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DISTRIBUTION ESTIMATES OF BARRIER-CROSSING PROBABILITIES OF THE YEH-WIENER PROCESS

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Let $Q = [0, S] \times [0, T]$ be a rectangle and $\{X(s, t): s, t \geqq 0\}$ be the two-parameter Yeh-Wiener process. This paper finds probabilities of $X(s, t)$ crossing barriers of the type $ast + bs + ct + d$ on the boundary ∂Q. These probabilities give lower bounds for the yet unknown probabilities of $X(s, t)$ crossing $ast + bs + ct + d$ on Q. The paper also discusses sharper bounds for the latter probabilities.

1. **Introduction.** Let $\{X(s, t): s, t \geqq 0\}$ be the standard Yeh-Wiener process of two parameters such that it is a separable real Gaussian stochastic process satisfying:

$$X(s, t) = 0 \text{ a.s. if } s \text{ or } t \text{ is } 0\,, \tag{1.1}$$

$$\text{the expected value } E\{X(s, t)\} = 0 \text{ at every } s, t \geqq 0\,, \tag{1.2}$$

$$E\{X(s, t)X(s', t')\} = \min(s, s')\cdot\min(t, t')\,. \tag{1.3}$$

Further properties of the process are found in Yeh's [8] and [9].

For the square $D = [0, 1] \times [0, 1]$ and its boundary ∂D, Paranjape and Park [6] showed that the probability

$$P\left\{\sup_{\partial D} X(s, t) \geqq \lambda\right\} = 3N(-\lambda) - e^{4\lambda^2}N(-3\lambda)\,, \quad \lambda \geqq 0\,, \tag{1.4}$$

where $N(\cdot)$ stands for the standard normal distribution function. This probability is a lower bound of the yet unknown probability, $P\{\sup_D X(s, t) \geqq \lambda\}$. It is known (see [4] or [7]) that

$$P\left\{\sup_{D} X(s, t) \geqq \lambda\right\} \leqq 4P\{X(1, 1) \geqq \lambda\} = 4N(-\lambda)\,. \tag{1.5}$$

Recently Chan [1] showed that, for every $\varepsilon > 0$,

$$P\left\{\sup_{D} X(s, t) \geqq \lambda\right\} \leqq N(\varepsilon)^{-1}P\left\{\sup_{D} X(s, t) \geqq \lambda - \varepsilon\right\}\,. \tag{1.6}$$

By the same technique as he used in his paper, the upper bound can easily be improved to $N(\varepsilon)^{-1}P\{\sup X(1, t) \geqq \lambda - \varepsilon: 0 \leqq t \leqq 1\} = 2N(-\lambda + \varepsilon)/N(\varepsilon)$. However it turns out to be that even this improved upper bound is not as good as $4N(-\lambda)$ for any $\varepsilon > 0$. In fact

$$4N(-\lambda) < N(\varepsilon)^{-1}P\left\{\sup_{0\leqq t\leqq 1} X(1, t) \geqq \lambda - \varepsilon\right\}\,, \quad \varepsilon > 0\,,$$

and

$$\lim_{\varepsilon\to 0^+} N(\varepsilon)^{-1} P\left\{\sup_{0\leqq t\leqq 1} X(1, t) \geqq \lambda - \varepsilon\right\} = 4N(-\lambda) .$$

More recently Goodman [3] showed that for $\lambda \geqq 0$,

$$2\left\{N(-\lambda) + \lambda \int_\lambda^\infty N(-s)ds\right\} \leqq P\left\{\sup_D X(s, t) \geqq \lambda\right\} . \tag{1.7}$$

Obviously the left-hand side of (1.7) is a much better lower bound of $P\{\sup_D X(s, t) \geqq \lambda\}$ than (1.4). He subsequently proves that

$$\lim_{\lambda\to\infty} \frac{2\{N(-\lambda) + \lambda \int_\lambda^\infty N(-s)ds}{4N(-\lambda)} = 1 , \tag{1.8}$$

thus showing that both $2\left\{N(-\lambda) + \lambda \int_\lambda^\infty N(-s)ds\right\}$ and $4N(-\lambda)$ are very good approximations of $P\{\sup_D X(s, t) \geqq \lambda\}$ for all sufficiently large λ.

