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A NASH-MOSER-TYPE IMPLICIT FUNCTION THEOREM AND NON-LINEAR BOUNDARY VALUE PROBLEMS

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The main objective of this paper is to formulate an implicit function theorem for Frechet spaces which is suitable for nonlinear systems of partial differential equations with prescribed boundary conditions. The applications are discussed in connection with deformation theory where such problems arise naturally and are of fundamental importance. Furthermore, their linearizations are certain second-order perturbations of second-order elliptic noncoercive boundary value problems. The last part of the paper deals with developing a general theory which covers these cases.

In [12] J. Moser gives a rather general method for the construction of solutions of nonlinear differential equations whose linearizations lose derivatives. His result is similar to that first formulated by J. Nash in [12] in connection with the isometric embedding of Riemannian manifolds. The recent progress in the theory of pseudo-complex structures has further emphasized the importance of the Nash-Moser technique. Various generalizations of this approach have been successfully used by R. Hamilton for the study of certain nonlinear complexes of partial differential operators (cf. [2]). Hamilton's version of a Nash-Moser-type inverse function theorem has enabled M. Kuranishi to construct a finite-dimensional universal family of deformations of pseudo-complex structures on a strongly pseudo-convex pseudo-complex compact manifold (cf. [10]).

One distinctive feature of the nonlinear problems which appear in the areas mentioned so far is that they are free of boundary conditions, although, as it has been demonstrated by R. Hamilton, the construction of inverses of the linearizations is very often achieved by considering linear boundary value problems. For example, the deformation theory developed by M. Kuranishi in [10] takes place on a compact C^∞ manifold M_0 which is the boundary of a complex manifold M. Hence the relevant nonlinear systems of partial differential equations naturally have no boundary conditions imposed on them. On the other hand, it is shown in [2] that if $H^1(M, T') = 0$, where T' is the holomorphic tangent bundle, and the Levi form on M_0 never has exactly one negative eigenvalue, then for any complex structure M_ω on M sufficiently close to the given structure one can find a C^∞ diffeomorphism f of M into the ambient manifold M' so that $f\colon M_\omega \to M'$ is complex analytic. Here ω is a

(0, 1) T'-valued C^∞ form on the closure $\bar{M}$. This shows that every sufficiently small complex structure on M can be obtained by wigging M_0 into M'. If, in terms of local coordinates $z = (z^1, \cdots, z^n)$ we write $f: M \to M'$ as $f(z) = (f'(z), \cdots, f^n(z))$ and set $\partial f = (\partial f^\alpha / \partial z^\beta)_{1 \leq \alpha, \beta \leq n}$, $\bar{\partial} f = (\partial f^\alpha / \partial \bar{z}^\beta)_{1 \leq \alpha, \beta \leq n}$, then th corresponding nonlinear problem can be stated as follows: given an integrable form ω find a diffeomorphism f such that $\omega = (\partial f)^{-1} \cdot \bar{\partial} f$. Except for the fact that f is required to smooth up to M_0, there are no other boundary conditions imposed on the solution of this nonlinear system.

In [3] the author considered the following question: Is it possible to extend every small deformation of M_0 to a complex structure on M? This means that one has to solve the nonlinear system $\bar{\partial}\omega - [\omega, \omega] = 0$ where the solution ω must satisfy some prescribed conditions on M_0. Here $\bar{\partial}$ is the exterior differentiation operator with respect to conjugates of holomorphic coordinates and [,] is the Poisson bracket.

The main result appears in § 1. As an application, in § 2 we solve the system $\square\omega - \bar{\partial}^*[\omega, \omega] = f - Hf$, $t(\omega) = \varphi$, $t(\bar{\partial}^*\omega) = \psi$, $H\omega = 0$ where f, φ, ψ are given T'-valued $C^\infty(0, 1)$ forms with sufficiently small Sobolev k-norms for some positive integer k, $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ with respect to some hermitian metric on M', $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, $t(\omega)$ is the complex tangential part of $i^*\omega$ ($i: M_0 \to M'$ is the given embedding and H is the projection on the space of harmonic forms satisfying $t(\omega) = 0$). The first important implication of the solvability of this system is an alternate method of deriving the extension problem for CR structures. A new feature of this approach is the fact that one can extend to a complex structure represented by a form ω for which $H\omega = \bar{\partial}^*\omega = 0$. That makes the extension of a small deformation of M_0 unique and part of the universal family for the set of all small deformations of a manifold with boundary (cf. [6]).

The second implication is the solvability of the system $\omega - N\bar{\partial}^*[\omega, \omega] = \psi$, $t(\omega) = t(\psi) = 0$, where ψ is given and N is the dual Neumann operator. In the case of compact manifolds without boundary Kuranishi [10] has already demonstrated the importance of this system (with N replaced by the classical Green's operator) in deformation theory. The fact that one can also solve it for manifolds with boundary under appropriate assumptions on the eigenvalues of the Levi form is crucial (as in Kuranishi's case) to the construction of universal families of complex structures on $\bar{M}$ which have M_0 fixed. The general theory of such deformations will be treated in a future publication.

It is worth noting that the theorem in § 1 is also applicable to nonlinear problems which have already been treated by Hamilton and

Kuranishi (cf. [1] and [10]). As an example, we prove Hamilton's theorem in § 2.

The linearization at ω of the operator $\square\omega - \bar{\partial}^*[\omega, \omega]$ is $\mathscr{L}u = \square u - 2\bar{\partial}^*[\omega, u]$. One has to solve the linear boundary value problem $\mathscr{L}u = g - Hg$, $t(u) = t(\bar{\partial}^* u) = 0$ with the appropriate estimate on u (cf., (1.6) of this paper) in order to apply the general theorem of § 1. This problem is very similar to the elliptic noncoercive linear boundary value problems considered by J. J. Kohn and L. Nirenberg in [8]. The difference is that $\mathscr{L} - \mathscr{L}^*$ is no longer an operator of order lower than the order of $\mathscr{L}$. In § 3 we show that this type of system can also be handled by the Kohn-Nirenberg methods.

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1. The main result. Let E and F be graded Frechet spaces with fundamental systems of norms which will be denoted by $\|\ \|_s$, $s \in \boldsymbol{Z}^+$, for both E and F. Here $\boldsymbol{Z}^+$ is the set of positive integers. For $s \geqq r$ we have $\|\ \|_s \geqq \|\ \|_r$. We assume that for each real positive number t there are smoothing operators on E and F denoted by the same symbol $S(t)$, i.e., $S(t): E \to E$ is an endomorphism such that for $r \leqq s$ there exists a constant $c_{s,r} > 0$ for which the following inequalities hold for all $x \in E$:

$$(1.1) \qquad \|S(t)x\|_s \leqq c_{s,r} t^{s-r} \|x\|_r$$

$$(1.2) \qquad \|x - S(t)x\|_r \leqq c_{s,r} t^{r-s} \|x\|_s \,.$$

If M is a compact manifold (with or without boundary) and V is a vector bundle over M with a metric along the fibers, then the space of C^∞ sections of V over M is an example of such a Frechet space where $\|\ \|_s$ is the Sobolev s-norm. In this case the smoothing operators were first introduced by J. Nash in [11]: if $v \in C_0^\infty(\boldsymbol{R}^m)$, then

$$S(t)v = t^m \int \chi(ty) v(x - y) dy$$

where $\chi(y)$ is a function whose Fourier transform $\hat{\chi}(\xi)$ is identically equal to one for $|\xi| < 1/2$ and identically equal to zero for $|\xi| > 1$.

For $s_1 \in \boldsymbol{Z}^+$ we define $U(s_1, a) = \{x \in E: \|x\|_{s_1} < a\}$. Let $G: U(s_1, a) \to F$ be a map for which the derivative $G'(x): E \to F$, $G'(x)(y) = \lim_{u \to 0} u^{-1}[G(x + uy) - G(x)]$, exists for all $x \in U(s_1, a)$. We now assume that $G(x)$ has some smoothness properties, i.e., there are integers m and q and real numbers d and l with $l > 1$, $d < l$ and for each $s \in \boldsymbol{Z}^+$ there are continuous increasing functions $\alpha_s(\tau) > 0$, $\beta_s(\tau) > 0$

on the positive real line $\boldsymbol{R}^+$ with $\beta_s(\tau) \leqq C_s \tau^d$ for some constant $C_s > 0$ such that the following conditions are satisfied:

$$\| G(x) \|_s \leqq \alpha_s(\| x \|_k)(\| x \|_{s+m} + 1) \tag{1.3}$$

for all s and k with $2k > s + q$

$$\| G(x + y) - G(x) - G'(x)(y) \|_s \leqq \alpha_s(\| x \|_{s+m}) \| y \|_{s+m}^l \tag{1.4}$$

for all $s \in \boldsymbol{Z}^+$ and $x \in U(s_1, a)$, $y \in E$ with $x + y \in U(s_1, a)$

$$\| G'(x)(y) \|_s \leqq \alpha_s(\| x \|_{s+m}) \| y \|_{s+m} \tag{1.5}$$

for all $s \in \boldsymbol{Z}^+$, $x \in U(s_1, a)$ and $y \in E$.

(1.6) There is a subspace $B \subset E$ such that for all $x \in U(s_1, a)$ the equation $G'(x)(y) + G(x) = 0$ has a solution $y \in B$ with

$$\| y \|_s \leqq \alpha_s(\| x \|_k)(\| G(x) \|_{s+m} + \beta_s(\| x \|_{s+m}) \| G(x) \|_{k-m})$$

for all s and k with $2k > s + q$.

REMARK 1.1. It is easy to see that (1.3)-(1.6) are based on properties satisfied by nonlinear partial differential operators. In this case m serves as a bound on the order of G and the number of derivatives lost by the solutions of the corresponding linearized differential equation; q is the dimension of the manifold over which G is defined, $l = 2$ and k is chosen so large that one can use the Sobolev inequalities. Condition (1.3) in the general case of Frechet spaces is the definition of a tame map. This concept has been introduced by R. Hamilton.

THEOREM. *If the map G has the properties* (1.3)-(1.6) *then for each $z \in U(s_1, b)$, $b = \min\{a, c_{k,k}^{-1} a\}$, there exists $w \in U(k, a)$ with $G(w) = 0$ and $w - z \in B$ provided $\| G(z) \|_k$ is sufficiently small and k is a sufficiently large integer depending only on m, q, d, and l.*

Proof. We will construct w as the limit of a sequence $x_0, x_1, \cdots, x_n, \cdots$ of approximate solutions.

Let $t_0 > 1$, $t_1, \cdots, t_n, \cdots$ be a sequence of real numbers defined by $t_{n+1} = t_n^\delta$, $\max\{1, l-1, d\} < \delta < l$. Let t_0 be so large that $\|z\|_{\lambda+k} \leqq t_0^\lambda$ for $\nu = (1/\min\{\delta - 1, \delta - d)[((d + 2)m + \delta)/\delta] < \lambda \leqq \lambda_0 < k - m - q$. The choice of λ_0 will be specified later.

Set $x_0 = z$. Assume that $x_0, x_1, \cdots, x_n$ have been determined in such a way that the following conditions hold for $p \leqq n$:

$$\| x_p \|_{\lambda+k} \leqq t_p^\lambda \quad \text{for} \quad \nu < \lambda \leqq \lambda_0 \tag{1.7}$$

$$\| x_p \|_k < b \tag{1.8}$$

(1.9) $x_p = x_{p-1} + y_{p-1}$ where $y_{p-1} \in B$ is the solution of (1.6) with

$$x = S(t_p)x_{p-1} .$$

For $p = 0$ (1.7) and (1.8) are satisfied by the choice of x_0 and t_0 and (1.9) is vacuous. Let $y_n \in B$ be the solution of (1.6) with $x = S(t_{n+1})x_n$ and let $x_{n+1} = x_n + y_n$. Observe that $x_p - x_0 \in B$ for all $p \leqq n + 1$. We proceed to verify (1.7) and (1.8) for $p = n + 1$.

From (1.1), (1.3), (1.6), (1.7), and (1.8) we obtain the inequalities

$$\begin{aligned}
||x_{n+1}||_{\lambda+k} &\leqq ||x_n||_{\lambda+k} + ||y_n||_{\lambda+k} \\
&\leqq t_n^\lambda + \alpha_{\lambda+k}(||S(t_{n+1})x_n||_k)(||G(S(t_{n+1})x_n)||_{\lambda+k+m}) \\
&\quad + \beta_{\lambda+k}(||S(t_{n+1})x_n||_{\lambda+k+m})||G(S(t_{n+1})x_n||_{k-m}) \\
&\leqq t_n^\lambda + \alpha_{\lambda+k}(c_{k,k}b)[\alpha_{\lambda+k+m}(c_{k,k}b)(c_{\lambda+k+2m,\lambda+k+2m}t_n^\lambda t_{n+1}^{2m} + 1) \\
&\quad + c_{k,k}bC_{\lambda+k}c_{\lambda+k+m,\lambda+k+m}t_n^{d\lambda}t_{n+1}^{dm}] .
\end{aligned}$$

Since $\nu < \lambda$, $\lambda + 2m/\delta \leqq \delta\lambda - 1$ and $d(\lambda + m/\delta) \leqq \delta\lambda - 1$. Hence the above expression is clearly dominated by t_{n+1}^λ if t_0 is sufficiently large. This shows that (1.7) holds for $p = n + 1$.

