PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 2, 1978

SOME RADICAL PROPERTIES OF RINGS WITH $(a, b, c) = (c, a, b)$

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A ring is an s-ring if (for fixed s) A^s is an ideal whenever A is. We show that at least two different definitions for the prime radical are equivalent in s-rings. If R satisfies $(a, b, c) = (c, a, b)$ then R is a 2-ring. In this note we investigate various properties of the prime and nil radicals of R. In addition, if R is a finite dimensional algebra over a field of characteristic $\neq 2$ of 3 we show that the concepts of nil and nilpotent are equivalent.

In [1] Brown and McCoy studied a collection of prime radicals and nil radicals in an arbitrary nonassociative ring. In light of their treatment we will consider these radicals in rings which satisfy the identity

$$(a, b, c) = (c, a, b) . \tag{1}$$

While these rings may be viewed as an extension of alternative rings, they are in general not even power associative. Examples of (not power associative) rings satisfying (1) appear in [2] and [4].

1. s-rings and the prime radical.

Prime radicals for an arbitrary ring R were treated in [1] in the following way. Let $\mathscr{A}$ be the set of all finite nonassociative products of at least two elements from some countable set of indeterminates $x_1, x_2, x_3, \cdots$. Then if $u \in \mathscr{A}$ we call an ideal P u-prime if $u(A_1, A_2, \cdots, A_n) \subseteqq P$ implies some $A_i \subseteqq P$ for ideals $A_1, A_2, \cdots, A_n$. For example if $u = (x_1x_2)x_3$ then P is u-prime if whenever $(A_1A_2)A_3 \subseteqq P$ we have one of the A_i's in P. The u-prime radical R^u is then the intersection of all u-prime ideals in R. It was shown that if u^* is the word having the same association as u, but in only one variable, then $R^u = R^{u^*}$. For example if $u = (x_1x_2)x_3$ then $u^* = (xx)x$, and R^{u^*} is the intersection of ideals P with the property that if $(AA)A \subseteqq P$ for an ideal A, then $A \subseteqq P$.

Another theory of the prime radical was given in [9]. Call a ring R an s-ring if for some fixed positive integer s, A^s is an ideal whenever A is. Call an ideal P prime if $A_1A_2 \cdots A_s \subseteqq P$ implies some $A_i \subseteqq P$ for ideals $A_1, \cdots, A_s$. The prime radical $P(R)$ of an s-ring R is then the intersection of all prime ideals.

In the case of s-rings we see that these approaches are essentially the same:

THEOREM 1. *Let R be an s-ring. Then for each $u \in \mathscr{A}$ having degree $\geqq s$, R^u coincides with $P(R)$.*

Proof. If A is an ideal of R, consider the two descending chains: $A^{(0)} = A_0 = A$, $A^{(n+1)} = A^{(n)}A^{(n)}$, and $A_{n+1} = (A_n)^s$. It is easily seen that $\langle A_n \rangle$ is a chain of ideals in R and for each n, $A_n \subseteqq A^{(n)}$. Next choose $u \in \mathscr{A}$. We first show that there is an integer r such that $A^{(r)} \subseteqq u^*(A, A, \cdots, A)$. We induct on $\deg u^*$. When $u^* = x^2$, take $r = 1$. Assuming $\deg u^* > 2$, write $u^* = v_1v_2$ where each v_i has degree less than that of u^*. Then there exists r_1, r_2 such that $A^{(r_i)} \subseteqq v_i(A, A, \cdots, A)$. Letting $r = \max\{r_1, r_2\}$, $A^{(r+1)} = A^{(r)}A^{(r)} \subseteqq A^{(r_1)}A^{(r_2)} \subseteqq v_1(A)v_2(A) \subseteqq u^*(A)$, which completes the induction. Now assume P is prime (in the sense of [9]). Then P is also u^*-prime. For if A is any ideal with $u^*(A, A, \cdots, A) \subseteqq P$ we may choose r such that $A_r \subseteqq A^{(r)} \subseteqq u^*(A) \subseteqq P$. Using repeatedly the fact that P is prime we see that $A \subseteqq P$. We have shown $R^u = R^{u^*} \subseteqq P(R)$.