The main purpose of this paper is to generalize the above results for more general barriers, namely, to find a formula for

$$P\left\{\sup_{\partial D} X(s, t) - (ast + bs + ct + d) \geqq 0\right\} , \quad a, b, c, d \geqq 0 ,$$

and then find a lower bound for $P\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0\}$ for which (1.7) is a special case. It is apparent that for all $a, b, c, d \geqq 0$

$$P\left\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0\right\} \leqq 4N(-d) . \tag{1.9}$$

In addition we obtain a formula for

$$P\{\sup_{\partial D} |X(s, t)| - (ast + bs + ct + d) \geqq 0\} , \quad a, b, c, d \geqq 0 .$$

Some results on two-parameter Brownian bridge are also included.

2. Some lemmas.

To avoid unnecessary repetitions in the proofs of the theorems, the following lemmas are given. Throughout this paper $W(t)$ and $X(s, t)$ will denote the standard Wiener process and the Yeh-Wiener process, respectively.

LEMMA 1. (*Doob* [2: p. 398]). *If* $a \geqq 0$, $b > 0$, $\alpha \geqq 0$, $\beta > 0$, *then*

$$\begin{aligned} P&\left\{\sup_{0\leqq t<\infty} [W(t) - (at + b)] \geqq 0 \quad or \quad \inf_{0\leqq t<\infty} [W(t) + \alpha t + \beta] \leqq 0\right\} \\ &= \sum_{m=1}^\infty \exp\{-2[m^2ab + (m-1)^2\alpha\beta + m(m-1)(a\beta + \alpha b)]\} \\ &\quad + \exp\{-2[(m-1)^2ab + m^2\alpha\beta + m(m-1)(a\beta + \alpha b)]\} \end{aligned}$$

$$- \exp\{-2[m^2(ab + \alpha\beta) + m(m-1)\alpha\beta + m(m+1)\alpha b]\}$$
$$- \exp\{-2[m^2(ab + \alpha\beta) + m(m-1)a\beta + m(m-1)\alpha b]\} .$$

LEMMA 2. *Let $f(t)$ be a Borel measurable function. Then for each Borel set E of real numbers,*

$$\begin{aligned} &P\{W(t) - f(t) \in E,\ 0 < t \leqq 1 \mid W(1) = u\} \\ &\quad = P\left\{W(t) + u - (t+1)f\left(\frac{1}{t+1}\right) \in \frac{1}{t}E,\ 0 < t < \infty\right\} . \end{aligned} \tag{2.1}$$

Proof. The basic technique used here is the same as the one used by Malmquist in [5]. Observe that $W(t)$ and $tW(1/t)$ are equivalent processes for $t > 0$. Thus, the left-hand side of (2.1) reduces to

$$\begin{aligned} &P\left\{W\left(\frac{1}{t}\right) - \frac{1}{t}f(t) \in \frac{1}{t}E,\ 0 < t \leqq 1 \,\middle|\, W(1) = u\right\} \\ &\quad = P\left\{W\left(\frac{1}{t}\right) - W(1) - \left[\frac{1}{t}f(t) - u\right] \right. \\ &\qquad \left. \in \frac{1}{t}E,\ 0 < t \leqq 1 \,\middle|\, W(1) = u\right\} . \end{aligned}$$

Upon using the fact that $W(1/t - 1)$ and $W(1/t) - W(1)$ are equivalent processes for $t > 0$, and $W(1/t) - W(1)$ and $W(1)$ are independent for $1 \geqq t > 0$, we have the result by the transformation $1/t - 1 \to t$.

LEMMA 2.a. *If $f(t)$ is a Borel measurable function on $[0, 1]$, then*

$$\begin{aligned} &P\left\{\sup_{0 \leqq t \leqq 1} |X(1, t)| - f(t) \geqq 0 \mid X(1, 1) = u\right\} \\ &\quad = P\left\{\sup_{0 \leqq t < \infty} |X(1, t) + u| - (t+1)f\left(\frac{1}{t+1}\right) \geqq 0\right\} , \end{aligned}$$

and the same holds for $X(t, 1)$.