For any $x \in U(k, b)$ and any t we have by (1.4) and (1.5)

$$\begin{aligned}
||G(S(t)x)||_s &\leqq ||G(x)||_s \\
&\quad + \alpha_s(||x||_{s+m})||x - S(t)x||_{s+m}(1 + ||x - S(t)x||_{s+m}^{l-1}) .
\end{aligned} \tag{1.10}$$

From (1.1) and (1.2) we also obtain

$$\begin{aligned}
||G(S(t)x)||_{k+m} &\leqq ||S(t)G(S(t)x)||_{k+m} \\
&\quad + ||G(S(t)x) - S(t)G(S(t)x)||_{k+m} \\
&\leqq c_{k+m,k-m}t^{2m}||G(S(t)x)||_{k-m} \\
&\quad + c_{k+m,k+m+\lambda}t^{-\lambda}||G(S(t)x)||_{k+m+\lambda} .
\end{aligned} \tag{1.11}$$

Then for some constant c_k' and for λ between ν and λ_0 (1.2), (1.3), (1.6), (1.7), (1.8), (1.10), and (1.11) imply the inequality

$$\begin{aligned}
||x_{n+1} - x_n||_k &= ||y_n||_k \\
&\leqq c_k'(t_{n+1}^{\tau m}||G(x_n)||_{k-m} + t_{n+1}^{-\lambda l}t_n^{l\lambda} + t_{n+1}^{-\lambda}t_n^{\tau m+\lambda}) , \\
&\qquad \tau = \max(2, d) .
\end{aligned} \tag{1.12}$$

We observe that in order to apply (1.7) λ must satisfy $\nu < \lambda + \tau m \leqq \lambda_0$.

On the other hand, if c_k' stands for various constants depending on k, we obtain from (1.2), (1.4) (with $s = k - m$, $x = S(t_n)x_{n-1} + y_{n-1}$, $y = x_{n-1} - S(t_n)x_{n-1}$), (1.5) (with the same data), (1.7) and (1.8) the estimate

$$\begin{aligned}
\|G(x_n)\|_{k-m} &= \|G(x_{n-1} + y_{n-1})\|_{k-m} \\
&= \|G(S(t_n)x_{n-1} + y_{n-1} + x_{n-1} - S(t_n)x_{n-1})\|_{k-m} \\
&\leqq c'_k(\|G(S(t_n)x_{n-1} + y_{n-1})\|_{k-m} + t_n^{-\lambda}t_{n-1}^{\lambda} + t_n^{-\lambda l}t_{n-1}^{\lambda l}) .
\end{aligned} \tag{1.13}$$

Again by (1.4) with $s = k - m$, $x = S(t_n)x_{n-1}$, $y = y_{n-1}$ and by (1.9) we have

$$\begin{aligned}
&\|G(S(t_n)x_{n-1} + y_{n-1})\|_{k-m} \\
&\leqq c'_k(\|G(x_{n-1}) - G(S(t_n)x_{n-1})\|_{k-m} + \|y_{n-1}\|_k^l) .
\end{aligned} \tag{1.14}$$

Another application of (1.4) and (1.5) with $s = k - m$, $x = x_{n-1}$ and $y = x_{n-1} - S(t_n)x_{n-1}$ together with (1.2) and (1.7) implies that one can estimate the first term of the right-hand side of (1.14) by $t_n^{-\lambda}t_{n-1}^{\lambda}$ since $t_n^{-l\lambda}t_{n-1}^{l\lambda} = t_{n-1}^{-l(\delta-1)\lambda} \leqq t_{n-1}^{-(\delta-1)\lambda} = t_n^{-\lambda}t_{n-1}^{\lambda}$. By combining (1.12), (1.13), and (1.14) we obtain the inequality

$$\|x_{n+1} - x_n\|_k \leqq c'_k(t_{n+1}^{\tau m}t_n^{-\lambda}t_{n-1}^{\lambda} + t_{n+1}^{2m}\|x_n - x_{n-1}\|_k^l) . \tag{1.15}$$

We set $\varepsilon_p = t_p^{\mu}\|x_p - x_{p-1}\|_k$ for $p \leqq n + 1$ where the number μ will be determined below. Then (1.15) becomes

$$\varepsilon_{n+1} \leqq c'_k(t_{n+1}^{\tau m+\mu}t_n^{-\lambda}t_{n-1}^{\lambda} + t_{n+1}^{\tau m+\mu}t_n^{-l\mu}\varepsilon_n^l) . \tag{1.16}$$

We first choose μ so that $\tau m\delta + (\delta - l)\mu \leqq 0$. This can be done since $\delta < l$. With the choice of μ we can choose λ in such a way that $(\tau m + \mu)\delta - \delta^{-1}(\delta - 1)\lambda \leqq -1$. We observe that λ and μ do not depend on k. So if k is sufficiently large we still have $\lambda_0 < k - m - q$. Let $c_k > \max\{1, c'_k\}$ and let $\rho = (1 - l)^{-1}$. It is clear from (1.12) that $\|G(z)\|_{k-m}$ is sufficiently small and t_0 sufficiently large; then $\varepsilon_1 \leqq c_k^{\rho} = c''_k$. From (1.16) we can conclude by induction that $\varepsilon_{n+1} \leqq c''_k$, i.e.,

$$\|x_{n+1} - x_n\|_k \leqq c''_k t_{n+1}^{-\mu} . \tag{1.17}$$

Furthermore, (1.13) and (1.14) imply

$$\|G(x_n)\|_{k-m} \leqq c''_k t_n^{-l\mu} \tag{1.18}$$

if λ is sufficiently large.

We are now in a position to verify (1.8) for $p = n + 1$.

$$\|x_{n+1} - x_0\|_k \leqq \sum_{p=0}^{n+1} \|x_p - x_{p-1}\|_k \leqq c''_k \sum_{p=0}^{n+1} t_p^{-\mu} \leqq c''_k t_0^{-\mu} \sum_{p=0}^{\infty} (t_p t_0^{-1})^{-\mu} .$$

If t_0 is sufficiently large the series on the right-hand side converges and $c''_k t_0^{-\mu} \sum_{p=0}^{\infty} (t_r t_0^{-1})^{-\mu} < b/2$. Since $x_0 = z$ we also have $\|x_{n+1}\|_k < b$. This completes the induction step. The sequence $x_0, x_1, \cdots, x_n, \cdots$ satisfies (1.7), (1.8), (1.9), (1.17), and (1.18). In particular, (1.17) shows that $x_0, x_1, \cdots, x_n, \cdots$ is Cauchy in $\|\ \|_k$ since

$$\|x_{n+p} - x_n\|_k \leqq \sum_{\nu=n}^{n+p-1} \|x_{\nu+1} - x_\nu\|_k \leqq c_k'' \sum_{\nu=n}^{n+p-1} t_{\nu+1}^{-\mu} \leqq \text{const. } t_{n+1}^{-\mu} .$$

Our next task is to show that $x_0, x_1, \cdots, x_n, \cdots$ is a Cauchy sequence in every $\|\ \|_s$-norm. In fact, we will prove that for any $s \in \boldsymbol{Z}^+$ there is a constant c_s such that

$$(1.19)_s \qquad \|x_{n+1} - x_n\|_s \leqq c_s t_{n+1}^{-\mu}$$

$$(1.20)_s \qquad \|G(x_n)\|_{s-m} \leqq c_s t_n^{-l\mu}$$

if n is sufficiently large.

We note that these statements for $s = k$ are given by (1.17) and (1.18). In the sequel all constants depending on s will be denoted by c_s.

First of all, we show that $(1.19)_s$ implies

$$(1.21)_s \qquad \|x_n\|_{2s-\gamma} \leqq c_s t_n^{\sigma s} , \quad \gamma = m + q + 1 , \quad \sigma = \max\{1, d\} .$$

By (1.7) we see that $(1.21)_s$ holds for $s = k$. Since $\|x_n - x_0\|_{2s-\gamma} \leqq \sum_{j=1}^{n} \|y_{j-1}\|_{2s-\gamma}$ it is enough to estimate y_{j-1} in the $\|\ \|_{2s-\gamma}$-norm. We apply (1.6) with s replaced by $2s - \gamma$, k by s and x by $S(t_j)x_{j-1}$. Since we assume $(1.19)_s$, the sequence $\|x_0\|_s, \cdots, \|x_n\|_s, \cdots$ is bounded. Hence we obtain the inequality

$$(1.22) \qquad \begin{aligned} \|y_{j-1}\|_{2s-\gamma} \leqq c_s(&\|G(S(t_j)x_{j-1})\|_{2s-\gamma+m} \\ &+ \|S(t_j)x_{j-1}\|_{2s-\gamma+m}^{d} \|G(S(t_j)x_{j-1})\|_{s-m}) . \end{aligned}$$

Next, the term $\|G(S(t_j)x_{j-1})\|_{2s-\gamma+m}$ is estimated by

$$c_s(\|S(t_j)x_{j-1}\|_{2s-\gamma+2m} + 1) .$$

For this we use (1.3) with s replaced by $2s - \gamma$ and k by s. Observe that $2s > 2s - \gamma + q$. Another application of (1.3) shows that $\|G(S(t_j)x_{j-1})\|_{s-m}$ is dominated by $c_s(\|S(t_j)x_{j-1}\|_s + 1)$. Finally, (1.1) and $(1.19)_s$ imply that $\|S(t_j)x_{j-1}\|_{2s-\gamma+2m}$ is bounded by $c_s t_j^{s-\gamma+2m}$, $\|S(t_j)x_{j-1}\|_{2s-\gamma+m}^{d}$ by $c_s t_j^{d(s-\gamma+m)}$ and $\|S(t_j)x_{j-1}\|_s$ by a constant. If k is so large that $k \geqq (l-1)^{-1}(q - m - 1)$ then for $s \geqq k$ we have $s - \gamma + 2m \leqq \sigma s$. Also, $d(s - \gamma + m) = d(s - q - 1) \leqq \sigma s$ since $d < l$. Hence $\|y_{j-1}\|_{2s-\gamma} \leqq c_s t_j^{\sigma s}$. This proves $(1.21)_s$.

We further claim that $(1.19)_s$ and $(1.20)_s$ imply $(1.19)_{s+1}$. We again use (1.3) to estimate $x_{n+1} - x_n = y_n$. This gives (1.22) with $2s - \gamma$ replaced by $s + 1$ and j by $n + 1$. An application of (1.11) with $s + 1$ instead of k and t_{n+2} instead of t shows that $\|G(S(t_{n+1})x_n)\|_{s+m+1}$ can be estimated by

$$c_s(t_{n+2}^{2m+1} \|G(S(t_{n+1})x_n)\|_{s-m} + t_{n+2}^{-\lambda} \|G(S(t_{n+1})x_n)\|_{s+m+\lambda+1}) .$$

The second term can be estimated by $c_s t_{n+2}^{-\lambda} t_{n+1}^{2m+\lambda+1}$ by using (1.1) and (1.3) with s replaced by $s+m+\lambda+1$ and k by s povided $s > m+\lambda+1+q$ for $s \geqq k$. This can be achieved if k is sufficiently large. Moreover, if this is the case then λ can be chosen so that $(\delta-1)\lambda + 2m + 1 \geqq \mu$ which shows that $t_{n+2}^{-\lambda} t_{n+1}^{2m+\lambda+1} \leqq t_{n+1}^{-\mu}$. From (1.10) it follows that

$$\|G(S(t_{n+1})x_n)\|_{s-m} \leqq c_s(\|G(x_n)_{s-m}\| + t_{n+1}^{-\lambda'}\|x_n\|_{s+\lambda'}(1 + t_{n+1}^{-l\lambda'}\|x_n\|_{s+\lambda'}^{l}))\ .$$

We note that in all of the arguments given so far we have also used $(1.19)_s$ which implies the existence of a bound for $\|x_n\|_s$. If we set $\lambda' = s - \gamma$ then $\|x_n\|_{s+\lambda'} \leqq c_s t_n^{\sigma s}$. Moreover, if k is sufficiently large then for $s \geqq k$ we have $(\delta - \sigma)s \geqq \delta\mu + \delta^2(2m+1) + \gamma$ since $\delta > \sigma$. Thus $t_{n+2}^{2m+1} t_{n+1}^{-\lambda'}\|x_n\|_{s+\lambda'} \leqq c_s t_{n+1}^{-\mu}$. Hence by $(1.20)_s$ we obtain $\|G(S(t_{n+1})x_n)\|_{s+m+1} \leqq c_s t_{n+1}^{-\mu}$ if μ is sufficiently large with respect to m. Finally, $\|S(t_{n+1})x_n\|_{s+m+1}^{d}$ in (1.22) with j replaced by $n+1$ and $2s-\gamma$ by $s+1$ is estimated by $c_s t_{n+1}^{d(m+1)}\|x_n\|_s^d$. Since this term is multiplied by $\|G(x_n)\|_{s-m}$ it follows from $(1.19)_s$ and $(1.20)_s$ that $\|y_n\|_{s+1} \leqq c_s t_{n+1}^{-\mu}$. Note that $t_n^{-l\mu} \leqq t_{n+1}^{-\mu}$ because $\delta < l$. This finishes the verification of $(1.19)_{s+1}$.

Finally, we will show that $(1.19)_{s+1}$ implies $(1.20)_{s+1}$. If we rederive (1.13) and (1.14) with k replaced by $s+1$ we obtain

$$(1.23)\qquad \|G(x_n)\|_{s+1-m} \leqq c_s(t_n^{-(s+1-\gamma)}\|x_{n-1}\|_{2(s+1)-\gamma} + \|x_n - x_{n-1}\|_{s+1}^{l})\ .$$

Then $(1.20)_{s+1}$ follows from $(1.19)_{s+1}$ and $(1.21)_{s+1}$ for $s \geqq k$ if k is sufficiently large.

This completes the induction step and establishes $(1.19)_s$ and $(1.20)_s$ for all s. Since $x_0, x_1, \cdots, x_n, \cdots$ is a Cauchy sequence in every $\|\ \|_s$-norm the limit $w \in U(k, a)$ exists and $(1.20)_s$ shows that $G(w) = 0$. By construction $w - z \in B$.

REMARK 1.2. As we shall see in the next section it is sometimes necessary to replace (1.6) by a condition which only gives the existence of a suitable approximate solution of the linearized problem.