To see the other inclusion, assume $\deg u \geqq s$. Let P be u^*-prime. Then P is also prime. For if A is an ideal with $A^s \subseteqq P$ it follows that $u^*(A) \subseteqq A^{\deg u^*} \subseteqq A^s \subseteqq P$, and so $A \subseteqq P$. This shows $P(R) \subseteqq R^{u^*} = R^u$, which completes the proof.

COROLLARY. *If R is a 2-ring, the u-prime radicals all coincide.*

Rich has shown that in an s-ring the prime radical $P(R)$ is the intersection of all ideals Q such that R/Q has no nonzero nilpotent ideals [5]. However, if R/Q has no nonzero nilpotent ideals it also has no nonzero solvable ideals: For if $A^{(n)} \subseteqq Q$ for some ideal A, then $A_n \subseteqq A^{(n)} \subseteqq Q$ using the same notation as above. It follows that $A \subseteqq Q$. This shows that the word "nilpotent" may be replaced by "solvable" in Rich's characterization of $P(R)$.

2. **Nilalgebras.** In this section we let R denote a ring satisfying equation (1) and having characteristic not equal to 2 or 3. Outcalt showed that if R is simple then it is alternative (and hence a Caycley-Dickson algebra or associative) [3]. Sterling extended this result by showing that if R has no nonzero ideals whose square is zero then R is alternative [8].

We see that rings R which satisfy (1) are 2-rings. For if A is an ideal with $a_1, a_2 \in A$, then $(a_1a_2)x = (a_1, a_2, x) + a_1(a_2x) = (a_2, x, a_1) + a_1(a_2x) \in A^2$. In fact, it is easily shown that A^n is an ideal for each $n \geqq 2$.

Next recall that an element a is nilpotent if there is some association u^* such that $u^*(a) = 0$. An ideal A is a nil ideal if each element in A is nilpotent. We call A solvable if $A^{(n)}$ (defined above) is zero for some n. Finally, A is right nilpotent if the sequence $A, A^2, A^2A, (A^2A)A, \cdots$ reaches zero in a finite number of steps.

LEMMA. *Let R be a ring satisfying* (1). *Then R is nilpotent if and only if R is right nilpotent.*

Proof. The proof of this lemma, which appears in [4], only required identity (1) and is therefore valid.

We will need the following identity [8, eq. 4] which holds in R

$$9(((a, x, x), x, x), x, x) = (a, (x, x, x), (x, x, x)) . \tag{2}$$

LEMMA. *Let R be a finite dimensional algebra, satisfying* (1), *over a field F of characteristic $\neq 2, 3$. If R is solvable then R is nilpotent.*

Proof. We induct on $\dim R$. When $\dim R = 1$ the result is obvious, so assume $\dim R > 1$. By the previous lemma it is sufficient to show that R is right nilpotent. Let S_a denote the right multiplication operator $x \rightarrow xa$. Let $\hat{R}$ be the subalgebra of the multiplication algebra R^* which is generated by $\{S_a \mid a \in R\}$. Note that R is right nilpotent if and only if $\hat{R}$ is nilpotent. Now by the solvability of R we may write $R = B + Fx$ where B is an ideal containing R^2 and $B \subsetneqq R$. Since $\dim B < \dim R$, B is nilpotent by the induction assumption. Suppose $B^k = 0$. We claim $(\hat{R})^{6k^2} = 0$.

