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PROJECTIVE MODULES OVER SUBRINGS OF $k[X, Y]$ GENERATED BY MONOMIALS

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In this paper we study finitely generated projective modules over affine subrings A of $k[X, Y]$ generated by monomials. If A is normal, then all finitely generated projective A-modules are free. If A is not normal, we show that finitely generated projective A-modules stably have the form free $\oplus$ rank one

1. Introduction. In this paper we study projective modules over subrings A of $k[X, Y]$ generated by monomials. We study conditions on A so that all finitely generated projective A-modules have the form free $\oplus$ rank one. In §IV we use Seshadri's localization technique to show that all finitely generated projective A-modules are free when A is an affine normal subring of $k[X, Y]$ generated by monomials. If we drop the assumption that A is normal it need not be true that all finitely generated projective A-modules are free. However, in §V we show that finitely generated projective A-modules stably have the form free $\oplus$ rank one. We also give sufficient conditions on k for finitely generated projective A-modules to have the form free $\oplus$ rank one. These results do not generalize to arbitrary subrings of $k[X, Y]$.

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2. Preliminaries. All rings A will be commutative with 1. Spec (A) is the set of all prime ideals of A and max (A) is the subset of spec (A) consisting of maximal ideals. We give spec (A) the Zariski topology. If X is a topological space, the combinatorial dimension of X will be denoted by dim X. If A is a ring, the group of units of A is A^*. SL (n, A) is the group of $n \times n$ matrices over A with determinant 1, and $E(n, A)$ is the subgroup of SL (n, A) generated by elementary matrices. The Krull dimension of A will be denoted by dim A. k will always be a field. Let P be a finitely generated projective A-module and $Q \in$ spec (A). We define $\text{rank}_Q P$ to be $\dim_{A_Q/QA_Q} P_Q/QP_Q$. If $\text{rank}_Q P$ is constant, we will denote it by rank P. Our K-theory notation will follow Bass [4].

$\tilde{K}_0(A)$ is the subgroup of $K_0(A)$ generated by $[A^{\text{rank} P}] - [P]$ for finitely generated projective A-modules P, and Pic (A) is the group of isomorphism classes of finitely generated projective A-modules of rank one. There is a natural determinant epimorphism det: $\tilde{K}_0(A) \to \operatorname{Pic}(A)$ defined by $\det([P]) = \Lambda^n(P)$ where $n = \operatorname{rank} P$. We denote the kernel of this map by $SK_0(A)$. Clearly $SK_0(A) = 0$ iff every finitely generated projective A-module stably has the form free $\oplus$ rank one. In this case P is stably isomorphic to $A^{n-1} \oplus \Lambda^n(P)$.

A commutative square of rings

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ A_2 & \xrightarrow{g_2} & B \end{array}$$

is cartesian if $g_1(x) = g_2(y)$ implies there is a unique $z \in A$ with $f_1(z) = x$ and $f_2(z) = y$.

THEOREM 2.1 (*Milnor* [10]). *Given a cartesian square of rings with* g_1 *surjective, the following ("Mayer-Vietoris") sequences are exact*

$$(1)\qquad \begin{aligned} 0 \longrightarrow A^* \longrightarrow A_1^* \oplus A_2^* \longrightarrow B^* \xrightarrow{\partial} \operatorname{Pic}(A) \\ \longrightarrow \operatorname{Pic}(A_1) \oplus \operatorname{Pic}(A_2) \longrightarrow \operatorname{Pic}(B) \end{aligned}$$

$$(2)\qquad \begin{aligned} K_1(A) \longrightarrow K_1(A_1) \oplus K_1(A_2) \longrightarrow K_1(B) \xrightarrow{\partial} \tilde{K}_0(A) \\ \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow \tilde{K}_0(B)\ . \end{aligned}$$

Moreover, if $\mathrm{GL}(n, A_1) \to \mathrm{GL}(n, B)$ *is surjective for all* n *and all finitely generated projective* A_1 *and* A_2*-modules are free, then all finitely generated projective* A*-modules are free.*

