \varepsilon_k x_{i_k}'$ will appear in (37) 2^{r-k} times depending on the joint inclusion or exclusion of the pairs $x_{2j-1} + x_{2j}$ for $j \neq i_l$. If the primes are dropped, (36) becomes (39) and the result follows.

Another identity which is satisfied in $\mathscr{P}_{2n}$ is the following.

THEOREM 14. *If X is in $\mathscr{P}_{2n}$ and $4k > 2n$, then for all real a_i with $\sum_{i=1}^{m} a_i = 0$, equation* (40) *holds for all elements $x_1, \cdots, x_m$.*

$$\Sigma a_{i_1} \cdots a_{i_{2k}} \| x_{i_1} \pm x_{i_2} \pm \cdots \pm x_{i_{2k}} \|^{2n} = 0 . \tag{40}$$

In (40), the inner sum is taken over all choices of sign ± 1 as the i_j's range independently from 1 to m; the sum has $2^{2k-1} m^{2k}$ terms.

The proof is reserved for [11]. Krivine [9] introduced an inequality which determines whether a space X is isometrically isomorphic to a subspace of L_p. For technical reasons, the proof in [9] fails when $p = 2n$. Theorem 14 illustrates what happens to Krivine's inequality in this case. Further implications will be considered in [11].

7. **Koehler's G_{2n} spaces.** In 1970, Koehler [8] defined a G_{2n} space to be a complex Banach space on which a form $\langle x_1, \cdots, x_{2n} \rangle$ is defined satisfying (41)-(44).

$$\langle x, \cdots, x \rangle = \| x \|^{2n} \tag{41}$$

(42) $$\overline{\langle x_1, \cdots, x_{2n}\rangle} = \langle x_{2n}, \cdots, x_1\rangle$$

(43) $\langle\ \rangle$ is linear in $x_1, \cdots, x_n$

(44) $\langle\ \rangle$ is conjugate linear in $x_{n+1}, \cdots, x_{2n}$.

Various properties of G_{2n} spaces are then discussed which parallel the development of Hilbert spaces. Among them are a proof that any G_{2n} space satisfies (39) and the construction through a polarization formula of a form $\langle x_1, \cdots, x_{2n}\rangle$ on any complex Banach space which satisfies (39). Thus, by Theorem 6, we can identify G_{2n} and P_{2n} and deduce the following corollary. (Note that one direction is immediate upon consideration of $\langle x + ty, \cdots, x + ty\rangle$.)

COROLLARY 15. *A form $\langle x_1, \cdots, x_{2n}\rangle$ satisfying* (41)-(44) *can be defined on a complex Banach space if and only if $\|x + ty\|^{2n}$ is a polynomial in t for every x and y in X.*

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