Suppose that in addition to (1.3), (1.4), and (1.5) we also assume that the following condition holds:

(1.6′) There is a subspace $B \subset E$ such that for all $x \in U(s_1, a)$ there exists $G_1(x) \in F$ for which the equation $G'(x)(y) + G_1(x) = 0$ has a solution with

$$\|y\|_s \leqq \alpha_s(\|x\|_k)(\|G_1(x)\|_{s+m} + \beta_s(\|x\|_{s+m})\|G_1(x)\|_{k-m})$$
$$\|G_1(x) - G(x)\|_s \leqq \alpha_s(\|x\|_k)\|G(x)\|_{s+m}^{l}$$

for all s and k with $2k > s + q$.

Then the conclusion of Theorem 1 is still valid. This can be easily seen as follows.

First of all, $\|G(x_n)\|_{k-m} \leqq t_n^{-l\mu}$ for all n. For $n = 0$ this is true if $\|G(x)\|_k$ is sufficiently small. Assume that the claim holds for n. From (1.13) and (1.14) we obtain

$$\|G(x_{n+1})\|_{k-m} \leqq \tilde{c}_k(\|G(S(t_{n+1})x_n)\|_k^l + \|y_n\|_k^l) . \tag{1.24}$$

(Observe that the verification of (1.8) remains the same as before.)

By writing $G(S(t_{n+1})x_n) - S(t_{n+1})G(x_n)$ as $G(S(t_{n+1})x_n) - G(x_n) + G(x_n) - S(t_{n+1})G(x_n)$ we have the estimate

$$\begin{aligned}\|G(S(t_{n+1})x_n\|_k \leqq \|S(t_{n+1})G(x_n)\|_k + \|G'(x_n)(x_n - S(t_{n+1})x_n)\|_k \\ + \alpha_k(\|x_n\|_k)\|x_n - S(t_{n+1})x_n\|_{k+m}^l .\end{aligned}$$

From (1.1), (1.2), (1.5), (1.7), and (1.8) it follows that for some constant (denoted again by $\tilde{c}_k$)

$$\|G(S(t_{n+1})x_n\|_k \leqq \tilde{c}_k t_{n+1}^m(t_n^{-l\mu} + t_{n+1}^{-\lambda}t_n^{\lambda}) .$$

Here we have also used the induction hypothesis. Since we are free to choose λ to be larger than μ (more precisely, $(\delta - 1)\lambda \geqq l\lambda$) we get

$$\|G(S(t_{n+1})x_n)\|_k \leqq \tilde{c}_k' t_{n+1}^m t_n^{-l\mu} \tag{1.25}$$

for some constant $\tilde{c}_k'$.

Furthermore, from (1.6)$'$ we have

$$\begin{aligned}\|y_n\|_k \leqq \text{const.} [\|G(S(t_{n+1})x_n\|_k + \|G(S(t_{n+1})x_n)\|_{k+m}^l \\ + t_n^{dm}(\|G(S(t_{n+1})x_n)\|_{k+m} + \|G(S(t_{n+1})x_n)\|_k^l)] .\end{aligned}$$

Note that (1.25) can be derived with k replaced by $k + m$ and t_{n+1}^m by t_{n+1}^{2m}. Hence

$$\|y_n\|_k \leqq \text{const.}\ t_{n+1}^{2ml} t_n^{-l\mu} . \tag{1.26}$$

Combining (1.24), (1.25), and (1.26) we obtain from some constant $\tilde{c}_k$

$$\|G(x_{n+1})\|_{k-m} \leqq \tilde{c}_k t_{n+1}^{2ml^2} t_n^{-l^2\mu} .$$

Now the right-hand side can be dominated by $t_{n+1}^{-l\mu}$ if t_0 is sufficiently large and μ is chosen so that $l^2\mu - 2ml^2 - 1 \geqq \delta l\mu$. This can be done since $\delta < l$.

The rest of the proof remains unchanged. We note that in (1.6)$'$ the power does note have to be l but just any $\kappa > \max\{1, l - 1, d\}$. Then all that has to be done is to construct the sequence $t_0, t_1, \cdots, t_n, \cdots$ with $\max\{1, l - 1, d\} < \delta < \min\{\kappa, l\}$.

2. Some nonlinear problems in deformation theory. Let M be an open relatively compact subset of a complex manifold M' of dimension $n \geqq 3$. We assume that the boundary M_0 of M is a C^∞ manifold of real dimension $2n-1$. Let T' be holomorphic tangent bounle of M' and $C^{p,q}(\bar{M}, T')$ be the space of all C^∞ T'-valued (p, q) forms extendible to a neighborhood of $\bar{M}$. If h is a real-valued C^∞ function on M' defining M_0, i.e., $M_0 = \{x \in M' : h(x) = 0\}$ and $dh \neq 0$ on M_0, then, in a neighborhood of M_0, every $\omega \in C^{p,q}(\bar{M}, T')$ can be uniquely expressed as $\alpha + \beta \wedge \bar{\partial} h$ where $\alpha \in C^{p,q}(\bar{M}, T')$, $\beta \in C^{p,q-1}(\bar{M}, T')$, and $\bar{\partial}$ is the exterior differentiation operator with respect to the conjugates of the local holomorphic coordinates. Let $i: M_0 \rightarrow M'$ be the embedding of M_0. Then $t(w) = i^*\alpha$ is the complex tangential part of ω and we set $\nu(\omega) = i^*\beta$.

We now make the following assumption:
(2.1) For $q < n$ and for each $x \in M_0$ the Levi form either has q positive eigenvalues or $n - q + 1$ negative eigenvalues.

Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ with respect to an inner product given by some hermitian metric g on M'. Then the space $\mathscr{H}^{p,q} = \{\omega \in C^{p,q}(\bar{M}, T') : \bar{\partial}\omega = \bar{\partial}^*\omega = 0, t(\omega) = 0\}$ is finite-dimensional and for each $f \in C^{p,q}(\bar{M}, T')$ there exists a unique form $Nf \in C^{p,q}(\bar{M}, T')$ such that $HNf = 0$, $t(Nf) = t(\bar{\partial}^* Nf) = 0$ and $\Box Nf = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})Nf = f - Hf$. Here H is the orthogonal projection on $\mathscr{H}^{p,q}$. In particular, $\bar{\partial}Nf$ is a solution of $\bar{\partial}^* u = f$ with $t(u) = 0$ provided $\bar{\partial}^* f = Hf = 0$. (Note that for any u $t(\bar{\partial}u) = 0$ if $t(u) = 0$.)

All of the above statements follow from the dual $\bar{\partial}$-Neumann problem (cf., [6] and [8]). We recall that the $\bar{\partial}$-Neumann problem asks for the existence of an operator N satisfying the boundary conditions $\nu(\omega) = \nu(\bar{\partial}\omega) = 0$. Duality is obtained by considering $*\#: C^{p,q}(\bar{M}, T') \rightarrow C^{n-p,n-q}(\bar{M}, T'^*)$ where T'^* is the holomorphic cotangent bundle, $*$ is the Hodge star-operator, and if $(g_{\alpha\bar{\beta}})$ are the components of g and $(\omega^1, \cdots, \omega^n)$ is the local representation of ω as a vector of (p, q) scalar-valued forms, then the local representation of $\#\omega \in C^{q,p}(\bar{M}, T'^*)$ is given by $(\#\omega)_\sigma = \sum_\mu g_{\sigma\bar{\mu}}\omega^{\bar{\mu}}$. It is easily seen that $t(\omega) = 0$ if and only if $\nu(*\#\omega) = 0$ and $\bar{\partial}^* = - *\#\bar{\partial}\#*$.

We also mention the basic Kohn-Morrey estimate: there exists a constant $C_0 > 0$ such that for all ω with $t(\omega) = 0$

$$(2.2) \qquad \|\omega\|^2 + \int_{M_0} |\omega|^2 dS + \|\omega\|_z^2 \leqq C_0 D(\omega, \omega) ,$$

$$D(\omega, \omega) = \|\omega\|^2 + \|\bar{\partial}\omega\|^2 + \|\bar{\partial}^*\omega\|^2 , \quad \|\omega\|_z^2 = \sum_{\sigma,\tau} \sum_{\mu,\nu} \int_{U_\sigma \cap \bar{M}} |\partial\omega_\mu^\nu / \partial z^\tau|^2 dx ;$$

$\{U_\sigma\}$ is a finite cover of $\bar{M}$ by coordinate neighborhoods, $\| \ \|$ is the L_2-norm, and ω_μ^ν are the components of ω on U_σ. Observe that different coverings of $\bar{M}$ by coordinate patches give equivalent seminorms $\|\ \|_z$.

We now assume that $p = 0$. Then $t(\omega)$ and $\nu(\omega)$ are sections of the bundles $\Lambda^{q0}T'^* \otimes (T'|M_0)$ and $\Lambda^{q-10}T'^* \otimes (T'|M_0)$, resp., i.e., they are C^∞ forms on M_0 of type $(0, q)_b$ and $(0, q-1)_b$, resp., and with values in $T'|M_0$. We have set ${}^0T' = (T'|M_0) \cap CTM_0$ where CTM_0 is the complexification of the real tangent bundle of M_0. Let $[\ ,\ ]$ denote the Poisson bracket on $C^{2,1}(\bar{M}, T')$. One can then consider the following nonlinear problem: given $f \in C^{0,1}(\bar{M}, T')$ and C^∞ forms φ and ψ of type $(0, 1)_b$ and $(0, 0)_b$, resp., find $\omega \in C^{0,1}(\bar{M}, T')$ such that

$$\square\omega - \bar{\partial}^*[\omega, \omega] = f - Hf\ , \tag{2.3}$$

$$t(\omega) = \varphi\ ,\quad t(\bar{\partial}^*\omega) = \psi\ ,\quad H\omega = 0\ .$$

The purpose of this section is to establish the existence of such a solution ω provided f, φ, and ψ are sufficiently small in some Sobolev k-norm.

We will first show that given φ and ψ there exists $\tilde{\omega}$ with $H\tilde{\omega} = 0$ for which the boundary conditions are satisfied. Let $\theta_1 \in C^{0,1}(\bar{M}, T')$ be such that $t(\theta_1) = \varphi$. Moreover, θ_1 can be chosen so that $\|\theta_1\|_k \leqq \text{const.}\,|\varphi|_k$ where $\|\ \|_k$ and $|\ |_k$ are the Sobolev k-norms over M and M_0, resp. Next, let θ_2 be such that $\bar{\partial}^*\theta_2 = -\bar{\partial}^*\theta_1$ with $t(\theta_2) = 0$. Thus $\theta_3 = \theta_1 + \theta_2$ has the property $\bar{\partial}^*\theta_3 = 0$ and $t(\theta_3) = \varphi$. Now $\theta = \bar{\partial}N\theta_3$ is a form of type $(0, 2)$ for which $\bar{\partial}^*\theta = \theta_3$, i.e., $t(\bar{\partial}^*\theta) = \varphi$. Furthermore, the same arguments used for finding θ_3 show that there is $\tilde{\psi} \in C^{0,0}(\bar{M}, T')$ with $\bar{\partial}^*\tilde{\psi} = 0$ and $t(\tilde{\psi}) = \psi$. Then $\gamma = \bar{\partial}N\tilde{\psi}$ is such that $\bar{\partial}^*\gamma = \tilde{\psi}$ and $t(\gamma) = 0$. Now $\tilde{\omega} = \bar{\partial}^*\theta + \bar{\partial}N\tilde{\psi}$ has the required properties. Moreover, the construction and the inequality $\|Nu\|_{s+1} \leqq \text{const.}\,\|u\|_s$ for each real s imply that if φ and ψ are small in $|\ |_{k+1}$-norm, then $\tilde{\omega}$ is small in $\|\ \|_k$-norm.

We are now in a position to apply the theorem of the preceding section. Let $E = F = C^{0,1}(\bar{M}, T')$, $B = \{\omega \in E\colon H\omega = t(\omega) = t(\bar{\partial}^*\omega) = 0\}$, $G(\omega) = \square\omega - \bar{\partial}^*[\omega, \omega] - f$ (we have assumed $Hf = 0$). Since G is a differential operator (1.3), (1.4), and (1.5) are obviously satisfied.

$$G'(\omega)(u) = \square u - 2\bar{\partial}^*[\omega, u]\ .$$

This follows from the properties of the Poisson bracket. The equation $\square u - 2\bar{\partial}^*[\omega, u] = w$ (and more general type of equations) will be studied in § 3. It follows from Theorem 3.5 that if for some sufficiently large integer $k \in \boldsymbol{Z}^+$ $\|\omega\|_k$ is sufficiently small, then for each w with $Hw = 0$ there exists a unique $u \in B$ with $\|u\|_s \leqq C_s(\|w\|_s + \|\omega\|_{k+s}\|w\|_0)$ where C_s is a polynomial of $\|\omega\|_{k+s}$. This establishes (1.6). Having in mind the discussion in § 3, it is worthwhile to observe that if $Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^*u, \bar{\partial}^*v) - 2([\omega, u], \bar{\partial}v)$, then by (2.2) and the fact that $[\omega, u]$ contains only the $\partial/\partial z^j$-derivatives of u we have for some constants $C_1 > 0$, $C_2 > 0$

$$C_1D(u, u) \leqq |Q(u, u)| \leqq C_2D(u, u) .$$

Hence there exists $\omega \in C^{0,q}(\bar{M}, T')$ such that $G(\omega) = 0$ and $\omega - \tilde{\omega} \in B$, i.e., ω is a solution of (2.3). The proof of the theorem of the preceding section also shows that $||\omega||_k$ is small if $\tilde{\omega}$ is small in the $|| \, ||_k$-norm. With the aid of this fact we can conclude that the solution ω is also unique. Indeed, if ω_1 and ω_2 are two solutions of (2.3), then for $\theta = \omega_1 - \omega_2$ we have $\Box\theta = \bar{\partial}^*([\theta, \omega_1] + [\omega_2, \theta])$. Since $t(\theta) = t(\bar{\partial}^*\theta) = 0$ integration by parts gives $||\bar{\partial}\theta||^2 + ||\bar{\partial}^*\theta||^2 = ([\theta, \omega_1] + [\omega_2, \theta], \bar{\partial}\theta)$. By (2.2) $||\bar{\partial}\theta||^2 + ||\bar{\partial}^*\theta||^2 \leqq \text{const.}\,(||\omega_1||_k + ||\omega_2||_k)D(\theta, \theta)$. On the other hand, $H\theta = 0$ implies $||\theta||^2 \leqq \text{const.}\,(||\bar{\partial}\theta||^2 + ||\bar{\partial}^*\theta||^2)$. Thus $D(\theta, \theta) \leqq \text{const.}\,(||\omega_1||_k + ||\omega_2||_k)D(\theta, \theta)$ which shows that $\theta = 0$ if $||\omega_j||_k$ is sufficiently small, $j = 1, 2$.