Using the natural determinant maps, sequences (1) *and* (2) *may be connected to obtain the following commutative diagram with exact rows and columns.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & SK_1(A) & \longrightarrow & SK_1(A_1) \oplus SK_1(A_2) & \longrightarrow & SK_1(B) & \xrightarrow{\partial} & SK_0(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(B) & \xrightarrow{\partial} & \tilde{K}_0(A) \\
 & & \downarrow & & {\scriptstyle h}\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A^* & \longrightarrow & A_1^* \oplus A_2^* & \longrightarrow & B^* & \xrightarrow{\partial} & \operatorname{Pic}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

$$\begin{array}{ccccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
\longrightarrow & SK_0(A_1) \oplus SK_0(A_2) & \longrightarrow & SK_0(B) \\
 & \downarrow & & \downarrow \\
\longrightarrow & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) & \longrightarrow & \tilde{K}_0(B) \\
 & \downarrow & & \downarrow \\
\longrightarrow & \mathrm{Pic}\,(A_1) \oplus \mathrm{Pic}\,(A_2) & \longrightarrow & \mathrm{Pic}\,(B) \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}$$

The following lemma is obvious.

LEMMA 2.2. *Suppose that* $SK_0(A_1) = SK_0(A_2) = 0$, *then*
(1) $SK_0(A) = \partial(SK_1(B))$.
(2) $SK_1(B) = 0$ *implies* $SK_0(A) = 0$.
(3) *If* h *is an isomorphism, then* $SK_1(B) \approx SK_0(A)$.

We review a localization technique due to Seshadri which will be used in §IV. For details one may consult [4]. A set S of ideals of A is multiplicative if $I, J \in S$ implies $IJ \in S$. A prime ideal P is special if P is invertible and A/P is a PID for which $E(n, A/P) = \mathrm{SL}\,(n, A/P)$ for all n. A multiplicative set of ideals is special if it is generated by special prime ideals. If S is any multiplicative set of invertible ideals, we define $S^{-1}A = \bigcup_{I \in S} I^{-1}$. For M an A-module, $S^{-1}M = S^{-1}A \otimes_A M$.

THEOREM 2.3 (*Seshadri*). *Let* A *be a commutative noetherian ring and* S *a special multiplicative set of invertible ideals. Let* P *be a finitely generated projective* A*-module and suppose that* $S^{-1}P \approx L'_1 \oplus \cdots \oplus L'_n$ *where each* L'_i *is a finitely generated projective* $S^{-1}A$*-module of rank one. Then*

(1) *There are finitely generated projective* A*-modules* L_i *of rank one with* $L'_i \approx S^{-1}A \otimes_A L_i$ *for* $i = 1, \cdots, n$.

(2) *For each choice of* L_i *in* (1) *there is an* I *inthe group of invertible ideals generated by* S *such that* $P \approx IL_1 \oplus L_2 \oplus \cdots \oplus L_n$.

COROLLARY 2.4. *Let* A, S, *and* P *be as above. If* $S^{-1}P$ *is the direct sum of a free* $S^{-1}A$*-module and a projective* $S^{-1}A$*-module of rank one, then* P *is also the direct sum of a free* A*-module and a projective* A*-module of rank one.*

The next two lemmas will also be used in §IV. I do not know a reference for Lemma 2.6, however compare [16, p. 7].

LEMMA 2.5 ([11]). *Let* $A = A_0 \oplus A_1 \oplus \cdots$ *be a graded affine normal domain with* A_0 *a field, then* $\operatorname{Pic}(A) = 0$.

LEMMA 2.6. *Let* A *be a commutative ring with* $\max(A)$ *noetherian and* $V(I) = F \subset \max(A)$ *closed with* $\dim(\max(A)\backslash F) \leqq 1$. *Let* P *be a finitely generated projective* A*-module with rank* $P = n \geqq 2$, *and assume that* P/IP *is a free* A/I*-module. Then* $P \approx A^{n-1} \oplus \Lambda^n(P)$.

Proof. It is sufficient to show that if P is a finitely generated projective A-module with rank $P \geqq 2$ and P/IP a free A/I-module, then $P \approx A \oplus P'$ with $P'/IP'A/I$-free.

Let $\max(A)\backslash F = U_1 \cup \cdots \cup U_t$ be a decomposition into closed irreducible components and pick $M_i \in U_i$. For $s \in P$ and $N \in \max(A)$, let $s(N)$ be the image of s in P_N/NP_N. P/IP is free, so by the Chinese Remainder Theorem there is a $s \in P$ with $s(M_i) \neq 0, 1 \leqq i \leqq t$, and $\bar{s}$ a basis element for P/IP. Clearly $s(M) \neq 0$ for $M \supset I$.