LEMMA 3. *Let $f(s, t)$ be a Borel measurable function on D. Then for each Borel set E of real numbers,*

$$\begin{aligned} &P\{X(s, t) - f(s, t) \in E,\ (s, t) \in (0, 1]^2 \mid X(1, 1) = u\} \\ &\quad = P\left\{X(s+1, t+1) - X(1, 1) \right. \\ &\qquad \left. - \left[(s+1)(t+1)f\left(\frac{1}{s+1}, \frac{1}{t+1}\right) - u\right] \in \frac{E}{st},\ (s, t) \in (0, \infty)^2\right\} . \end{aligned}$$

Proof. This lemma is a two-parameter analogue of Lemma 2, and it can be proved similarly by observing that $X(s, t)$ and $stX(1/s, 1/t)$ are equivalent processes for $s, t > 0$.

LEMMA 4. *Let $f(t)$ and $g(t)$ be any Borel measurable functions on $[0, 1]$. Then for any Borel sets E_1 and E_2 of real numbers,*

$$\begin{aligned}(2.2)\qquad & P\{X(s, 1) - f(s) \in E_1, X(1, t) - g(t) \in E_2, (s, t) \in D \mid X(1, 1) = u\} \\ & = P\{X(s, 1) - f(s) \in E_1, 0 \leqq s \leqq 1 \mid X(1, 1) = u\} \\ & \cdot P\{X(1, t) - g(t) \in E_2, 0 \leqq t \leqq 1 \mid X(1, 1) = u\} .\end{aligned}$$

Proof. Observe first that $X(s, 1)$ and $sX(1/s, 1)$ are equivalent standard Wiener processes for $s > 0$, and so are $X(1, t)$ and $tX(1, 1/t)$ for $t > 0$. Now $s[X(1/s, 1) - X(1, 1) + u]$ and $t[X(1, 1/t) - X(1, 1) + u]$ are independent processes for $1 \geqq s, t > 0$, and they are also independent of $\{X(s, t): (s, t) \in D\}$. Hence (2.2) gives:

$$\begin{aligned}(2.3)\qquad & P\{X(s, 1) - f(s) \in E_1, X(1, t) - g(t) \in E_2, (s, t) \in D \mid X(1, 1) = u\} \\ & = P\{s[X(1/s, 1) - X(1, 1) + u] - f(s) \in E_1, 0 < s \leqq 1\} \\ & \cdot P\{t[X(1, 1/t) - X(1, 1) + u] - g(t) \in E_2, 0 < t \leqq 1\} .\end{aligned}$$

But the two probabilities on the right-hand side of (2.3) are equal to $P\{X(s, 1) - f(s) \in E_1, 0 \leqq s \leqq 1 \mid X(1, 1) = u\}$ and $P\{X(1, t) - g(t) \in E_2, 0 \leqq t \leqq 1 \mid X(1, 1) = u\}$ respectively, and hence the proof is complete.

3. Main results and proofs. In what follows $\{X(s, t): s, t \geqq 0\}$ will be used exclusively for the Yeh-Wiener process.

THEOREM 1. *If $a, b, c, d \geqq 0$, then with $\bar{a} = a + b + c + d$,*

$$\begin{aligned} & P\left\{\sup_{\partial D} X(s, t) - (ast + bs + ct + d) \geqq 0\right\} \\ & = N(-\bar{a}) + e^{-2(a+b)(c+d)} N(a + b - c - d) \\ & + e^{-2(a+c)(b+d)} N(a - b + c - d) \\ & - e^{2(d-a)(b+c+2d)} N(a - b - c - 3d) .\end{aligned}$$

Proof. First observe that

$$\begin{aligned} P_1 & \equiv P\left\{\sup_{\partial D} X(s, t) - (ast + bs + ct + d) \geqq 0\right\} \\ & = P\left\{\sup_{0 \leqq s \leqq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0\right\} \\ (3.1)\qquad & + P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0\right\}\end{aligned}$$

$$- P\left\{\sup_{0\leq s\leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0, \sup_{0\leq t\leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0\right\} .$$

Since $X(s, 1)$ and $X(1, t)$ are equivalent to the standard Wiener process $W(t)$, the first two probabilities on the right of (3.1) can be evaluated explicitly.