We now give some geometric applications of (2.3). First we recall some of the basic facts of deformation theory (cf. [10]).

Let $\boldsymbol{M}$ be the underlying differentiable structure of M and let $\boldsymbol{CTM}$ be the complexified tangent bundle of $\boldsymbol{M}$. An almost complex structure on $\boldsymbol{M}$ is given by a subbundle $T_1'' \subset \boldsymbol{CTM}$ such that $\boldsymbol{CTM} = T_1' \oplus T_1''$, $T_1' = \bar{T}_1''$. T_1'' is of finite distance from the given structure M if $\rho'': T_1'' \rightarrow T''$ is a bundle isomorphism where $T'' = \bar{T}'$ and $\rho'': \boldsymbol{CTM} \rightarrow T''$ is the projection. The set of all almost complex structures of finite distance from M is in one-to-one correspondence with the C^∞ bundle homomorphisms $\omega: T'' \rightarrow T'$, i.e., the set of C^∞ T'-valued forms of type (0, 1). This correspondence is given by $T_1'' \equiv T_\omega'' = \{X - \omega(X): X \in T''\}$. The almost complex structure T_ω'' is called integrable if for any two sections L_1, L_2 of T_ω'' over an open set $U \subset M$ the Lie bracket $[L_1, L_2]$ also belongs to T_ω''. A necessary and sufficient condition for integrability is $\Omega \equiv \bar{\partial}\omega - [\omega, \omega] = 0$. We will refer to integrable T_ω'''s as deformations of M. The term "small deformation" will be used if ω has a sufficiently small Sobolev k-norm for some $k \in \boldsymbol{Z}^+$.

There are similar facts for pseudo-complex or Cauchy-Riemann (or, simply, CR) structures on M_0 (cf. [3] and [9]).

An almost CR structure on M_0 is given by a C^∞ subbundle $E'' \subset \boldsymbol{CTM}_0$ of complex fiber dimension $n - 1$ such that $E' \cap E'' = \{0\}$, $E' = \bar{E}''$. As before, E'' is and (integrable) CR structure if the Lie bracket $[L_1, L_2]$ of any two sections of E'' over an open set $V \subset M_0$ is also a section of E''. The given complex structure on M' induces a CR structure on M_0 by the subbundle ${}^\circ T'' = (T''|M_0) \cap \boldsymbol{CTM}_0$.

In this case $CTM_0 = {}^\circ T' \oplus {}^\circ T'' \oplus CF$ where CF is the complexification of some one-dimensional real subbundle of TM_0. By normalization we may assume that CF is generated by a purely imaginary vector field $P = P' - P''$, $P'' = \bar{P}'$, $P' = \sum_j p^j \partial/\partial z^j$ and $P'(h) = P''(h) = 1$, i.e., if $h_j = \partial h/\partial z^j$, then $\sum_j p^j h_j = 1$. There is also a C^∞ bundle isomorphism τ: $T'|M_0 \to {}^\circ T' \oplus CF$.

We say that E'' is of finite distance from ${}^\circ T''$ if π'': $E'' \to {}^\circ T''$ is an isomorphism where π'': $CTM_0 \to {}^\circ T''$ is the projection. If φ_1: ${}^\circ T'' \to {}^\circ T' \oplus CF$ is the homomorphism defined by $\varphi_1 = -(id - \pi'')\circ(\pi''|E'')^{-1}$, then it is easy to see that $E'' = \{X - \varphi_1(X): X \in {}^\circ T''\}$. Set $\varphi = \tau^{-1}\circ\varphi_1$: ${}^\circ T'' \to T'|M_0$. Then φ is a C^∞ $T'|M_0$-valued form of type $(0, 1)_b$ and we can write

$$E'' = \{X - \tau\circ\varphi(X): X \in {}^\circ T''\}\,. \tag{2.4}$$

Conversely, if for a given φ the above formula defines an almost CR structure provided at each point $x \in M_0$ the map $\bar{\sigma}_x \circ \bar{\sigma}_x$: ${}^\circ T''_x \to {}^\circ T''_x$ $(\sigma = \pi'\circ\tau\circ\varphi)$ does not have eigenvalue 1. This is always true if φ is sufficiently small in some Sobolev k-norm. We will denote by ${}^\circ T''_\varphi$ the almost CR structure given by (2.4).

Let $\varphi = \sum \varphi^j/\partial z^j$ an a coordinate neighborhood $V \subset M_0$ and let $\theta^j = i^* dz^j + \varphi^j$, $1 \leqq j \leqq n$. Then, for each point $x \in M_0$, $\{\theta^1_x, \cdots, \theta^n_x\}$ is a base of $({}^\circ T''_{\varphi,x})^\perp = \{u \in CT^*_x M_0: u(X) = 0 \text{ for all } X \in {}^\circ T''_{\varphi,x}\}$. Using the formula $2d\theta(L_1, L_2) = L_1\cdot\theta(L_2) - L_2\cdot\theta([L_1, L_2])$ for any differential form θ of degree 1 and for all sections L_1, L_2 of CTM_0 one finds that ${}^\circ T''_\varphi$ is integrable if and only if $d\varphi^j \equiv 0 \,(\mathrm{mod}\, \theta^1, \cdots, \theta^n)$, $1 \leqq j \leqq n$.

A set of local generators for ${}^\circ T''$ is given by $Z_{\bar{j}} = \partial/\partial\bar{z}^j - h_{\bar{j}} P''(h_{\bar{j}} = \bar{h}_j)$, $\sum_j \overline{p^j} Z_{\bar{j}} = 0$. Then a set of local generators for the dual bundle ${}^\circ T''^*$ is given by $\bar{Z}^k = i^* d\bar{z}^k - \bar{p}^k i^*\bar{\partial} h$, $1 \leqq k \leqq n$, $\sum_k h_{\bar{k}} \bar{Z}^k = 0$. Thus every form ψ of type $(0, q)_b$ can be uniquely expressed as $\psi = \sum_{j_1<\cdots<j_q} \psi_{\bar{j}_1\cdots\bar{j}_q} \bar{Z}^{j_1} \wedge \cdots \wedge \bar{Z}^{j_q}$, $\sum_k \bar{p}^k \psi_{\bar{k}\bar{j}_2\cdots\bar{j}_q} = 0$. If $\bar{\partial}_b = \sum (\partial/\partial\bar{z}^k)\bar{Z}^k$ is the tangential Cauchy-Riemann operator, $\partial^\tau/\partial z^j \equiv \partial/\partial z^j - h_j P'' = \tau(\partial/\partial z^j)$ and $\varphi^k = \sum \varphi^k_{\bar{l}} \bar{Z}^l$, then a straightforward compution (cf. [9]) shows that ${}^\circ T''_\varphi$ is integrable if and only if the $T'|M_0$-valued form $\sum \Phi^k/\partial z^k = \Phi$ is equal to zero where

$$\Phi^k = \bar{\partial}_b \varphi^k - \sum_{j,l} (\partial^\tau \varphi^k_{\bar{l}}/\partial z^j)\varphi^j \wedge \bar{Z}^l + \sum_{i,j,l} h_i \varphi^i \wedge \varphi^k_{\bar{l}}(\bar{\partial}_b \bar{p}^l - (\partial^\tau \bar{p}^l/\partial z^j)\varphi^j)\,.$$

If a deformation of the fixed complex structure on M is given by T''_ω, $\omega \in C^{0,1}(\bar{M}, T')$, then a deformation of the fixed CR structure on M_0 is given by

$${}^\circ T''_\varphi = (T''_\omega | M_0) \cap CTM_0\,. \tag{2.5}$$

The extension problem discussed in [3] is the converse of the above statement for small deformations: if ${}^{\circ}T''_{\varphi}$ represents a small deformation of M_0 find $\omega \in C^{0,1}(\bar{M}, T')$ such that T''_{ω} is integrable and (2.5) holds. We will now indicate how (2.3) can be applied to this problem.

Let $\mathscr{C}$ be the set of all C^{∞} $(T'|M_0)$-valued forms φ of type $(0, 1)_b$ such that $\sum_i \varphi^i h_i = 0$. We list the following facts which have already been established in [3] and [4].

(i) Every CR structure ${}^{\circ}T''_{\varphi}$ can be extended to a complex structure T''_{θ} on a neighborhood $\mathscr{N}$ of M_0 (see also [1]), and there exists an embedding $f: M_0 \longrightarrow \mathscr{N}$ such that for the pull-back (under f) ${}^{\circ}T''_{\varphi}$ of the CR structure $[T''_{\theta}|f(M_0)] \cap CT[f(M_0)]$ on $f(M_0)$ one has $\varphi \in \mathscr{C}$. This means that after "wiggling" M_0 in a neighborhood in M' every CR structure is equivalent to one represented by a form in the set $\mathscr{C}$ (cf. [4]).

(ii) If $\omega \in C^{0,1}(\bar{M}, T')$ and $\varphi \in \mathscr{C}$ are any forms (not necessarily representing integrable almost complex and CR structures), then (2.5) holds if and only if $t(\omega) = \varphi$. Furthermore, if $t(\omega) = \varphi$, then $t(\Omega) = 0$ if and only if $\Phi = 0$ (cf. [3]).

Hence (i) and (ii) show that the extension problem for small deformations of the CR structure on M_0 reduces to solving the following first-order nonlinear system:

$$\begin{aligned} &\bar{\partial}\omega = [\omega, \omega] = 0 \\ (2.6) \qquad &t(\omega) = \varphi \qquad \text{(boundary condition)} \\ &\Phi = 0 \qquad \text{(compatibility condition).} \end{aligned}$$

By setting $f = 0$, $\psi = 0$ in (2.3) we find that if φ is sufficiently small in some Sobolev k-norm for a sufficiently large integer k, then there is a unique $\omega \in C^{0,1}(\bar{M}, T')$ such that $t(\omega) = \varphi$, $t(\bar{\partial}^*\omega) = 0$, $H\omega = 0$, and $\bar{\partial}\bar{\partial}^*\omega + \bar{\partial}^*(\bar{\partial}\omega - [\omega, \omega]) = 0$. This gives $\bar{\partial}^*\bar{\partial}\bar{\partial}^*\omega = 0$, and Stokes' theorem implies $(\bar{\partial}^*\bar{\partial}\bar{\partial}^*\omega, \bar{\partial}^*\omega) = \|\bar{\partial}^*\omega\|^2 = 0$. By another application of Stokes' theorem we have $(\bar{\partial}\bar{\partial}^*\omega, \omega) = \|\bar{\partial}^*\omega\|^2 = 0$. Thus we have found $\omega \in C^{0,1}(\bar{M}, T')$ such that $t(\omega) = \varphi$ and $\bar{\partial}^*\omega = \bar{\partial}^*\Omega = 0$, $t(\Omega) = 0$. The properties of the Poisson bracket imply $\bar{\partial}\Omega = \pm 2[\omega, \Omega]$.

If, in addition to (2.1) with $q = 2$, we also assume that $\mathscr{H}^{0,2} = H^2(M, \underline{T'^* \otimes K}) = 0$ (the section cohomology group with coefficients in the sheaf of germs of holomorphic sections of the bundle $T'^* \otimes K$, K the canonical bundle of M), then the basic estimate (2.2) and $\|\Omega\| \leqq \text{const.}(\|\bar{\partial}\Omega\|^2 + \|\bar{\partial}^*\Omega\|^2)$ for all Ω with $t(\Omega) = H\Omega = 0$ imply $\Omega = 0$ since $\|\omega\|_k$ can be estimate by $|\varphi|_k$. Furthermore, for a sufficiently small φ the solution of (2.6) is unique because $H\omega = \bar{\partial}^*\omega = 0$.

Before proceeding with the next application we briefly recall the basic features of Kuranishi's method of constructing universal families of deformations of a compact manifold M without boundary (cf. [10]). Let $\Psi = \{\omega \in C^{0,1}(M, T'): \bar\partial\omega = \bar\partial^*\omega = 0\}$. A necessary condition for a form ω to belong to the set Ψ is given by $F(\omega) = \omega - G\bar\partial^*[\omega, \omega] \in \mathscr{H}^{0,1}$, where G is the classical Green's operator. In order to find a sufficient condition one first observes that by the inverse function theorem for Banach spaces the map F can be inverted in a neighborhood $\mathscr{U}$ of zero in $C^{0,1}(M, T')$. Let $W = \mathscr{U} \cap \mathscr{H}^{0,1}$ and let (by abuse of notation) $\omega: W \to C^{0,1}(M, T')$ be the inverse of F, i.e., $F(\omega(s)) = s$ for all $s \in W$. Then $\omega(s) \in \Psi$ if and only if $s \in S = \{s \in W: H[\omega(s), \omega(s)] = 0\}$. The family $\{\omega(s): s \in S\}$ is universal since every deformation of M represented by a form θ is equivalent to a structure T''_ω with $\bar\partial^*\omega = 0$.