Let $Z(s) = \{J \in \max(A) \mid s(J) = 0\}$; then $Z(s) \subset \max(A)\backslash F$ and $Z(s)$ is closed [16, p. 6]. Each $M_i \notin Z(s)$, so $Z(s)$ is 0-dimensional and hence finite, say $Z(s) = \{I_1, \cdots, I_l\}$. Rank $P \geqq 2$, so as above we may choose $t \in P$ such that (1) $t(I_i) \neq 0$ for $1 \leqq i \leqq l$, (2) $\bar{t}$ and $\bar{s}$ form part of a basis for P/IP, and (3) $s(M_i)$ and $t(M_i)$ are linearly independent for $1 \leqq i \leqq t$.

As above $Z(s, t) = \{M \in \max(A) \mid s(M)$ and $t(M)$ are linearly dependent$\}$ is finite. Let $Y = Z(s, t)\backslash Z(s) = \{J_1, \cdots, J_m\}$, then pick $0 \neq a \in (J_1 \cap \cdots \cap J_m)\backslash(I_1 \cup \cdots \cup I_l)$, or let $a = 1$ if $Y = \phi$. Let $u = s + at$, then $u(M) \neq 0$ for all $M \in \max(A)$, so Au is a direct summand of P [16, p. 6]. Note that $\bar{u} = \overline{s + at}$ is part of a basis for P/IP.

$(a_0, \cdots, a_n) \in A^{n+1}$ is unimodular if $Aa_0 + \cdots + Aa_n = A$. $U_{n+1}(A)$ is the set of all unimodular elements in A^{n+1}. The stable range of A, denoted by $\operatorname{sr}(A)$, is $\leqq d$ if given any unimodular row $(a_0, \cdots, a_d)$, there exist $c_0, \cdots, c_{d-1} \in A$ so that $(a_0 + c_0a_d, \cdots, a_{d-1} + c_{d-1}a_d) \in A^d$ is unimodular. It is well-known that $\operatorname{sr}(A) \leqq 1 + \dim A$ and that $A^{n+1} \approx A \oplus P$ implies $A^n \approx P$ whenever $n \geqq \operatorname{sr}(A)$ ([4, p. 239]).

3. Subrings of $k[X, Y]$ generated by monomials. Subrings A of $B = k[X, Y]$ generated by monomials arise naturally as either the ring of invariants of an automorphism of B of finite order or as the kernel of a k-derivation of B when $\operatorname{char} k = p \neq 0$. Clearly if A is as above, then $A \subset B$ is integral, so not all affine normal subrings of B generated by monomials are one of these two types. However, over an algebraically closed field of char $k = 0$, any affine normal subring of B generated by monomials is isomorphic to B^G where G is the cyclic group generated by an automorphism of the form $\phi: X \mapsto aX$, $Y \mapsto bY$, $a, b \in k$. We state the following three propositions without proof; for details see [1] or [2].

PROPOSITION 3.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $A \approx A'$ where $A' = k[X, Y]$ or $A' = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$, $\gcd(j, n) = 1$, and "$-$" denotes* mod n.

PROPOSITION 3.2. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials. If* $\dim A = 1$, *then $A \approx A'$ where A' is an affine subring of $k[X]$ generated by monomials. If* $\dim A = 2$, *then $A \approx A''$ where A'' is an affine subring of B generated by monomials with $A'' \subset k[X, Y]$ integral.*

PROPOSITION 3.3. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $\bar{A}$, the integral closure of A, is also an affine subring of B generated by monomials. The conductor of $\bar{A}/A$ contains a nonzero monomial.*

We recall that $\operatorname{sing}(A) = \{P \in \operatorname{spec}(A) \mid A_p$ is not regular$\}$. If A is an affine normal domain of dim 2, then $\operatorname{sing}(A)$ is a closed subset of $\operatorname{spec}(A)$ of dim 0, and hence finite [9, p. 245]. If in addition $A \subset k[X, Y]$ is generated by monomials, we can explicitly describe $\operatorname{sing}(A)$.

PROPOSITION 3.4. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and A not regular. Then the origin is the only singularity of A, that is,* $\operatorname{sing}(A) = \{M = (X, Y)B \cap A\}$.

Proof. The proof of Proposition 3.1 [2, Thm. 2.5] shows that the isomorphism is just a change of variables which does not change the origin. Thus we may assume that $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. It is sufficient to show that for each of the generators $f_1 = X^n, f_2 = XY^j, \cdots, f_{n+1} = Y^n$ of M, $A[1/f_i]$ is regular. For if N is any other maximal ideal, then some $f_i \notin N$, and thus A_N is regular since it is a localization of the regular ring $A[1/f_i]$.