Now,

$$\begin{aligned} P_2 &\equiv P\left\{\sup_{0\leq s\leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0, \sup_{0\leq t\leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0\right\} \\ &= P\{X(1, 1) \geqq \bar{a}\} \\ &\quad + \int_{-\infty}^{a+b+c+d} P\left\{\sup_{0\leq s\leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0 , \sup_{0\leq t\leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0 \,|\, X(1, 1) = u\right\} dN(u) . \end{aligned}$$

Due to the fact that

$$\begin{aligned} &P\left\{\sup_{0\leq s\leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0, \sup_{0\leq t\leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0 \,|\, X(1, 1) = u\right\} \\ &= P\left\{\sup_{0\leq s\leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0 \,|\, X(1, 1) = u\right\} \\ &\quad \cdot P\left\{\sup_{0\leq t\leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0 \,|\, X(1, 1) = u\right\} , \end{aligned} \tag{3.2}$$

we may use Lemma 2 to get

$$\begin{aligned} P_2 &= N(-\bar{a}) \\ &\quad + \int_{-\infty}^{a+b+c+d} P\left\{\sup_{s\geq 0} X(s, 1) - [(c + d)s + (\bar{a} - u)] \geqq 0\right\} \\ &\quad \cdot P\left\{\sup_{t\geq 0} X(1, t) - [(b + d)t + (\bar{a} - u)] \geqq 0\right\} dN(u) \\ &= N(-\bar{a}) \\ &\quad + \int_{-\infty}^{a+b+c+d} e^{-2(c+d)(\bar{a}-u)} e^{-2(b+d)(\bar{a}-u)} dN(u) \\ &= N(-\bar{a}) + e^{2(d-a)(b+c+2d)} N(a - b - c - 3d). \end{aligned}$$

The result now readily follows.

COROLLARY. *If $d \geqq 0$, then*

$$P\left\{\sup_{\partial D} X(s, t) \geqq d\right\} = 3N(-d) - e^{4d^2}N(-3d) .$$

This corollary agrees with the result in [6: p. 877].

THEOREM 2. *If* $\{Y(s, t): (s, t) \in D\}$ *is the two-parameter Brownian bridge,* i.e., $\{Y(s, t): (s, t) \in D\} = \{X(s, t): (s, t) \in D \mid X(1, 1) = 0\}$ *and* $a, b, c, d \geqq 0$, *then*

$$\begin{aligned} &P\{\sup_{\partial D} Y(s, t) - (ast + bs + ct + d) \geqq 0\} \\ &\quad = e^{-2(b+d)\bar{a}} + e^{-2(b+d)\bar{a}} - e^{-2(b+c+2d)\bar{a}} . \end{aligned}$$

Proof. This follows from (3.2) by setting $u = 0$.

THEOREM 3. *If* $a, b, c \geqq 0$ *and* $d > 0$, *then with* $\bar{a} = a + b + c + d$ *and* $\bar{c} = c + d$,

$$P\left\{\sup_{\partial D} \frac{|X(s, t)|}{ast + bs + ct + d} \geqq 1\right\} = 2f(a, b, c, d) ,$$

where

$$\begin{aligned} f(a, b, c, d) = N(-\bar{a}) &+ \sum_{K=1}^{\infty} (-1)^{k+1}\Big[e^{-2(a+b)\bar{c}k^2} \int_{-\bar{a}-2\bar{c}k}^{\bar{a}-2\bar{c}k} dN(u) \\ &+ e^{-2(a+c)(b+d)k^2} \int_{-\bar{a}-2(b+d)k}^{\bar{a}-2(b+d)k} dN(u)\Big] \\ &- \sum_{j,k=1}^{\infty} (-1)^{j+k} e^{-2\bar{a}[\bar{c}j^2+(b+d)k^2]}\Big\{e^{2[\bar{c}j+(b+d)k]^2} \\ &\times \int_{-\bar{a}-2[\bar{c}j+(b+d)k]}^{\bar{a}-2[\bar{c}j+(b+d)k]} dN(u) + e^{2[\bar{c}j-(b+d)k]^2} \int_{-\bar{a}-2[\bar{c}j(b+d)k]}^{\bar{a}-2[\bar{c}j-(b+d)k]} dN(u)\Big\} . \end{aligned}$$