For a manifold M with boundary M_0 and for the deformations which leave M_0 fixed one replaces $C^{0,1}(M, T')$ by $\mathscr{B} = \{\omega \in C^{0,1}(\bar M, T'): t(\omega) = 0\}$ and G by the dual Neumann operator N. The map F: $\mathscr{B} \to \mathscr{B}$, $F(\omega) = \omega - N\bar\partial^*[\omega, \omega]$ can no longer be inverted by the inverse function theorem for Banach spaces. However, one can easily satisfy that for a given ψ in a sufficiently small neighborhood $\mathscr{U} \subset \mathscr{B}$ of zero in the $\|\ \|_k$-norm topology for some fixed integer k, a solution $\omega \in \mathscr{U}$ of $F(\omega) = \psi$ is of the form $\omega = u + H\psi$ where $\Box u - \bar\partial^*[u, u] - 2\bar\partial^*[u, H\psi] = \Box\psi + \bar\partial^*[H\psi, H\psi]$ and $Hu = t(u) = t(\bar\partial^* u) = 0$. The last problem has a solution because it is essentially the same as (2.3) since the addition of the linear perturbation $2\bar\partial^*[u, H\psi]$ is irrelevant. It turns out that the invertibility of F plays the same important role in the study of deformations which leave M_0 fixed as in the methods developed by Kuranishi in the compact case. The details will appear as part of a general theory in the work mentioned in § 0.

Another application of the results of the first section is Hamilton's theorem mentioned in the Introduction.

Let $\omega \in C^{0,1}(\bar M, T')$ be such that $\bar\partial\omega - [\omega, \omega] = 0$ and the Sobolev k-norm of ω is sufficiently small for some sufficiently large integer k. Let $\bar\partial_\omega: C^{0,q}(\bar M, T') \to C^{0,q+1}(\bar M, T')$ be the differential operator defined by $\bar\partial_\omega\theta = \bar\partial\theta - 2[\omega, \theta]$. Because of the integrability condition $\bar\partial_\omega \circ \bar\partial_\omega = 0$. Let $g_\omega = g + 0(\omega)$ be a variation of the given hermitian metric g on M'. We denote by $(\ ,\ )_\omega$ the L_2-inner product with respect to ω. Then, for a suitable choice of g_ω, the Hilbert space domain of the adjoint $\bar\partial_\omega^*$ of $\bar\partial_\omega$ is the set $C_\nu^{0,q}(\bar M, T') = \{\theta \in C^{0,q}(\bar M, T'): \nu(\theta) = 0\}$, i.e., $(\bar\partial_\omega\psi, \theta)_\omega = (\psi, \bar\partial_\omega^*\theta)_\omega$ for all $\psi \in C^{0,q-1}(\bar M, T')$ and $\theta \in C_\nu^{0,q}(\bar M, T')$. This can be seen as follows.

Let U be a coordinate neighborhood and let $\zeta^1, \cdots, \zeta^n$ be C^∞ forms of type (0, 1) on U such that $g(\zeta^i, \zeta^j) = \delta^{ij}$, $g(\zeta^i, \bar\zeta^j) = 0$, and $\bar\zeta^n = \bar\partial h$ if $U \cap M_0 \neq \varnothing$. Let $\zeta_1, \cdots, \zeta_n$ be the dual basis for vector fields on U of type (1, 0). Then $\zeta_1, \cdots, \zeta_{n-1}$ forms a basis for ${}^\circ T'$

on $U \cap M_0$, and $\zeta_n(h) = 1$. Any $\theta \in C^{0,q}(\bar{M}, T')$ can be locally written as

$$\theta = \sum_{i,J} \theta^i_{\bar{J}} \bar{\zeta}^J \otimes \zeta_i$$

where $J = (j_1, \cdots, j_q)$ with $j_1 < \cdots < j_q$ and $\bar{\zeta}^J = \bar{\zeta}^{j_1} \wedge \cdots \wedge \bar{\zeta}^{j_q}$.

Since $\overline{\zeta^\omega_k} = \bar{\zeta}_k - \sum_l \omega^l_{\bar{k}} \zeta_l$, $1 \leqq k \leqq n$, is a basis for T''_ω on U, $\bar{\partial}_\omega$ is given locally by

$$\bar{\partial}_\omega \theta = \sum_{k,i,K,J} (\varepsilon^{kJ}_K \overline{\zeta^\omega_k}(\theta^i_{\bar{J}}) + \cdots) \bar{\zeta}^K \otimes \zeta_i .$$

Here the dots stand for terms which contain no derivatives of the components of θ, $K = (k_1, \cdots, k_{q+1})$, $k_1 < \cdots < k_{q+1}$, and

$$\varepsilon^{kJ}_K = \begin{cases} 0 & \text{if } \{kJ\} \neq \{K\} \\ \text{sign of permutation } \begin{pmatrix} kJ \\ K \end{pmatrix} & \text{if } \{kJ\} = \{K\} . \end{cases}$$

Let $g_{\omega, i\bar{j}} = g(\zeta_i, \zeta_j) = \delta_{i\bar{j}} + s(\omega)_{i\bar{j}}$ be the components of the metric g with respect to the frame $\zeta_1, \cdots, \zeta_n$. Observe that $\delta_{i\bar{j}} = g(\zeta_i, \zeta_j)$. Let $g^{i\bar{j}}_\omega = \delta^{i\bar{j}} + s(\omega)^{i\bar{j}}$ be the components of the inverse matrix of $(g_{\omega, i\bar{j}})$. Then the formal adjoint $\bar{\partial}^*_\omega$ of $\bar{\partial}_\omega$ with respect to g_ω is locally expressed as

$$\bar{\partial}^*_\omega \theta = \sum_{\substack{i,k,j \\ L,J}} (\varepsilon^J_{jL} g^{k\bar{j}}_\omega \zeta^\omega_k(\theta^i_{\bar{J}}) + \cdots) \bar{\zeta}^L \otimes \zeta_i$$

$L = (l_1, \cdots, l_{q-1})$, $J = (j_1, \cdots, j_q)$.

Now by Stokes' theorem it is easy to see that $(\bar{\partial}_\omega \psi, \theta)_\omega = (\psi, \bar{\partial}^*_\omega \theta)_\omega$ for all $\psi \in C^{0,q-1}(\bar{M}, T')$ if and only if $\sum_{k,j,J} \varepsilon^J_{jL} \theta^i_{\bar{J}} g^{k\bar{j}}_\omega \zeta^\omega_k(h) = 0$ on M. Choose $s(\omega)^{i\bar{j}}$ to be such that on $U \cap M_0$ $s(\omega)^{k\bar{j}} = \overline{s(\omega)^{j\bar{k}}} = 0$ if $k, j < n$ or $k = j = n$ and $s(\omega)^{n\bar{j}} = \overline{s(\omega)^{j\bar{n}}} = \overline{\omega^n_{\bar{j}}}(1 - \overline{\omega^n_{\bar{n}}})^{-1}$ if $j < n$. The metric g_ω obtained in this way has the property thrt, in terms of the frame $\zeta_1, \cdots, \zeta_n$, $\sum_k g^{k\bar{j}}_\omega \zeta^\omega_k(h) = 0$ if $j < n$ and $\sum_k g^{k\bar{n}} \zeta^\omega_k(h) \neq 0$ (if ω is sufficiently small) on $U \cap M_0$. Hence θ is in the Hilbert space domain of $\bar{\partial}^*_\omega$ if and only if $\theta^i_{\bar{J}} = 0$ on $U \cap M_0$ whenever $n \in J$, i.e., if and only if $\theta \in C^{0,q}_\nu(\bar{M}, T')$.

From this point on we make the assumption that the Levi form of M_0 never has exactly one negative eigenvalue, i.e., at each point of M_0 there are either at least two negative eigenvalues or else they are all positive. The techniques of local integration by parts developed in [6] ahd [8] can be applied without any substantial changes to the frame $\zeta^\omega_1, \cdots, \zeta^\omega_n$ and the metric g_ω. One can then obtain a uniform Kohn-Morrey basic estimate, i.e., there exists a constant $C_0 > 0$ such that for all ω in a neighborhood of zero in the k-norm topology and all $\theta \in C^{0,1}_\nu(\bar{M}, T')$

$$|\theta| \leqq C_0(\|\theta\| + \|\bar{\partial}_\omega\theta\| + \|\bar{\partial}_\omega^*\theta\|)$$

where $|\ |$ is the L_2-norm on M_0 and $\|\ \|$ is the L_2-norm on M. Since the metrics g and g_ω are equivalent the above norms can be taken with respect to either of them.

As before, we let $\mathscr{H}_\omega^{0,1} = \{\theta \in C_\nu^{0,1}(\bar{M}, T'): \bar{\partial}_\omega\theta = \bar{\partial}_\omega^*\theta = 0\}$ and H_ω the harmonic projection. Then for each $\theta \in C^{0,1}(\bar{M}, T')$ there exists a unique $N_\omega\theta \in C^{0,1}(\bar{M}, T')$ such that

$$(2.8)\quad \begin{cases} \Box_\omega N_\omega\theta = (\bar{\partial}_\omega\bar{\partial}_\omega^* + \bar{\partial}_\omega^*\bar{\partial}_\omega)N_\omega\theta = \theta - H_\omega\theta \\ \nu(N_\omega\theta) = \nu(\bar{\partial}_\omega N_\omega\theta) = 0 \\ \|N_\omega\theta\|_s \leqq C_s\|\theta\|_{s-1} \text{ where } C_s \text{ is a polynomial of } \|\omega\|_{s+k} \end{cases}$$

(cf. § 3).

If $\psi \in C^{0,1}(\bar{M}, T')$ with $\bar{\partial}_\omega\psi = 0$ and $(\psi, \alpha)_\omega = 0$ for all $\alpha \in \mathscr{H}_\omega^{0,1}$, then $\theta = \bar{\partial}_\omega^* N_\omega\psi$ is the unique solution of

$$(2.9)\qquad \bar{\partial}_\omega\theta = \psi \quad \text{with} \quad \|\theta\|_s \leqq C_s\|\psi\|_s\,.$$

We also point out that $\mathscr{H}_0^{0,1} = H^1(M, \underline{T}')$, the first cohomology group with coefficients in the sheaf of holomorphic tangent vectors $\underline{T}'$. Thus by Lemma 11.1 of [6], p. 143, we have that if $H^1(M, \underline{T}') = 0$, then $\mathscr{H}_\omega^{0,1} = 0$ for all sufficiently small ω. In particular, (2.8) is solvable for all ψ with $\bar{\partial}_\omega\psi = 0$. From now on we also assume that $H^1(M, \underline{T}') = 0$.

Hamilton's theorem states that if M_ω is a complex structure sufficiently close to M, then there exists a diffeomorphism f of M into M' such that $f\colon M_\omega \to M'$ is complex analytic. If $z = (z', \cdots z^n)$ is a set of local coordinates and $f = (f'(z), \cdots, f^n(z))$, $\omega = \sum \omega_\alpha^\beta d\bar{z}^\alpha \otimes \partial/\partial z^\beta$ are the representations of f and ω in terms of z, then the analyticity of f with respect to M_ω means

$$(2.10)\quad \frac{\partial f^\alpha}{\partial \bar{z}^\beta} = \sum_{\gamma=1}^n \omega_\beta^\gamma \frac{\partial f^\alpha}{\partial z^\gamma}\,, \quad 1 \leqq \alpha, \beta \leqq n\,, \quad \text{or} \quad (\partial f)^{-1}\cdot\bar{\partial} f = \omega\,.$$

The C^∞ embeddings $f\colon M \to M'$ which are close to the identity can be parametrized by elements $\xi \in C^{0,0}(\bar{M}, T')$. This can be done, for example, by setting $f(p) = \exp_p(\xi(p) + \bar{\xi}(p))$ where $p \in \bar{M}$ and exp is the exponential map with respect to a Riemannian metric. In this case we denote f by $e(\xi)$ and in terms of local coordinates we have $e^\alpha(\xi) = z^\alpha + \xi^\alpha + O(|\xi|^2)$, $1 \leqq \alpha \leqq n$. Then $G(\xi) = \partial e(\xi)^{-1}\cdot\bar{\partial} e(\xi) - \omega$ is a mapping from a neighborhood of zero in $C^{0,0}(\bar{M}, T')$ into $C^{0,1}(\bar{M}, T')$. We observe that $C^{0,q}(\bar{M}, T')$ is a graded Fréchet space with respect to the Sobolev norms $\|\ \|_s$, $s \in \boldsymbol{Z}^+$. We now compute the derivative $G'(\xi) = R'(\xi)\colon C^{0,0}(\bar{M}, T') \to C^{0,1}(\bar{M}, T')$ where $R(\xi) = \partial e(\xi)^{-1}\cdot\bar{\partial} e(\xi)$.