Clearly $A[1/Y^n] = k[XY^j, Y^n][1/Y^n]$ which is regular. Similarly $A[1/X^n] = k[X^n, YX^i][1/X^n]$ where $YX^i \in A$. If $a, b \neq 0$, then $A[1/X^aY^b]$

contains $1/X^n$ and $1/Y^n$ and thus is a localization of $A[1/Y^n]$. So $A[1/X^a Y^b]$ is also regular.

We note that a subring of $B = k[X, Y]$ generated by monomials is a graded ring with the natural grading it inherits from B.

4. Projective modules over affine normal subrings of $k[X, Y]$ generated by monomials.

THEOREM 4.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials, then all projective A-modules are free.*

Proof. Let P be a projective A-module. If P is not finitely generated, then P is free by a result of Bass [5] or Hinohara [8]. So we may assume that P is finitely generated.

If $\dim A = 1$, then by Serre's theorem [4, p. 173], P has the form free $\oplus$ rank one. But $\mathrm{Pic}(A) = 0$ by Lemma 2.5, so P is free. Thus, we may assume that $\dim A = 2$. By Propositions 3.1 and 3.2 we may assume that $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$.

Let $\bar{B} = \bar{k}[X, Y]$ where $\bar{k}$ is the algebraic closure of k. The maximal ideals of $\bar{B}$ are of the form $M_{a,b} = (X - a, Y - b)$, and thus the maximal ideals of A are of the form $A \cap M_{a,b}$ because $A \subset \bar{B}$ is integral.

For each $0 \neq b \in \bar{k}$ let $Q_b = (Y - b)\bar{B} \cap A$. Clearly

$$A/Q_b \approx k[T^n, b^jT, \overline{b^{2j}}T^2, \cdots, b^{\overline{(n-1)j}}T^{n-1}, b^n] .$$

$b \in \bar{k}$ is algebraic over k, so $k' = k[b^n]$ is a field. Each $b^{\overline{ij}}T^i = (b^n)^l(b^jT)^i$ for some integer l. Thus $A/Q_b \approx k'[b^jT] = k'[S]$ is a euclidean ring because $S = b^jT$ is transcendental over k'.

Next we show that $Q_b (b \neq 0)$ is invertible. It is sufficient to show that $(Q_b)_N$ is principal for each maximal ideal N of A. If $Q_b \not\subset N$, then $(Q_b)_N = A_N$. If $Q_b \subset N$, then clearly $N \neq (X, Y)\bar{B} \cap A$. So by Proposition 3.4, A_N is a regular local ring and hence factorial. Thus $(Q_b)_N$ is a *ht* one prime ideal in a factorial ring and hence principal. So Q_b is locally principal and thus invertible. Q_b is actually principal because $\mathrm{Pic}(A) = 0$ by Lemma 2.5. In fact $Q_b = (Y - b)\bar{B} \cap A = fA$ where $f \in k[Y^n]$ is the polynomial of least degree satisfying $f(b) = 0$.

For each $0 \neq b \in \bar{k}$ Q_b is an invertible ideal such that A/Q_b is a euclidean ring, and thus $E(n, A/Q_b) = \mathrm{SL}(n, A/Q_b)$ for all n. Thus each Q_b is a special prime ideal. Let S be the multiplicative set generated by the Q_b for $0 \neq b \in \bar{k}$. We show that all finitely generated projective $S^{-1}A$-modules have the form free $\oplus$ rank one.

Let $I = (Y\bar{B} \cap A)S^{-1}A$ and $Z = \max(S^{-1}A \backslash V(I))$. S kills all the maximal ideals $(X - a,\ Y - b) \cap A$ when $b \neq 0$; so $\dim Z \leqq 1$.

$$A/(Y\bar{B} \cap A) \approx k[T] \quad \text{and} \quad S^{-1}A/I \approx S^{-1}(A/Y\bar{B} \cap A)\ ,$$

so $\operatorname{Pic}(S^{-1}A/I) = 0$ also. Thus all finitely generated projective $S^{-1}A/I$-modules are free. By Lemma 2.6 all finitely generated projective $S^{-1}A$-modules have the form free $\oplus$ rank one. Thus all finitely generated projective A-modules have the form free $\oplus$ rank one by Corollary 2.4. But $\operatorname{Pic}(A) = 0$, so all finitely generated projective A-modules are free.