Proof. Observe that

$$\begin{aligned} P_3(u) &\equiv P\left\{\sup_{\partial D} \frac{|X(s, t)|}{ast + bs + ct + d} \geqq 1 \,\middle|\, X(1, 1) = u\right\} \\ &= P\Big\{\sup_{0 \leqq s \leqq 1} \frac{|X(s, 1)|}{(a + b)s + (c + d)} \geqq 1 \quad \text{or} \\ &\qquad \sup_{0 \leqq t \leqq 1} \frac{|X(1, t)|}{(a + c)t + (b + d)} \geqq 1 \,\Big|\, X(1, 1) = u\Big\} . \end{aligned}$$

Upon applying Lemma 4, we obtain

$$\begin{aligned} P_3(u) = &P\left\{\sup_{0 \leqq s \leqq 1} \frac{|X(s, 1)|}{(a + b)s + (c + d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\} \\ &+ P\left\{\sup_{0 \leqq t \leqq 1} \frac{|X(1, t)|}{(a + c)t + (b + d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\} \end{aligned}$$

$$- P\left\{\sup_{0\leqq s\leqq 1}\frac{|X(s, 1)|}{(a+b)s+(c+d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\}$$
$$\cdot P\left\{\sup_{0\leqq t\leqq 1}\frac{|X(1, t)|}{(a+c)t+(b+d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\} .$$

Due to Lemma 2.a., it follows

$$P\left\{\sup_{0\leqq s\leqq 1}\frac{|X(s, 1)|}{(a+b)s+(c+d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\}$$
$$= P\left\{\sup_{0\leqq s<\infty}\frac{|X(s, 1)+u|}{(c+d)s+\bar{a}} \geqq 1\right\}$$
$$= P\left\{\sup_{0\leqq s<\infty} X(s, 1) - [(c+d)s + (\bar{a}-u)] \geqq 0\right.$$
$$\left.\text{or } \inf_{0\leqq s<\infty} X(s, 1) + [(c+d)s + (\bar{a}+u)] \leqq 0\right\} .$$

Lemma 1 applied to the last expression gives:

$$P\left\{\sup_{0\leqq s<\infty} X(s, 1) - [(c+d)s(\bar{a}-u)] \geqq 0\right.$$
$$\left.\text{or } \inf_{0\leqq s<\infty} X(s, 1) + [(c+d)s + (\bar{a}+u)] \leqq 0\right\}$$
$$= \sum_{m=1}^{\infty}\{e^{-2\bar{a}(c+d)(2m-1)^2}[e^{2(c+d)(2m-1)u} + e^{-2(c+d)(2m-1)u}]$$
$$- e^{-2\bar{a}(c+d)(2m)^2}[e^{2(c+d)(2m)u} + e^{-2(c+d)(2m)u}]\} .$$

Therefore

$$P\left\{\sup_{0\leqq s\leqq 1}\frac{|X(s, 1)|}{(a+b)s+(c+d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\}$$
$$= \sum_{j=1}^{\infty}(-1)^{j+1}e^{-2\bar{a}(c+d)j^2}[e^{2(c+d)ju} + e^{-2(c+d)ju}]$$

and

$$P\left\{\sup_{0\leqq t\leqq 1}\frac{|X(1, t)|}{(a+c)t+(b+d)} \geqq 1 \,\middle|\, X(1, 1) = u\right\}$$
$$= \sum_{k=1}^{\infty}(-1)^{k+1}e^{-2\bar{a}(b+d)k^2}[e^{2(b+d)ku} + e^{-2(b+d)ku}] .$$