Treating a as the independent variable and expanding (2) it becomes apparent that $(S_x)^6$ may be written as the sum of 15 terms each containing S_{x^2}, S_{x^2x}, or S_{xx^2}. These factors are in $(R^2)^* \subseteqq B^*$. This implies that $(S_x)^{6k}$ can be expressed as a sum of terms each containing at least k factors from B^*. Since B^n is as ideal for each n, it follows that $(S_x)^{6k} = 0$. Now choose $T \in (\hat{R})^{6k2}$. Then T is a sum of terms each containing a factor of the form

$$(S_{y_1}S_{y_2} \cdots S_{y_{6k}})(S_{z_1}S_{z_2} \cdots S_{z_{6k}}) \cdots (S_{w_1}S_{w_2} \cdots S_{w_{6k}}) ,$$

where each subscript is either equal to x or is a member of B. Note there are k "blocks" each having length $6k$. If k of the S's have elements from B attached to them then the above expression is 0 since B^n is always an ideal. On the other hand if there are not k such S's, then one of the blocks must be of the form $S_xS_x \cdots S_x$, or $(S_x)^{6k} = 0$. In any case $T = 0$, so R is nilpotent completing the proof.

THEOREM 2. *If R is a finite dimensional nilalgebra, satisfying* (1), *over a field of characteristic $\neq 2, 3$, then R is nilpotent.*

Proof. We induct on dim R. Assume dim $R > 1$. If R is alternative we are done. If not, by Sterling's result [8], there exists an ideal $J \neq 0$ such that $J^2 = 0$. Then R/J is solvable by the induction assumption. Since J is solvable it follows that R must be. By the previous lemma R is nilpotent.

3. **Radicals.** If v is a word in one variable, then a is called v-nilpotent if the sequence $a, v(a), v(v(a)), \cdots$ ends in 0. An ideal is v-nil if each of its elements is v-nilpotent. Every ring has a maximal v-nil ideal N_v and a maximal nil ideal N[1]. We shall call N_v the v-nil radical and N the nil radical. The Jacobson radical J is the set of all elements which generate quasi-regular ideals. It is shown in [1] that for each word $u^* = v$ we have

$$R^u \subseteqq N_v \subseteqq N \subseteqq J .$$

THEOREM 3. *Let R be a ring of characteristic $\neq 2, 3$ and satisfying* (1). *Then all of the u-prime radicals coincide and each of the v-nil radicals coincides with N.*

Proof. The first statement follows from the corollary to Theorem 1 and the fact that R is a 2-ring. The second statement follows from Sterling's theorem: The ring R/R^u contains no nonzero ideals whose square is zero (since $A^2 \subseteqq R^u$ implies $A \subseteqq R^u$). Hence R/R^u is alternative, and so R/N_v is alternative. Since R/N_v is power associative, N/N_v is a v-nil ideal in R/N_v, and so N must be a v-nil ideal in R. This means $N_v = N$.

THEOREM 4. *If R is a finite dimensional algebra, satisfying* (1) *over a field of characteristic $\neq 2, 3$, then the Jacobson radical R is nilpotent.*

Proof. By the reasoning in the proof of Theorem 3 we may conclude that R/N is alternative. A result of Slater's says that in an alternative ring with d.c.c. on right ideals, the nil radical equals the Jacobson radical]7]. Hence $0 = N(R/N) = J(R/N)$. It follows that $J \subseteqq N$ so J is nilpotent.

We will add one final note. If R is a ring the attached ring R^+ is the ring where multiplication is redefined by $a \cdot b = ab + ba$. Rich has shown that if R is alternative and having characteristic $\neq 2, 3$, then the (Jordan) ring R^+ has the same prime radical as R [6]. That is, $P(R) = P(R^+)$ using the notation of §1. This result may be generalized slightly: If R satisfies (1) and has characteristic

$\neq 2, 3$, then the prime radical of R coincides with each of the u-prime radicals $(R^+)^u$ in R^+. This is interesting because while Jordan rings are 3-rings, it does not seem likely that in general R^+ will be an s-ring. The proof (which we omit) is similar to the one found in [6].

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Received December 6, 1976.

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