Seshadri [17] first showed that all finitely generated projective $k[X, Y]$-modules are free. Murthy and Pedrini [12] showed that all finitely generated projective A-modules are free if $A = k[X^n, XY, Y^n]$ or $A = k[X^n, XY^{n-1}, \cdots, X^{n-1}Y, Y^n]$. Our result generalizes these. Quillen [15] and Suslin have recently, and independently, proved Serre's problem. That is, all finitely generated projective $k[X_1, \cdots, X_n]$-modules are free. The following conjecture thus seems reasonable.

Conjecture. Let A be an affine normal subring of $k[X_1, \cdots, X_n]$ generated by monomials, then all finitely generated projective A-modules are free.

We can however prove a weaker version of this conjecture. First a result which follows from Quillen's work [15]. The author learned of this result in a course given by R. G. Swan.

PROPOSITION 4.2. *Let A be a commutative ring and $f \in A[X]$ a monic polynomial. Let P and Q be finitely generated projective $A[X]$-modules with*

(1) *Q is extended from A.*

(2) *$fQ \subset P \subset Q$.*

Then P and Q are isomorphic.

Proof (sketch). The proof is similar to that of [15, Thm. 3]. Let $A(X)$ denote the localization of $A[X]$ with respect to the multiplicative system of monic polynomials. Let $Q \approx Q_0 \otimes_A A[X]$. Since $f \in A[X]$ is monic, by (2), $P \otimes_{A[X]} A(X) \approx Q \otimes_{A[X]} A(X) \approx Q_0 \otimes_A A(X)$. Then as in [15, Thm. 3], P is extended from A, say $P \approx P_0 \otimes_A A[X]$. Thus $P_0 \approx Q_0$, and so $P \approx Q$.

THEOREM 4.3. *Let A be an affine normal subring of $k\ [X, Y]$ generated by monomials, then all finitely generated projective $A[X_1, \cdots, X_n]$-modules are free.*

Proof. By induction on n, the case $n = 0$ is just Theorem 4.1. Suppose $A = k[f_1, \cdots, f_r]$ with $f_i \in k[X, Y]$ and let $B = A[X_1, \cdots, X_n]$. Let S be the set of monic polynomials in $k[X_{n+1}]$. Then $B[X_{n+1}]_S = k(X_{n+1})[f_1, \cdots, f_r][X_1, \cdots, X_n]$. Clearly $k(X_{n+1})[f_1, \cdots, f_r]$ is still an affine normal subring of $k(X_{n+1})[X, Y]$ generated by monomials. So by induction all finitely generated projective $B[X_{n+1}]_S$-modules are free. Let P be a finitely generated projective $B[X_{n+1}]$-module; then P_S is free, say $P_S \approx F_S$ where F is a finitely generated free $B[X_{n+1}]$-module. Thus there exists a $g \in S$ so that $P_g \approx F_g$ and hence $g^m F \subset P \subset F$ for some m. By Proposition 4.2 $P \approx F$, and hence is free.

Thus affine normal subrings A of $k[X, Y]$ generated by monomials are nontrivial examples of nonregular rings for which $NK_0(A) = 0$, where $NK_0(A) = \ker(K_0(A[T]) \to K_0(A))$, induced by $T \mapsto 0$. In fact, it is an open question if $NK_0(A) = 0$ for all normal domains.

5. **Projective modules over subrings of $k[X, Y]$ generated by monomials.** If we drop the assumption that A is normal, all finitely generated projective A-modules need not be free.

EXAMPLE 5.1 ([7]). Let $A = k[X^2, X^3, Y]$, then not all finitely generated projective A-modules are free. $P = (1 + XY, 1 + XY + X^2Y^2)$ is a rank one projective A-module (invertible ideal in $k(X, Y)$) which is not free. In fact, $K_0(A) \approx \boldsymbol{Z} \oplus \operatorname{Pic}(A) \approx \boldsymbol{Z} \oplus k[Y]$. That $K_0(A) \approx \boldsymbol{Z} \oplus \operatorname{Pic}(A)$ is just Theorem 5.5. We show that $\operatorname{Pic}(A) \approx k[Y]$. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, X^3, Y] & \hookrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[Y] & \hookrightarrow & B/I = k[\varepsilon][Y] \end{array}$$

Here $I = (X^2, X^3)B$ is contained in the conductor ideal and $\varepsilon^2 = 0$. By (1) of Theorem 2.1, $\operatorname{Pic}(A) \approx k[\varepsilon][Y]^*/k^*$. But, as abelian groups, $k[\varepsilon][Y]^*/k^* \approx k[Y]$.