Since $X(1, 1)$ is the standard normal random variable, the result now follows by:

$$P\left\{\sup_{\partial D}\frac{|X(s, t)|}{ast+bs+ct+d} \geqq 1\right\}$$
$$= P\{|X(1, 1)| \geqq \bar{a}\} + \int_{-\bar{a}}^{\bar{a}} P_3(u)dN(u)$$

$$= 2N(-\bar{a}) + \int_{-\bar{a}}^{\bar{a}} P_3(u) dN(u) .$$

THEOREM 4. *If* $a, b, c, d \geqq 0$ *and* $\bar{a} = a + b + c + d$, $\bar{b} = b + c + d$, $\bar{c} = c + d$, *then for* $u < \bar{a}$,

$$P_4 \equiv P\left\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0 \middle| X(1, 1) = u\right\}$$
$$\geqq \begin{cases} \dfrac{\bar{b}}{b}[e^{2\bar{c}(u-\bar{a})} - e^{-2\bar{b}(u-\bar{a})}] + e^{2\bar{b}(u-\bar{a})} , & b > 0 \\ e^{2\bar{c}(u-\bar{a})}[1 + 2\bar{c}(\bar{a} - u)] , & b = 0 . \end{cases}$$

Proof. Upon applying Lemma 3, we obtain

$$\begin{aligned} P_4 = P\Big\{\sup_{s,t \geqq 0} X(s + 1, t + 1) - X(1, t + 1) + X(1, t + 1) - X(1, 1) \\ - [d(s + 1)(t + 1) + c(s + 1) + b(t + 1) + a - u] \geqq 0\Big\} . \end{aligned} \tag{3.3}$$

Consider the fact that $X(s + 1, t + 1) - X(1, t + 1)$ and $X(1, t+1) - X(1, 1)$ are independent processes equivalent to $X(s, t + 1)$ and $X(1, t)$, respectively. The latter $X(1, t)$ will be denoted by $X^*(1, t)$ to signify that it is independent of $X(s, t + 1)$. Due to the fact that $c(s + 1) \leqq c(s + 1)(t + 1)$ for all $c, s, t \geqq 0$, it follows from (3.3)

$$\begin{aligned} P_4 &\geqq P\Big\{\sup_{s,t \geqq 0} X(s, t + 1) + X^*(1, t) - [\bar{c}(t + 1)s + \bar{b}t + \bar{a} - u] \geqq 0\Big\} \\ &\geqq \int_{u-\bar{a}}^{\infty} P\Big\{\sup_{s \geqq 0} X(s, t + 1) - [\bar{c}(t + 1)s - r] \geqq 0 \Big| \sup_{t \geqq 0} X^*(1, t) \\ &\qquad -(\bar{b}t + \bar{a} - u) = r\Big\} p(r, u) dr , \end{aligned} \tag{3.4}$$

where $p(r, u)$ is the probability density of

$$P\left\{\sup_{t \geqq 0} X^*(1, t) - (\bar{b}t + \bar{a} - u) \leqq r\right\}$$
$$= \begin{cases} 1 - e^{-2\bar{b}(\bar{a}+r-u)} , & u - \bar{a} \leqq r \\ 0 , & \text{otherwise} . \end{cases}$$

Thus

$$p(r, u) = \begin{cases} 2\bar{b}e^{-2\bar{b}(\bar{a}+r-u)} , & u - \bar{a} \leqq r \\ 0 , & \text{otherwise} . \end{cases} \tag{3.5}$$

Observe that the probability in the integrand of (3.4) becomes

$$P\left\{\sup_{s\geqq 0} X(s,\, t+1) - [\bar{c}(t+1)s - r] \geqq 0\right\} \tag{3.6}$$
$$= \begin{cases} e^{2\bar{c}r}, & r \leqq 0 \\ 1, & r > 0\,. \end{cases}$$

Therefore, (3.5) and (3.6) together with (3.4) give

$$P_4 \geqq \int_{u-\bar{a}}^{0} e^{2\bar{c}r} 2\bar{b} e^{-2\bar{b}(\bar{a}+r-u)} dr + \int_0^\infty 2\bar{b} e^{-2\bar{b}(\bar{a}+r-u)} dr\,,$$

from which the result readily follows.