The linear transformation $e'(\xi): C^{0,0}(\bar{M}, T') \to C^{0,0}(\bar{M}, T')$ is of the form $e'(\xi) = id + O(|\xi|)$; hence it is invertible for small ξ. In (2.10) we replace f^α by $e^\alpha(\xi + u\eta)$ and differentiate with respect to u at $u = 0$. This gives (if we set $\sigma = e'(\xi)\eta$ and $\rho = R'(\xi)\eta$)

$$\frac{\partial \sigma^\alpha}{\partial \bar{z}^\beta} = \sum_\gamma R(\xi)^\gamma_{\bar\beta} \frac{\partial \sigma^\alpha}{\partial z^\gamma} + \rho^\gamma_{\bar\beta} \frac{\partial e^\alpha(\xi)}{\partial z^\gamma} .$$

Define $\tau \in C^{0,0}(\bar{M}, T')$ by $\sigma^\alpha = \sum_\mu \partial e^\alpha(\xi)/(\partial z^\mu)\tau^\mu$. Hence

$$\sum_\mu \left(\frac{\partial e^\alpha(\xi)}{\partial z^\mu} \cdot \frac{\partial \tau^\mu}{\partial \bar{z}^\beta} + \frac{\partial^2 e^\alpha(\xi)}{\partial z^\mu \partial \bar{z}^\beta} \tau^\mu \right) = \sum_{\gamma,\mu} \left[R(\xi)^\gamma_{\bar\beta} \left(\frac{\partial e^\alpha(\xi)}{\partial z^\mu} \frac{\partial \tau^\mu}{\partial z^\gamma} + \frac{\partial^2 e^\alpha(\xi)}{\partial z^\gamma \partial z^\mu} \tau^\mu \right) \right]$$
$$+ \sum_\gamma \rho^\gamma_{\bar\beta} \frac{\partial e^\alpha(\xi)}{\partial z^\gamma} .$$

A differentiation of $\partial e^\alpha(\xi)/\partial \bar{z}^\beta = \sum_\gamma R(\xi)^\gamma_{\bar\beta}(\partial e^\alpha(\xi)/\partial z^\gamma)$ with respect to z^μ and a substitution in the above equality yields

$$\sum_\mu \frac{\partial e^\alpha(\xi)}{\partial z^\mu} \left\{ \frac{\partial \tau^\mu}{\partial \bar{z}^\beta} - \sum_\gamma R(\xi)^\gamma_{\bar\beta} \frac{\partial \tau^\mu}{\partial z^\gamma} + \sum_\gamma \frac{\partial R(\xi)^\gamma_{\bar\beta}}{\partial z^\mu} \tau^\mu \right\} = \sum_\gamma \frac{\partial e^\alpha(\xi)}{\partial z^\mu} \rho^\mu_{\bar\beta} .$$

Since the matrix $\partial e(\xi)$ is invertible we get

$$\frac{\partial \tau^\mu}{\partial \bar{z}^\beta} - \sum_\gamma \left(R(\xi)^\gamma_{\bar\beta} \frac{\partial \tau^\mu}{\partial z^\gamma} - \frac{\partial R(\xi)^\gamma_{\bar\beta}}{\partial z^\mu} \tau^\mu \right) = \rho^\mu_{\bar\beta} .$$

But the terms on the left-hand side give the local expression of $\bar\partial_{R(\xi)}\tau$. Hence

(2.11) $$G'(\xi)(\eta) = \bar\partial_{R(\xi)}(\partial e(\xi)^{-1} \cdot e'(\xi)(\eta)) .$$

Observe that the linear map $a(\xi) = \partial e(\xi)^{-1} \cdot e'(\xi): C^{0,0}(\bar{M}, T') \to C^{0,0}(\bar{M}, T')$ is of the form $id + O(|j^1\xi|)$ where $j^1\xi$ stands for terms involving the components of ξ and their first derivatives. Thus $a(\xi)$ is also invertible for small ξ. Since $R(\xi)$ determines an integrable almost complex structure on M (the pull-back of the complex structure on M' by the diffeomorphism $e(\xi)$) $\bar\partial_{R(\xi)} \circ \bar\partial_{R(\xi)} = 0$.

Conditions (1.3), (1.4), and (1.5) of Theorem 1.1 are obviously satisfied because G is a differential operator. We now proceed to verify (1.6)$'$.

The equation $\bar\partial_{R(\xi)}\nu = \bar\partial_{R(\xi)}G(\xi)$ has a solution $\nu = \nu(\xi)$ of the form $\bar\partial^*_{R(\xi)}\bar\partial_{R(\xi)}N_{R(\xi)}G(\xi)$. This follows from (2.8). Moreover, $||\nu(\xi)||_s \leqq C_s||\bar\partial_{R(\xi)}N_{R(\xi)}G(\xi)||_{s+1}$ where C_s is a polynomial of $||\xi||_k$ if $2k > s + 2n$. We claim that for a sufficiently large s_0 and for each $s \geqq s_0$ there exists a constant d_s such that for all $\psi \in C^{0,1}(\bar{M}, T')$

(2.12) $$||\bar\partial_{R(\xi)}N_{R(\xi)}\psi||_s \leqq d_s||\bar\partial_{R(\xi)}\psi||_s .$$

Assume that the assertion is false. Then one can find a sequence $\psi_1, \cdots, \psi_m, \cdots$ in $C^{0,1}(\bar{M}, T')$ such that $||\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi_m||_s = 1$ and $||\bar{\partial}_{R(\xi)}\psi_m||_s \to 0$. We may assume that $\{||\psi_m||_{s+1}\}$ is a bounded sequence because otherwise we may replace ψ_m by $\psi_m/||\psi_m||_{s+1}$. Then there exists a subsequence, again denoted by $\psi_1, \cdots, \psi_m, \cdots$, which converges in $C_s^{0,1}$, the completion of $C^{0,1}(\bar{M}, T')$ in the $|| \; ||_s$-norm. Let $\psi \in C_s^{0,1}$ be the limit of $\{\psi_m\}$. Then ψ is also in the domain of the operator $\bar{\partial}_{R(\xi)}$ and $\bar{\partial}_{R(\xi)}\psi = 0$. Since the theory of the $\bar{\partial}$-Neumann problem implies the existence of the operator $N_{R(\xi)}$ on the space of L_2-integrable forms, (2.8) holds on $C_s^{0,1}$, too. The equation $\square_{R(\xi)}N_{R(\xi)}\psi = \psi$ is satisfied not only in the distributional sense but also in the classical sense because s_0 is sufficiently large. Moreover, $\nu(N_{R(\xi)}\psi) = \nu(\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi) = 0$. By the integrability of $R(\xi)$ we have $\bar{\partial}_{R(\xi)}\bar{\partial}^*_{R(\xi)}\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi = 0$. Then by applying Stokes' theorem twice and using the boundary conditions we obtain $0 = (\bar{\partial}_{R(\xi)}\bar{\partial}^*_{R(\xi)}N_{R(\xi)}\psi, \bar{\partial}_{R(\xi)}N_{R(\xi)}\psi) = ||\bar{\partial}^*_{R(\xi)}\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi||^2$ and $0 = (\bar{\partial}^*_{R(\xi)}\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi, N_{R(\xi)}\psi) = ||\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi||^2$. On the other hand, (2.8) implies that $||\bar{\partial}_{R(\xi)}N_{R(\xi)}(\psi_m - \psi)||_s \leqq C_s||\psi_m - \psi||_s$. Hence $\lim_{m\to\infty} \bar{\partial}_{R(\xi)}N_{R(\xi)}\psi_m = \bar{\partial}_{R(\xi)}N_{R(\xi)}\psi$ in $C_s^{0,1}$. But this is a contradiction since $||\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi_m||_s = 1$ and $\bar{\partial}_{R(\xi)}N_{R(\xi)}\psi = 0$. Thus (2.12) is verified and we have

$$(2.13) \qquad ||\nu(\xi)||_s \leqq C_s||\bar{\partial}_{R(\xi)}G(\xi)||_{s+1} \leqq C_s||G(\xi)||^2_{s+2}$$

since by the integrability of $R(\xi)$ and ω one obtains $\bar{\partial}_{R(\xi)}G(\xi) = \bar{\partial}_{R(\xi)}(R(\xi) - \omega) = -[G(\xi), G(\xi)]$.

We can now find an approximate solution of $G'(\xi)(\eta) + G(\xi) = 0$ by setting $\eta = a(\xi)^{-1}\bar{\partial}^*_{R(\xi)}N_{R(\xi)}(\nu(\xi) - G(\xi))$. Then (2.9) and (2.13) show that (1.6)′ is satisfied with $B = C^{0,1}(\bar{M}, T')$. We can now apply Theorem 1.1 by taking $z = 0$ and find a solution $\xi \in C^{0,0}(\bar{M}, T')$ of the nonlinear equation $G(\xi) = 0$. This finishes the proof of Hamilton's theorem.

3. A class of boundary value problems. The nonlinear problem considered in the previous section has an obvious generalization. Let $\mathscr{M}u = f$ be an elliptic noncoercive system of the type investigated in [7] of order $2m$ subject to the boundary conditions $u \in B$. Let $\mathscr{M} + \mathscr{K}$ be a $2m$th order nonlinear perturbation of $\mathscr{M}$. Then one would like to solve the system $(\mathscr{M} + \mathscr{K})u = f$ with u in B. The theorem in § 1 shows that this can be done for small f if, for each v in a neighborhood of zero, $\mathscr{L}u = (\mathscr{M} + \mathscr{K}'(v))u = g$ has a solution u in B.

As in the example of § 2 it may happen that $\mathscr{L}$ satisfies all of the requirements imposed in [7] except the condition of essential self-adjointness, i.e., $\mathscr{L} - \mathscr{L}^*$ is of order at most $2m - 1$. We now proceed to show that even if the order of $\mathscr{L} - \mathscr{L}^*$ turns out to be $2m$ one can still obtain existence and regularity by applying the Kohn-Nirenberg methods. In particular, this will give us existence and regularity for $\square u - 2\bar{\partial}^*[\omega, u] = g - Hg$ with $t(u) = t(\bar{\partial}u) = H(u) = 0$.

For the purposes of this section it is enough to consider M as an open submanifold of an $n+1$-dimensional C^∞ manifold M' with compact closure $\bar{M}$ and a smooth boundary bM. Let $\mathscr{V}$ be a vector bundle over M' of fiber dimension p. We denote by $C^\infty(\bar{M}, \mathscr{V})$ the space of smooth sections of $\mathscr{V}$. With respect to a Hermitian inner product $\langle\,,\rangle$ along the fibers of $\mathscr{V}$ and a Riemannian metric on M' the L^2-inner product on $C^\infty(\bar{M}, \mathscr{V})$ is given by

$$(u, v) = \int_M \langle u, v\rangle dM .$$

Let ∇^α be the covariant differential with respect to a connection on $\mathscr{V}$ of order α. The Sobolev s-norm is defined by

$$\|u\|_s^2 = \sum_{|\alpha|\leqq s} \int_M \langle \nabla^\alpha u, \nabla^\alpha u\rangle dM , \quad \|u\|_0 = \|u\| .$$

This norm is equivalent to the norm given by

$$\sum_{k,j} \sum_{|\alpha|\leqq s} \int_{U_j \cap \bar{M}} |D^\alpha(\rho_j u^k)|^2 dx$$

where $\{U_j\}$ is a finite coordinate covering of $\bar{M}$, $\{\rho_j\}$ is a partition of unity with respect to $\{U_j\}$, $D^\alpha = D_1^{\alpha_1} \cdots D_{n+1}^{\alpha_{n+1}}$, $D_j = -\sqrt{-1}\partial/\partial x^j$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}$, and $\{u^k\}$ is the local component-wise expression of $u \in C^\infty(\bar{M}, \mathscr{V})$.

Let $\mathscr{L}: C^\infty(\bar{M}, \mathscr{V}) \to C^\infty(\bar{M}, \mathscr{V})$ be a differential operator of order $2m$ which arises from a quadratic form

$$Q(u, v) = \int_M \sum_{i,j} \sum_{\substack{|\alpha|\leqq m \\ |\beta|\leqq m}} a_{ij}^{\alpha\beta} D^\alpha u^i D^\beta v^j dM , \quad u, v \in C^\infty(\bar{M}, \mathscr{V}) ,$$

i.e., $(\mathscr{L}u, v) = Q(u, v)$ for all $u, v \in C_0^\infty(\bar{M}, \mathscr{V})$, the space of C^∞ sections with compact support in M.

Let $B \subset C^\infty(\bar{M}, \mathscr{V})$ be a subspace of certain homogeneous boundary conditions. More specifically, we require that

(3.1) $$C_0^\infty(\bar{M}, \mathscr{V}) \subset B .$$

(3.2) if U is a boundary coordinate neighborhood and $\zeta \in C_0^\infty(U)$, $\zeta = 1$ on $V \cap \bar{M}$, $V \subset \bar{V} \subset U$, then $\zeta B \subset B$.

(3.3) if T represents a translation or differentiation parallel to bM, then $\zeta T u \in B$ for $u \in B$.

Set $2Q_0(u, v) = Q(u, v) + \overline{Q(v, u)}$ and $2\sqrt{-1}Q'(u, v) = Q(u, v) - \overline{Q(v, u)}$. The following conditions will be imposed on Q:

(3.4) $$Q_0(u, u) \geqq \|u\|^2_{m-1} \quad \text{for} \quad u \in B .$$

(3.5) If Q' contains terms of the form (Lu, Kv) where both L and K involve covariant derivatives of order m, then for some constant $C > 0$ $\|Lu\|^2 \leqq CQ_0(u, u)$, and $\|Ku\|^2 \leqq CQ_0(u, u)$ for all $u \in B$.
(3.6) $|Q'(u, v)| \leqq C(Q_0(u, u)Q_0(v, v))^{1/2}$, $u, v \in B$.
(3.7) bM is noncharacteristic with respect to Q_0, i.e., if $x_0 \in bM$ and $\{x^1, \cdots, x^{n+1}\}$ are local coordinates in $U \ni x_0$ such that $U \cap bM = \{x^{n+1} = 0\}$, then for any nonzero vector $X = \xi\partial/\partial x^{n+1} \in T^x{}_0M'$ which is normal to bM the matrix $(a_{ij} = \sum_{|\alpha|+|\beta|=m} a^{\alpha\beta}_{ij}(x_0)\xi^{\alpha+\beta})$ is positive definite.
(3.8) The norm $Q_0(u, u)^{1/2}$ is compact with respect to $\| \|_{m-1}$ on $\hat{B}$, the completion of B with respect to $Q_0(u, u)^{1/2}$.