Of course all finitely generated projective A-modules may be free even though A is not normal.

EXAMPLE 5.2. Let $A = k[X^2, XY, Y]$, then all finitely generated projective A-modules are free. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, XY, Y] & \longrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[X^2] & \longrightarrow & B/I = k[X]\,. \end{array}$$

Here $I = YB$ is the conductor ideal. All finitely generated projective B and A/I-modules are free and all $\mathrm{GL}(n, B) \to \mathrm{GL}(n, B/I)$ are surjective. Thus by Theorem 2.1, all finitely generated projective A-modules are free.

We show that if A is an affine subring of $B = k[X, Y]$ generated by monomials, then $SK_0(A) = 0$, that is, the natural map det: $\widetilde{K}_0(A) \to \mathrm{Pic}(A)$ is an isomorphism. This just means that stably any finitely generated projective A-module has the form free $\oplus$ rank one.

Let A be an affine subring of $B = k[X, Y]$ generated by monomials, $\bar{A}$ the integral closure of A, and I the conductor ideal. $I \neq 0$, so let J be any nonzero ideal contained in I. Thus $\dim A/J$, $\dim \bar{A}/J \leqq 1$, so $SK_0(A/J) = SK_0(\bar{A}/J) = 0$ by Serre's theorem [4, p. 173]. By Theorems 3.3 and 4.1 $SK_0(\bar{A}) = 0$, so by Lemma 2.2 it is sufficient to show that $SK_1(\bar{A}/J) = 0$. Again we may assume that $\dim A = 2$, and thus by Propositions 3.1, 3.2, and 3.3 we may assume that $\bar{A} = k[X^n, XY^j, X^2Y^{\overline{2j}}, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. Also I contains a nonzero monomial $f = X^aY^b$ with $a, b \neq 0$; let $J = f\bar{A}$.

LEMMA 5.3. *Let A and B be commutative rings with $A \subset B$ integral. If $I \subset A$ is an ideal of A, then* $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.

Proof. Clearly $\sqrt[A]{I} \subset \sqrt[B]{IB} \cap A$. Let $P \in \mathrm{spec}(A)$ with $I \subset P$. $A \subset B$ is integral, so there is a $\bar{P} \in \mathrm{spec}(B)$ with $P = \bar{P} \cap A$. $I \subset P \subset \bar{P}$, so $IB \subset \bar{P}$. Thus $\sqrt[B]{IB} \subset \bar{P}$ and $\sqrt[B]{IB} \cap A \subset \bar{P} \cap A = P$. So $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.

LEMMA 5.4. $SK_1(\bar{A}/f\bar{A}) = 0$.

Proof. By above $\bar{A}/f\bar{A} = k[X^n, XY^j, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]/X^aY^b\bar{A}$. Let $B = k[X, Y]$, then $\sqrt[B]{fB} = XYB$. By Lemma 5.3 $\sqrt[\bar{A}]{f\bar{A}} = XYB \cap \bar{A}$. By [4, p. 469], $SK_1(\bar{A}/f\bar{A}) \approx SK_1((\bar{A}/f\bar{A})/(\sqrt[\bar{A}]{f\bar{A}}/f\bar{A})) \approx SK_1(\bar{A}/\sqrt[\bar{A}]{f\bar{A}})$. Clearly $\bar{A}/\sqrt[\bar{A}]{f\bar{A}} \approx k[X, Y]/(XY)$. But $SK_1(k[X, Y]/(XY)) = 0$, so $SK_1(\bar{A}/f\bar{A}) = 0$.

THEOREM 5.5. *Let A be an affine subring of $k[X, Y]$ generated by monomials, then $SK_0(A) = 0$. Thus all finitely generated projective A-modules stably have the form free $\oplus$ rank one.*

COROLLARY 5.6. *Let A be a subring of $k[X, Y]$ generated by monomials, then all finitely generated projective A-modules stably have the form free $\oplus$ rank one.*

Proof. This follows from the following well-known result. Let M be a finitely presented A-module, then there is a noetherian subring R of A and a finitely presented R-module M' with $M \approx M' \otimes_R A$. If M is projective, M' may also be chosen to be projective.