The following is a special case ($u = 0$) of Theorem 4, which has broad application in Kolmogorov-Smirnov statistics.

THEOREM 4.a. *If $\{Y(s, t): (s, t) \in D\}$ is the two-parameter Brownian bridge and if $a, b, c, d \geqq 0$, then*

$$P\{\sup_D Y(s, t) - (ast + bs + ct + d) \geqq 0\}$$
$$\geqq \begin{cases} \dfrac{\bar{b}}{b}(e^{-\bar{2a}\bar{c}} - e^{-\bar{2a}\bar{b}}) + e^{-2\bar{a}\bar{b}}, & b > 0 \\ (1 + 2\bar{a}\bar{c})e^{-\bar{2a}\bar{b}}, & b = 0\,. \end{cases}$$

THEOREM 5. *If $a, b, c, d \geqq 0$, then*

$$P\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0\}$$
$$\geqq \begin{cases} N(-\bar{a}) + \dfrac{\bar{b}}{b}[N(\bar{a} - 2\bar{c})e^{-2\bar{c}(a+b)} - N(\bar{a} - 2\bar{b})e^{-2a\bar{b}}] \\ \qquad + N(\bar{a} - 2\bar{b})e^{-2a\bar{b}}, & b > 0 \\ N(-\bar{a}) + \dfrac{2\bar{c}}{\sqrt{2\pi}} e^{-\bar{a}^2/2} + N(\bar{a} - 2\bar{c})(1 + 2\bar{a}\bar{c} - 4\bar{c}^2)e^{-2\bar{a}\bar{c}}, \\ & b = 0\,. \end{cases}$$

In particular,

$$P\{\sup_D X(s, t) - \lambda \geqq 0\} \geqq 2\left[(1 - \lambda^2)N(-\lambda) + \frac{\lambda}{\sqrt{2\pi}} e^{-\lambda^2/2}\right]$$
$$= 2\left[N(-\lambda) + \lambda \int_\lambda^\infty N(-s)ds\right], \quad \lambda \geqq 0\,.$$

Proof. The theorem now can be established by integrating lower estimates of the conditional probability P_4 in Theorem 4 with respect to $dP\{X(1, 1) \leqq u\} = dN(u) = (2\pi)^{-1/2} \exp(-u^2/2)du$. The special case when $a = b = c = 0$ and $d = \lambda$ agrees with Goodman's result (Theorem 3 in [3]).

In order to find sharper upper bounds for the barrier-crossing probabilities we introduce the following: Let $f(s, t)$ be a continuous function on D. If $\sup_D X(s, t) - f(s, t) \geqq 0$, then define $\tau_f = (s_0, t_0)$ where

$$s_0 = \inf \{s \in [0, 1] \mid X(s, t) = f(s, t) \text{ for some } t \in [0, 1]\} ,$$
$$t_0 = \inf \{t \in [0, 1] \mid X(s_0, t) = f(s_0, t)\} ,$$

while if $\sup_D X(s, t) - f(s, t) < 0$, then set $\tau_f = (\infty, \infty)$. Thus with the convention that $(s_1, t_1) \leqq (s_2, t_2)$ if and only if $s_1 \leqq s_2$ and $t_1 \leqq t_2$, we have that

$$P\left\{\sup_D X(s, t) - f(s, t) \geqq 0\right\} = P\left\{\tau_f \leqq (1, 1)\right\} .$$

THEOREM 6. *If* $c, d \geqq 0$, *then*

$$\begin{aligned} &P\left\{\sup_D X(s, t) - (ct + d) \geqq 0\right\} \\ &\leqq 2P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (ct + d) \geqq 0\right\} \\ &= 2[1 - N(c + d) + \exp(-2cd)N(a - b)] . \end{aligned}$$