It is shown in [7] that (3.8) is implied by the inequality $\int_{bM} \sum_{|\alpha|\leqq m-1} \langle \nabla^\alpha u, \nabla^\alpha u\rangle ds \leqq CQ_0(u, u)$, $u \in B$, where ds is the volume element on bM. In the case of complex manifolds M and M' and $\mathscr{L} = \Box$, the complex Laplace operator (in particular, $m = 1$), this is the basic Kohn-Morrey estimate.

The Lax-Milgram representation theorem immediately implies thot if (3.4) and (3.6) are satisfied, then for any $f \in L_2(M, \mathscr{V})$ (the space of sections integrable with respect to $\| \|$) there exists a unique $u \in \hat{B}$ such that $Q(u, v) = (f, v)$ for all $v \in \hat{B}$. We point out that in order to prove regularity [7] makes use of a condition which is slightly weaker than (3.5), namely, Q' contains no products of mth-order derivatives. In particular, if $\mathscr{L}u = f$ is already a solvable noncoercive boundary value problem, then one can perturb $\mathscr{L}$ by an operator whose order equals the order of $\mathscr{L}$ and still get a solvable problem with the same boundary conditions as long as the highest order derivatives can be controlled in the manner specified by (3.5).

A priori estimates and regularity

Let $\boldsymbol{R}^{n+1}_- = \{(x^1, \cdots, x^n, y) : y \leqq 0\}$. For $u \in C^\infty_0(\boldsymbol{R}^{n+1}_-)$ the partial Fourier transform $\tilde{u}$ is defined by

$$\tilde{u}(\xi, y) = \int_{\boldsymbol{R}^n} e^{-\sqrt{-1}x\cdot\xi} u(x, y) dx$$

where $x = (x^1, \cdots, x^n)$, $\xi = (\xi_1, \cdots, \xi_n)$, $x \cdot \xi = \sum_j x^j\xi_j$, $dx = dx^1 \cdots dx^n$.

For real s the operator T_s is given by

$$(\widetilde{T_s u})(\xi, y) = (1 + |\xi|^2)^{s/2}\tilde{u}(\xi, y) .$$

If $\|\ \|$ is the L_2-norm in $\boldsymbol{R}_-^{n+1}$, then

$$|||u|||_s = \|T_s u\|$$

is the tangential s-norm of u. If s is a nonnegative integer, this norm is equivalent to $(\sum_{\substack{|\alpha|\leqq s\\ \alpha_{n+1}=0}} \|D^\alpha u\|^2)^{1/2}$.

We will adopt the following convention: $V_1(u) \lesssim V_2(u)$ means that there exists a constant $C > 0$ such that for all u $V_1(u) \leqq CV_2(u)$. In the sequel the constants in inequalities involving $\lesssim$ will depend on n, various integers s, r, etc., and some fixed functions with compact support.

For a nonnegative integer s we let $|u|_{C^s}$ denote the supremum of u and its derivatives up to order s. The following interpolation inequalities are standard: if $0 \leqq s_1 \leqq s_2 \leqq s_3$, then

$$|u|_{C^{s_2}} \lesssim |u|_{C^{s_1}}^{(s_3-s_2)/(s_3-s_1)} |u|_{C^{s_1}}^{(s_2-s_1)/(s_3-s_1)} \tag{3.9}$$

$$\|u\|_{s_2} \lesssim \|u\|_{s_1}^{(s_3-s_2)/(s_3-s_1)} \|u\|_{s_3}^{(s_2-s_1)/(s_3-s_1)} . \tag{3.10}$$

LEMMA 3.1. *Let $P(a)$ be a nonlinear partial differential operator of degree μ in a. Then for all a with $|a|_{C^\mu} \leqq \rho$ we have the estimate*

$$|P(a)|_{C^s} \lesssim |a|_{C^{s+\mu}} + 1 . \tag{3.11}$$

(The constant in $\lesssim$ depends on ρ, too.)

Proof. We have $P(a) = \phi(a, \cdots, D^\alpha a, \cdots)$, $|\alpha| \leqq \mu$, where ϕ is a smooth function $\phi(y, \cdots, y^\alpha, \cdots)$ defined in a neighborhood of some compact set $K = \{|y^\alpha| \leqq \rho_\alpha\}$ where the ρ_α's are such that $|a|_{C^\mu} \leqq \rho$ implies $|D^\alpha a| \leqq \rho_\alpha$.

By the chain rule every derivative of $P(a)$ is a product of a derivative of ϕ (with respect to the y^α's) and derivatives D^β of the argument $D^\alpha a$. Since every derivative of ϕ is uniformly bounded on K, we have to estimate only products of derivatives of a which occur in the form $D^{\beta_1+\alpha_1}a \cdot D^{\beta_2+\alpha_2}a \cdots D^{\alpha_k+\beta_k}a$ with $|\alpha_j| \leqq \mu$ and $\sum_{j=1}^k |\beta_j| \leqq s$. The supremum of the products is the product of the suprema. Hence one has to consider $|a|_{C^{\gamma_1+\delta_1}} \cdots |a|_{C^{\gamma_k+\delta_k}}$ with $\max \gamma_j \leqq \mu$ and $\sum \delta_j \leqq s$. If $\gamma_j + \delta_j \geqq \mu$, then by (3.9)

$$|a|_{C^{\gamma_j+\delta_j}} \lesssim |a|_{C^{s+\mu}}^{(\gamma_j+\delta_j-\mu)/s} |a|_{C^\mu}^{(s-\gamma_j-\delta_j+\mu)/s} .$$

Since $|a|_{C^\mu} \leqq \rho$ and $\sum (\gamma_j + \delta_j - \mu) \leqq s$, $\prod_{j=1}^k |a|_{C^{\gamma_j+\delta_j}} \lesssim |a|_{C^{s+\alpha}} + 1$. This gives (3.11).

Let $\zeta \in C_0^\infty(\boldsymbol{R}_-^{n+1})$ and let A_j be the tangential self-adjoint operator $\zeta D_j \zeta$, $1 \leqq j \leqq n$. Let $D^l = D_1^{l_1} \cdots D_{n+1}^{l_{n+1}}$, $|l| = l_1 + \cdots + l_{n+1}$.

LEMMA 3.2. *For all* $a, u \in C_0^\infty(\boldsymbol{R}_-^{n+1})$

$$\|[aD^l, A_j]u\| \lesssim |a|_{C^1}\|u\|_{|l|} \tag{3.12}$$

$$\|[[aD^l, A_j], A_j]u\| \lesssim |a|_{C^2}\|u\|_{|l|} . \tag{3.13}$$

Proof. We have

$$[aD^l, A_j]u = a[D^l, \zeta]D_j\zeta u + a\zeta D_j[D^l, \zeta]u - \zeta D_j(a)\zeta D^l u . \tag{3.14}$$

Now (3.12) follows since $[D^l, \zeta]$ is a differential orerator of order $|l| - 1$. Furthermore, (3.14) shows that $[aD^l, A_j]$ is a sum of operators of the type bD^l where the b coefficients depend linearly on a and the first derivatives of a. By (3.12) $\|[bD^l, A_j]u\| \lesssim |b|_{C^1}\|u\|_l \lesssim |a|_{C^2}\|u\|_{|l|}$. This proves (3.13).

We now take coordinate neighborhoods U and V such that $\bar{V} \subset U$ and $U \cap M$ is identified with an open set in $\boldsymbol{R}_-^{n+1}$. Let ζ be a C^∞ function which is identically equal to one on $\bar{V} \cap M$ and identically equal to zero outside $U \cap M$. The operators A_j are constructed with the aid of the local coordinates on U. Set $|a|_{C^s} = \sum_{i,j,\alpha,\beta} |a_{ij}^{\alpha\beta}|_{C^s}$, $\|a\|_s = \sum_{i,j,\alpha,\beta} \|a_{ij}^{\alpha\beta}\|_s$ and let A stand for any of the operators A_j.

PROPOSITION 3.3. *For all* $u \in B$

$$|\mathrm{Re}\, Q(A^s u, A^s u)| \lesssim (\|u\|_{s+m-1} + |a|_{C^{s+1}}\|u\|_m)^2 + |\mathrm{Re}\, Q(u, A^{2s}u)| \tag{3.14}$$

where Re *stands for the real part and the constant in* $\lesssim$ *also depends on* $|a|_{C^2}$.

Proof. We consider the bilinear form $AQ(u, v) = Q(Au, v) - Q(u, Av)$. The form AQ is again of degree m in u and v and its coefficients are obtained by differentiating the coefficients of Q by A. By induction we can define the bilinear forms $A^\mu Q$ which are of degree m in u and v and their coefficients depend linearly on the $a_{ij}^{\alpha\beta}$'s and their derivatives up to order μ. Therefore, we have

$$|A^\mu Q(u, v)| \lesssim (|a|_{C^\mu} + 1)\|u\|_m\|v\|_m . \tag{3.15}$$

We can write $\mathrm{Re}\, Q(A^s u, A^s u) - \mathrm{Re}\, Q(u, A^{2s}u)$ as a sum of terms $A^\mu Q(A^\nu u, A^\sigma u)$ with $2 \leqq \mu \leqq s + 1$, $\nu, \sigma \leqq s - 1$, $\mu + \nu + \sigma = 2s$, and terms $A^\mu Q'(A^\nu u, A^\sigma u)$ with $1 \leqq \mu \leqq s$, $\mu + \nu + \sigma = 2s$, and either $\nu \leqq s$, $\sigma \leqq s - 1$ or $\nu \leqq s - 1$, $\sigma \leqq s$. Recall that Q' is the skew-hermitian part of Q.

We first consider the terms $A^\mu Q(A^\nu u, A^\sigma u)$, $2 \leqq \mu \leqq s + 1$, $\nu, \sigma \leqq s - 1$. By applying (3.15) we get

$$|A^\mu Q(A^\nu u, A^\sigma u)| \lesssim (|a|_{C^\mu} + 1)\|u\|_{\nu+m}\|u\|_{\sigma+m} . \tag{3.16}$$

By (3.9) and (3.10)

$$|a|_{C_\mu} \lesssim |a|_{C^{s+1}}^{(\mu-2)/(s-1)} \cdot |a|_{C^2}^{(s+1-\mu)/(s-1)}$$
$$\|u\|_{\nu+m} \lesssim \|u\|_{s+m-1}^{\nu/(s-1)} \cdot \|u\|_m^{(s-\nu-1)/(s-1)}$$
$$\|u\|_{\sigma+m} \lesssim \|u\|_{s+m+1}^{\sigma/(s-1)} \cdot \|u\|_m^{(s-\sigma-1)/(s-1)} .$$

If we treat $|a|_{C^2}$ as a constant and use the fact that $\nu + \sigma = 2s - \mu$ we have

$$|a|_{C^\mu}\|u\|_{\nu+m}\|u\|_{\nu+m} \lesssim (|a|_{C^{s+1}}\|u\|_m)^{(\mu-2)/(s-1)} \cdot \|u\|_{s+m-1}^{(2s-\mu)/(s-1)}$$
$$\lesssim (|a|_{C^{s+1}}\|u\|_m + \|u\|_{s+m-1})^2 .$$

The above inequality and (3.16) imply

$$(3.17) \qquad |A^\mu Q(A^\nu u, A^\sigma u)| \lesssim (|a|_{C^{s+1}}\|u\|_m + \|u\|_{s+m-1})^2 .$$

Next we turn our attention to the terms $A^\nu Q'(A^\nu u, A^\sigma u)$, $1 \leqq \mu \leqq s$. We will assume that $\nu \leqq s$ and $\sigma \leqq s-1$; the case $\nu \leqq s-1$, $\sigma \leqq s$ can be treated in exactly the same way.

(i) $\nu, \sigma \leqq s-1$:

Again by (3.15)

$$(3.18) \qquad |A^\mu Q'(A^\nu u, A^\sigma u)| \lesssim (|a|_{C^\mu} + 1)\|u\|_{\nu+m}\|u\|_{\sigma+m} .$$

Since $1 \leqq \mu \leqq s$, by interpolation we have

$$|a|_{C^\mu} \lesssim |a|_{C^s}^{(\mu-1)/(s-1)} \cdot |a|_{C^1}^{(s-\mu)/(s-1)} \lesssim |a|_{C^s}^{(\mu-1)/(s-1)} .$$

($|a|_{C^1} \leqq |a|_{C^2}$ and $|a|_{C^2}$ is treated as a constant.)

We can now proceed as in the derivation of (3.17) and obtain

$$(3.19) \qquad |A^\mu Q'(A^\nu u, A^\sigma u)| \lesssim (|a|_{C^s}\|u\|_m + \|u\|_{s+m-1})^2 .$$

(ii) $\nu = s, \sigma \leqq s-2$:

We observe that $A^\mu Q'(A^s u, A^\sigma u) = A^{\mu+1}Q'(A^{s-1}u, A^\sigma u) - A^\mu Q'(A^{s-1}u, A^{\sigma+1}u)$. Since $2 \leqq \mu + 1 \leqq s+1$, $\sigma + 1 \leqq s-1$, $\mu + s + \sigma = 2s$, the first term is estimated by (3.17) and the second by (3.18). Hence, in this case we again have

$$(3.20) \qquad |A^\mu Q'(A^s u, A^\sigma u)| \lesssim (|a|_{C^{s+1}}\|u\|_m + \|u\|_{s+m-1})^2 .$$

(iii) $\nu = s, \sigma = s-1$:

A typical term in the expression of $Q'(A^s u, A^{s-1}u)$ is of the form $(LA^s u, KA^{s-1}u)$ where L and K are mth- order differential operators satisfying (3.5). Since $\nu = s$ and $\sigma = s-1$, we have $\mu = 1$ and a typical term in the expression of $AQ'(A^s u, A^{s-1}u)$ is of the form

$$(LA^{s+1}u, KA^{s-1}u) - (LA^s u, KA^s u)$$
$$= ([[L, A], A]A^{s-1}u, KA^{s-1}u) + ([L, A]A^{s-1}u, [A, K]A^{s-1}u)$$
$$+ (LA^s u, [A, K]A^{s-1}u) + ([L, A]A^{s-1}u, KA^s u) .$$

We now observe that since A^s is a tangential operator $u \in B$ implies $A^s u \in B$. Therefore, by (3.5) $\|LA^s u\|^2 \lesssim Q_0(A^s u, A^s u)$, $\|KA^s u\|^2 \lesssim Q_0(A^s u, A^s u)$. Then (3.12) and (3.13) together with Schwarz's inequality give the estimate

$$\begin{aligned} &|(LA^{s+1}u, KA^{s+1}u) - (LA^s u, KA^s u)| \\ &\quad \lesssim C_\varepsilon |a|_{C^2}^2 \|u\|_{s+m-1}^2 + \varepsilon Q_0(A^s u, A^s) \end{aligned} \tag{3.21}$$

where ε is an arbitrary positive real number and C_ε is a constant depending only on ε.