Theorem 5.5 is rather unsatisfying because it does not say that any finitely generated projective A-module has the form free $\oplus$ rank one, but only that this is stably true. Since $\dim A \leqq 2$, by Bass' Cancellation Theorem [4, p. 184], if $\operatorname{rank} P = n \geqq 3$, then actually $P \approx A^{n-1} \oplus \Lambda^n(P)$. If rank $P = 2$, we only have $P \oplus A \approx A^2 \oplus \Lambda^2(P)$. If k is algebraically closed, by a cancellation theorem of Murthy and Swan [13], $P \approx A^{n-1} \oplus \Lambda^n(P)$. I know of no examples where $P \not\approx A^{n-1} \oplus \Lambda^n(P)$.

If $\operatorname{Pic}(A) = 0$, then all finitely generated projective A-modules are stably free. If $\operatorname{sr}(A) \leqq 2$, then $E(3, A)$ acts transitively on $U_3(A) = \{(a_0, a_1, a_2) \in A^3 \mid (a_0, a_1, a_2) \text{ unimodular}\}$, so all finitely generated projective A-modules are free. This happens when k is algebraic over a finite field [18, p. 45]. We next show that this also happens whenever $1/2 \in k$.

We recall a few definitions. $KSp_0(A)$ is the Grothendieck group with generators $[P]$ for each symplectic A-module P and relations $[P] = [Q]$ if $P \approx Q$ and $[P \perp Q] = [P] + [Q]$. $W(A)$ is the kernel of the natural map $KSp_0(A) \to K_0(A)$ given by $[P] \mapsto [P]$ which forgets the symplectic structure. W is a functor from rings to abelian groups. For more details one is referred to [6] or [18].

LEMMA 5.7 (*C. Weibel and R. G. Swan*). *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring and F a functor on rings. If the natural map induces an isomorphism $F(A) \to F(A[T])$, then the natural map $A_0 \to A$ also induces an isomorphism $F(A_0) \to F(A)$.*

Proof. Define $f: A \to A[T]$ by $f: \Sigma a_i \to \Sigma a_i T^i$. By hypothesis the two maps $F(A[T]) \to F(A)$ induced by $T \mapsto 0$ and $T \mapsto 1$ are both isomorphisms. Consider the composition $A \xrightarrow{f} A[T] \to A$. If $T \mapsto 1$ we obtain the identity, while $T \mapsto 0$ gives the natural augmentation $A \to A_0$. Thus the natural map $A \to A[T] \to A_0$ induces a monomorphism $F(A) \to F(A_0)$. But this map is always surjective, so $F(A_0) \approx F(A)$.

PROPOSITION 5.8. *Let A be an affine subring of $k[X, Y]$ generated by monomials. If* $\operatorname{Pic}(A) = 0$ *and* $1/2 \in k$, *then all finitely generated projective A-modules are free.*

Proof. By a theorem of Karoubi [6, p. 8], when R is a commutative ring with $1/2 \in R$, $R \to R[T]$ induces an isomorphism $W(R) \to$

$W(R[T])$. A is a graded ring with $A_0 = k$, so $W(A) \approx W(k)$ by Lemma 5.7. It is well-known than $W(k) = 0$ [6, p. 8], so also $W(A) = 0$.

By a result of Vaserstein [6, p. 7], there is a natural map

$$\phi\colon \mathrm{SL}(3, R)\backslash U_3(R) \longrightarrow W(R)$$

which is bijective if $E(r, R)$ acts transitively on $U_r(R)$ for all $r \geqq 4$.

In our case $W(A) \approx W(k) = 0$ and $E(r, A)$ acts transitively on $U_r(A)$ for all $r \geqq 4$ since $\mathrm{sr}(A) \leqq 3$. Thus $\mathrm{SL}(3, A)\backslash U_3(A) = 1$; that is, $\mathrm{SL}(3, A)$ acts transitively on $U_3(A)$. So all finitely generated projective A-modules which are stably free are actually free. But $\mathrm{Pic}(A) = 0$, so all finitely generated projective A-modules are free by Theorem 5.5.

6. **Subrings A of $k[X, Y]$ with $\mathrm{Pic}(A) = 0$.** It is not hard to determine precisely which subrings A of $k[X, Y]$ generated by monomials have $\mathrm{Pic}(A) = 0$. If $\dim A = 1$, clearly $A \approx k[X]$ iff $\mathrm{Pic}(A) = 0$. If $\dim A = 2$, by Proposition 3.2 we may assume that $A \subset k[X, Y]$ is integral.