Proof. Let τ stand for τ_f when $f(s, t) = ct + d$. Define

$$F(s, t) \equiv P\{\tau \leqq (s, t)\} .$$

Then

$$\begin{aligned} F(1, 1) &= P\left\{\sup_D X(s, t) - (ct + d) \geqq 0\right\} \\ &= P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (ct + d) \geqq 0\right\} \\ &\quad + P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (ct + d) < 0, \sup_D X(s, t) - (ct + d) \geqq 0\right\} \\ &= P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (ct + d) \geqq 0\right\} \qquad (3.7) \\ &\quad + \int_0^1 P\left\{\sup_{0 \leqq t' \leqq 1} X(1, t') - (ct' + d) < 0 \,\middle|\, \tau = (s, t)\right\} dF(s, t) \\ &\leqq P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (ct + d) \geqq 0\right\} \\ &\quad + \int_0^1 P\left\{X(1, t) - (ct + d) < 0 \,\middle|\, \tau = (s, t)\right\} dF(s, t) . \end{aligned}$$

On account of the fact that $\tau = (s, t)$ implies $X(s, t) = ct + d$ and $X(1, t) - X(s, t)$ is independent of the conditioning $\tau = (s, t)$, it follows that

$$\begin{aligned}(3.8)\qquad &\int_0^1 P\left\{X(1, t) - (ct + d) < 0 \,\middle|\, \tau = (s, t)\right\} dF(st) \\ &= \int_0^1 P\left\{X(1, t) - X(s, t) < 0\right\} dF(s, t) = \frac{1}{2} F(1, 1) .\end{aligned}$$

The theorem now follows readily from (3.7) and (3.8).

COROLLARY 6.1. *If* $b, c, d \geqq 0$, *then*

$$\begin{aligned}(3.9)\qquad &P\left\{\sup_D X(s, t) - (bs + ct + d) \geqq 0\right\} \\ &\leqq 2P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (b^*t + d) \geqq 0\right\} , \quad b^* = \max\{b, c\} .\end{aligned}$$

Proof. The result follows immediately by observing that

$$\begin{aligned}&P\left\{\sup_D X(s, t) - (bs + ct + d) \geqq 0\right\} \\ &\leqq \min\left\{ P\left[\sup_D X(s, t) - (bs + d) \geqq 0\right] , \right. \\ &\qquad \left. P\left[\sup_D X(s, t) - (ct + d) \geqq 0\right]\right\} .\end{aligned}$$

The right-hand side of (3.9) can also serve as an upper bound of $P\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0\}$, and it is certainly a substantial improvement over (1.9). We state this fact formally as a corollary.

COROLLARY 6.2. *If* $a, b, c, d \geqq 0$, *then*

$$\begin{aligned}&P\left\{\sup_D X(s, t) - (ast + bs + ct + d) \geqq 0\right\} \\ (3.10)\qquad &\leqq 2P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - (b^*t + d) \geqq 0\right\} \\ &\leqq 2P\left\{\sup_{0 \leqq t \leqq 1} X(1, t) - d \geqq 0\right\} = 4N(-d) ,\end{aligned}$$

where $b^* = \max\{b, c\}$.

4. **Supremum over rectangular regions.** Some adjustments are needed to apply the results for the more general rectangular region $Q = [0, S] \times [0, T]$. The conversion formulas are given by:

$$\begin{aligned}(4.1)\qquad &P\left\{\sup_{\partial Q} X(s, t) - (ast + bs + ct + d) \geqq 0\right\} \\ &= P\left\{\sup_{\partial D} X(s, t) - (a'st + b's + c't + d') \geqq 0\right\} ,\end{aligned}$$

where $a' = a\sqrt{ST}$, $b' = b\sqrt{S/T}$, $c' = c\sqrt{T/S}$, and $d' = d/\sqrt{ST}$.

$$\begin{aligned}(4.2)\quad & P\left\{\sup_Q X(s,\,t) - (ast + bs + ct + d) \geqq 0 \mid X(S,\,T) = u\right\} \\ & = P\left\{\sup_D X(s,\,t) - (a'st + b's + c't + d') \geqq 0 \,\middle|\, X(1,\,1) = u'\right\},\end{aligned}$$

where a', b', c', d' are as in (4.1) and $u' = u/\sqrt{ST}$. In (4.1), if ∂Q is replaced by Q, then D replaces ∂D.

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