Since $\mathrm{Re}\, Q(A^s u, A^s u) = Q_0(A^s u, A^s u)$ and $\mathrm{Re}\, Q(A^s u, A^s u) - \mathrm{Re}\, Q(u, A^{2s} u)$ is a sum of terms estimated by (3.17), (3.18), (3.19), and (3.20), the desired inequality (3.15) is obtained by combining these inequalities and taking a sufficiently small $\varepsilon > 0$.

Let $\{U_\alpha\}$ be a finite cover of $\bar{M}$ by coordinate neighborhoods. Let V_α be an open subset of U_α with $\bar{V}_\alpha \subset U_\alpha$ such that V_α's still cover $\bar{M}$. Let $A_{j,\alpha} = \zeta_\alpha D_j \zeta_\alpha$, $1 \leqq j \leqq n$, be defined with respect to the local coordinates on U_α where $\zeta_\alpha \equiv 1$ on $V_\alpha \cap \bar{M}$ and outside $U_\alpha \cap \bar{M}$. Then for all $u \in C^\infty(\bar{M}, \mathscr{V})$ we have $|||u|||_r \lesssim \sum_{s \leq r} \sum_{j,\alpha} \|A_{j,\alpha}^s u\| \lesssim |||u|||_r$.

Since $Q_0(u, u)^{1/2}$ is compact with respect to $\|\ \|_{m-1}$ on $\hat{B}$ for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $u \in \hat{B}$

$$\|u\|_{m-1}^2 \leqq \varepsilon Q_0(u, u) + C_\varepsilon \|u\|_{m-2}^2 \,. \tag{3.22}$$

Set

$$|||D^{m-1}u|||_r = \sum_{k,\alpha} \sum_{\substack{|l| \leq m-1+r \\ l_{n+1} \leq m-1}} \int_{V_\alpha \cap \bar{M}} |D^l(\rho_\alpha u^k)|^2 dx$$

where $\{\rho_\alpha\}$ is a partition of unity with respect to $\{V_\alpha\}$.

Then (3.4), (3.14), and (3.21) imply

$$\begin{aligned} |||D^{m-1}u|||_r^2 &\lesssim \varepsilon\Big[\sum_{j,\alpha} \sum_{s \leq r} |\mathrm{Re}\, Q(u, A_{j,\alpha}^{2s} u)| \\ &\quad + (\|u\|_{r+m-1} + |a|_{C^{r+1}} \|u\|_m)^2\Big] + C_\varepsilon \|u\|_{r+m-2}^2 \end{aligned} \tag{3.23}$$

for all $u \in B$, $r \in \boldsymbol{Z}^+$, $\varepsilon > 0$.

We are now in a position to obtain regularity of solutions up to the boundary.

THEOREM 3.4. *Assume that the quadratic form $Q(u, v)$ satisfies conditions* (3.1)-(3.8). *Then for each $f \in C^\infty(\bar{M}, \mathscr{V})$ there exists a unique $u \in B$ such that $Q(u, v) = (f, v)$ for all $v \in B$,* i.e., *$\mathscr{L} u = f$. Moreover, for each $r \in \boldsymbol{Z}^+$ there exists a constant C_r depending only on r and $|a|_{C^{m+1}}$ such that*

$$\|u\|_{r+m-1} \leqq C_r(\|f\|_r + |a|_{C^{r+m}}\|f\|) . \tag{3.24}$$

Proof. By the elliptic regularization method devised in [7] it is enough to prove the *a priori* estimate (3.23) under the assumption that a smooth solution $u \in B$ already exists.

Since $Q(u, v)=(f, v)$ for all $v \in B$, we have $Q(u, A^{2s}_{j,\alpha}u)=(f, A^{2s}_{j,\alpha}u)=(A^s_{j,\alpha}f, A^s_{j,\alpha}u)$. Thus (3.22) implies

$$|||D^{m-1}u|||_r^2 \lesssim \varepsilon\|f\|_r^2 + \varepsilon(\|u\|_{r+m-1} + |a|_{C^{r+1}}\|u\|_m)^2 + C_\varepsilon\|u\|_{r+m-2}^2 . \tag{3.25}$$

We now turn to the equation $\mathscr{L}u = f$ on a coordinate neighborhood V_α. Because of (3.7) the matrix of coefficients of $D^{2m}_{n+1}u$ is positive definite. We can write

$$D^{2m}_{n+1}u = \sum_{\substack{|\sigma|\leqq 2m \\ \sigma_{n+1}<2m}} P_\sigma(a)D^\sigma u + P(a)f \tag{3.26}$$

where $P_\sigma(a)$ and $P(a)$ are mth-order nonlinear differential operators in the $a^{\alpha\beta}_{ij}$'s. (Observe that the coefficients of $\mathscr{L}$ depend smoothly on $a^{\alpha\beta}_{ij}$ and their derivatives up to order m.)

$$\begin{aligned} D^{r+m-1}_{n+1}u = &\sum_{\substack{|\sigma|\leqq 2m \\ \sigma_{n+1}<2m}} P_\sigma(a)D^{r-m-1}_{n+1}D^\sigma u \\ &+ \sum_{\substack{|\sigma|\leqq 2m \\ \sigma_{n+1}<2m}} \sum_{1\leqq|\beta|\leqq r-m-1} b_\beta D^\beta_{n+1}P_\sigma(a)D^{r-m-1-\beta}_{n+1}D^\sigma u \\ &+ \sum_{\gamma\leqq r-m-1} d_\gamma D^\gamma_{n+1}P(a)D^{r-m-1-\gamma}_{n+1}f . \end{aligned} \tag{3.27}$$

By (3.11) we have

$$\begin{aligned} \int_{V_\alpha}|D^{r+m-1}_{n+1}u|^2dx \lesssim &(|a|_{C^m}+1)^2 \sum_{\substack{|\gamma|=r+m-1 \\ \sigma_{n+1}<r+m-1}} \int_{V_\alpha}|D^\sigma u|^2dx \\ &+ \sum_{1\leqq\beta\leqq r-m-1}(|a|_{C^{\beta+m}}+1)^2\|u\|^2_{r-m-1-\beta} \\ &+ \sum_{1\leqq\gamma\leqq r-m-1}(|a|_{C^{\gamma+m}}+1)^2\|f\|^2_{r-m-1-\gamma} . \end{aligned} \tag{3.28}$$

The above inequality will not change if we extend our summation over β to r. Then it follows from (3.9) and

$$|a|_{C^{\beta+m}} \lesssim |a|^{(r-\beta)/(r-1)}_{C^{m+1}}|a|^{(\beta-1)/(r-1)}_{C^{r+m}} \tag{3.10}$$

and

$$\|u\|_{r+m-1-\beta} \lesssim \|u\|^{(\beta-1)/(r-1)}_{m-1}\cdot\|u\|^{(r-\beta)/(r-1)}_{r+m-2} .$$

Thus

$$|a|_{C^{\beta+m}}\|u\|_{r+m-1-\beta} \lesssim |a|_{C^{r+m}}\|u\|_{m-1} + \|u\|_{r+m-2}|a|_{C^{m+1}} .$$

Similarly

$$|a|_{C^{\gamma+m}}\|f\|_{r-m-1-\gamma} \lesssim |a|_{C^{m+1}}\|f\|_{r-m-2} + |a|_{C^{r-1}}\|f\| \,.$$

Hence

$$\int_{V_\alpha}|D_{n+1}^{r+m-1}u|^2dx \lesssim (|a|_{C^m}+1)^2\sum_{\substack{|\sigma|=r+m-1\\ \sigma_{n+1}<r+m-1}}\int_{V_\alpha}|D^\sigma u|^2dx + |a|^2_{C^{m+1}}(\|u\|^2_{r+m-2}+\|f\|^2_r) + |a|^2_{C^{r+m}}(\|u\|^2_{m-1}+\|f\|^2) \,. \tag{3.29}$$

Next, for any $\eta > 0$ there is a constant $C_\eta > 0$ such that

$$\int_{V_\alpha}\sum_{\substack{|\sigma|=r+m-1\\ \sigma_{n+1}<r+m-1}}|D^\sigma u|^2dx \leqq \eta\int_{V_\alpha}|D_{n+1}^{r+m-1}u|^2dx + C'_\eta\int_{V_\alpha}\sum_{\substack{|\sigma|=r+m-1\\ \sigma_{n+1}\leqq m-1}}|D^\sigma u|^2dx \,. \tag{3.30}$$

This can easily be established by extending n outside V_α to be with compact support and then using Fourier transform (see [7]).

Combining (3.28) and (3.29) we obtain

$$\begin{aligned}\int_{V_\alpha}\sum_{\substack{|\sigma|=r+m-1\\ \sigma_{n+1}<r+m-1}}|D^\sigma u|^2dx + \int_{V_\alpha}|D_{n+1}^{r+m-1}u|^2dx \\ \lesssim [(|a|_{C^m}+1)^2+1]\Big[\eta\int_{V_\alpha}|D_{n+1}^{r+m-1}u|^2dx + C'_\eta\int_{V_\alpha}\sum_{\substack{|\sigma|=r+m-1\\ \sigma_{n+1}\leqq m-1}}|D^\sigma u|^2dx\Big] \\ + |a|^2_{C^{m+1}}(\|u\|^2_{r+m-2}+\|f\|^2_r) + |a|^2_{C^{r+m}}(\|u\|^2_{m-1}+\|f\|^2) \,.\end{aligned} \tag{3.31}$$

By choosing a sufficiently small η and summing over α we have

$$\|u\|_{r+m-1} \lesssim |||D^{m-1}u|||_r + \|u\|_{r+m-2} + \|f\|_r + |a|_{C^{r+m}}(\|u\|_{m-1}+\|f\|) \tag{3.32}$$

where the constant in $\lesssim$ depends on $|a|_{C^{m+1}}$.

We combine (3.24) and (3.31) and choose a sufficiently small ε in (3.24). This gives the inequality

$$\|u\|_{r+m-1} \lesssim \|f\|_r + \|u\|_{r+m-2} + |a|_{C^{r+m}}(\|u\|_{m-1}+\|f\|) \,. \tag{3.33}$$

By (3.4) $\|u\|^2_{m-1} \leqq Q_0(u, u) = \mathrm{Re}\, Q(u, u) = \mathrm{Re}\,(u, f)$. Thus $\|u\|^2_{m-1} \leqq |(u, f)| \leqq \|u\|\cdot\|f\| \leqq \|u\|_{m-1}\|f\|$ or $\|u\|_{m-1} \leqq \|f\|$. Therefore,

$$\|u\|_{r+m-1} \lesssim \|f\|_r + \|u\|_{r+m-2} + |a|_{C^{r+m}}\|f\| \,. \tag{3.34}$$

Finally, the desired estimate (3.23) follows from the fact that for every $\varepsilon > 0$ there is a constant C_ε (which also depends on r and m) such that $\|u\|_{r+m-2} \leqq \varepsilon\|u\|_{r+m-1} + C_\varepsilon\|u\|$.

REMARK 3.5. The above theorem is a global regularity result. For a quadratic form $Q(u, v)$ satisfying (3.4), (3.5), (3.6) the regularity result in [7] (cf. Theorem 4, p. 458) also holds.

REMARK 3.6. The careful analysis of the dependence of (3.23) on the coefficients of $\mathscr{L}$ is needed in order to obtain condition (1.6) in cases of application of the implicit function theorem of § 1 to solving nonlinear systems of partial differential equations. Observe that by the Sobolev inequalities $|a|_{C^{r+m}} \lesssim \|a\|_{n_1+r+m}$, $n_1 > (1/2)n + 1$, and the dependence of C_r on $|a|_{C^{m+1}}$ in (3.23) is consistent with the fact that by the methods of § 1 one solves such a nonlinear problem in a neighborhood of zero in some Sobolev norm.

REMARK 3.7. If the basic Kohn-Morry estimate (2.2) hold on (0, 1) forms, then the operator $\square_\omega u = \square u - 2\bar{\partial}^*[\omega, u]$ satisfies (3.1)-(3.3) and (3.5)-(3.7). Furthermore, if $Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^*u, \bar{\partial}^*v)_\perp - (2[\omega, u], \bar{\partial}v)$, then $Q_0(u, u)^{1/2}$ is compact with respect to $\| \|_0$ on $\hat{B} \cup \mathscr{H}^1$ where $B = \{u_\perp \in C^{0,1}(M; T'): \tau(u) = 0\}$ and $\mathscr{H}^1 = \operatorname{Ker} \square \cap B$. Hence, (3.4) holds on $B \cap \mathscr{H}^1$. Since $\mathscr{H}^1$ is finite-dimensional, all norms on $\mathscr{H}^1$ are equivalent, so that (3.21) still holds. Thus, the argument in Theorem 3.4 now applies to show that if $f \in C^{0,1}(\bar{M}, T') \cap \mathscr{H}^{1\perp}$, there is a unique solution $u \in B \cap \mathscr{H}^{1\perp}$ of $\square_\omega u = f$ and u satisfies (3.23) with $m = 1$.

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