PROPOSITION 6.1. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and let $\bar{A}$ be the integral closure of A. Then* $\mathrm{Pic}(A) = 0$ *iff*

(1) *Let X^m and Y^n be the lowest powers of X and Y in A, then X^i, $Y^j \in A$ imply $m \mid i$ and $n \mid j$.*

(2) *$XYB \cap \bar{A}$ is contained in the conductor of $\bar{A}/A$.*

Proof. We prove the notationally easier case with $\bar{A} = B = k[X, Y]$. Otherwise we may assume $\bar{A} = k[X^n, XY^j, \cdots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ and the proof is similar.

($\Leftarrow$) Let $I = XYB$, then $A/I \approx k[X, Y]/(XY)$. Clearly $A^* = B^* = (A/I)^* = (B/I)^* = k^*$. Also $\mathrm{Pic}(A/I) = 0$. This follows from Theorem 2.1 applied to following cartesian square.

$$\begin{array}{ccc} k[X, Y]/(XY) & \longrightarrow & k[X, Y]/(X) \\ \downarrow & & \downarrow \\ k[X, Y]/(Y) & \longrightarrow & k[X, Y]/(X, Y) \end{array}$$

Thus also $\mathrm{Pic}(A) = 0$ by Theorem 2.1.

($\Rightarrow$) Conversely assume that $\mathrm{Pic}(A) = 0$. Suppose that (1) fails. Say that not all powers of X are multiples of m. There is a retract of rings $R \rightarrow A \xrightarrow{\theta} R$ where R is the image in $k[X]$ of the map

$\theta: X \mapsto X,\ Y \mapsto 0$. Thus $\operatorname{Pic}(R) \subset \operatorname{Pic}(A)$. By Theorem 2.1 it is easy to see that $\operatorname{Pic}(R) \neq 0$, and hence also $\operatorname{Pic}(A) \neq 0$. So we may assume that (1) holds.

Pick $f = X^i Y^i$ in the conductor of B/A with $i > m, n$; this is possible by Proposition 3.3. Since $\operatorname{Pic}(A) = 0$, also $(A/fB)^* = (B/fB)^*$ by Theorem 2.1. For each $g = X^a Y^b$ with $1 \leqq a \leqq m$ and $1 \leqq b \leqq n$, $1 + g + fB$ is a unit in B/fB, and hence also in A/fB. But thus $X^a Y^b \in A$, so XYB is contained in the conductor, and the proposition is proved.

For example, the affine subrings A of $B = k[X, Y]$ generated by monomials with integral closure B for which $\operatorname{Pic}(A) = 0$ are precisely those of the form

$$A = k[X^m, \{X^i Y^j \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n\}, Y^n] .$$

For these rings $K_0(A) \approx \boldsymbol{Z}$. Since this does not depend on the field k, by an argument similar to that of Theorem 4.3 we see that all finitely generated projective $A[X_1, \cdots, X_n]$-modules are stably free. So these rings provide many examples of nonnormal rings for which $NK_0(A) = 0$.

We note that even though these rings are not normal, they are "power closed" in the sense that if f is in the quotient field of A and $f^n \in A$ for all large n, then actually $f \in A$. This condition is in fact necessary, for if A is not "power closed", then $\operatorname{Pic}(A) \to \operatorname{Pic}(A[T])$ is not an isomorphism (see Example 5.1).

One can also see that $NK_0(A) = 0$ for the rings of Proposition 6.1 by using the Mayer-Vietoris K-theory sequence for NK_1 and NK_0 ([14]). So the rings of Proposition 6.1 are precisely the nonnormal affine subrings of $k[X, Y]$ generated by monomials for which $NK_0(A) = 0$. Thus if A is an affine subring of $k[X, Y]$ generated by monomials, $\operatorname{Pic}(A) = 0$ iff $NK_0(A) = 0$.

7. **The general case.** One can ask if these results generalize to more general subrings of $k[X, Y]$. This is studied in more detail in [1] or [3]. *The analogue of Theorem 5.5 fails in general because* there exist $f \in k[X, Y]$ with $SK_1(k[X, Y]/(f)) \neq 0$. This also depends on the field k. We close with one example.

EXAMPLE 7.1. Let $A = k[X, Y(X^2 - Y), Y^2(X^2 - Y)]$, then

(1) If k is algebraic over a finite field all finitely generated projective A-modules are free.

(2) If k is an algebraically closed field of char 0, then $SK_0(A) \approx \Omega^1_{k/Z} \neq 0$ and $\operatorname{Pic}(A) = 0$. Thus there exist indecomposable finitely generated projective A-modules of rank 2.

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