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CONNECTIVITY PROPERTIES OF METRIC SPACES

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We discuss various connectivity properties of a metric space, and investigate how far their equivalence carries over from the classical to the constructive setting. In passing, we obtain interesting relations between connectivity and convexity for subsets of R, and a result on preservation of connectivity by continuous mappings.

1. The primary object of this note is a constructive examination of the relationship between several, classically equivalent connectivity properties of a metric space (E, d). In order to make sense of the statements of these properties, we recall that a subset A of E is *located* (in E) if

$$\operatorname{dist}(x, A) \equiv \inf \{d(x, a): a \in A\}$$

is computable for each x in E; in which case the *metric complement* of A in E is defined to be

$$E - A \equiv \{x \in E: \operatorname{dist}(x, A) > 0\} .$$

Note that a located set A is nonvoid, in the sense that we can construct at least one of its elements. For further properties of located sets, and general background material in constructive analysis, we refer the reader to [1] and [2].

In [3], we introduced the following types of connectivity of a metric space:

C-connectivity: if A is a closed, located subset of E with nonvoid metric complement, then there exists a point ξ in $A \cap (E - A)^-$;

0-connectivity: if A is an open, located subset of E with nonvoid metric complement, then there exists ξ in $\bar{A}$ such that $d(\xi, x) > 0$ for each x in A;

Connectivity: if A is an open, closed and located subset of E, then $A = E$.

We then showed that

$$C\text{-connectivity} \Longrightarrow 0\text{-connectivity} \Longrightarrow \text{connectivity} .$$

In this section, we shall show that these implications cannot be reversed within a constructive proof-theoretic framework. To do this, we first characterize located C- and 0-connected subsets of the real line R, and prove connectivity of subsets of R of the form $[a, b] \cup]b, c]$, where $a < b < c$.

LEMMA 1. *Let S be a subset of R with the property: $S \cap]a, b[$ is dense in $[a, b]$ whenever a, b belong to S and $a < b$. Let A be a located subset of the metric space S and let $b \in S - A$. Then there exist a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leqq \xi < b$ or $b < \xi \leqq a$.*

We first note that if $x \in S - A$, then

$$\min(\operatorname{dist}(x - \operatorname{dist}(x, A), A), \operatorname{dist}(x + \operatorname{dist}(x, A), A)) = 0 .$$

In particular, it follows that, if $r \equiv \operatorname{dist}(b, A)$, then either $\operatorname{dist}(b - r, A) < (1/2)r$ or $\operatorname{dist}(b + r, A) < (1/2)r$. Taking, for example, the former case (the latter produces the second alternative of the conclusion of the lemma), we compute a in $]b - 3r/2, b - r] \cap A$. As $a \in S$, $b \in S$, and $a < b$, there exists x_1 in $S \cap]b - r, b - (1/2)r]$. Let $\rho \equiv \operatorname{dist}(x_1, A)$ and $\xi \equiv x_1 - \rho$. Then $0 < \rho \leqq x_1 - a < r$; so that $x_1 + \rho$ belongs to $]x_1, b + (1/2)r[$, and therefore

$$\operatorname{dist}(x_1 + \rho, A) \geqq \min(\rho, r) > 0 .$$

Hence $\operatorname{dist}(\xi, A) = 0$, and $\xi \in \bar{A}$. On the other hand, as $\xi \geqq a$, $S \cap]a, b[$ is dense in $[a, b]$, and $|x_1 - \xi| = \operatorname{dist}(x_1, A)$, it follows that $]\xi, x_1] \subset S - A$, and therefore that $\xi \in (S - A)^-$.

THEOREM 1. *A necessary and sufficient condition that a located subset S of R be C-connected is that $S \supset [a, b]$ whenever a, b are points of S and $a < b$.*

If S is C-connected, and a, b are points of S with $a < b$, and $x \in [a, b]$, we have either $a < x$ or $x < b$. Without loss of generality, we suppose the latter. Then $A \equiv S \cap] - \infty, x]$ is a closed, located subset of S such that $b \in S - A$. Thus there exists ξ in $\bar{A} \cap S \cap (S - A)^-$. It is easy to see that $\xi = x$, whence $x \in S$.

Conversely, suppose the stated condition holds, and let A be a closed, located subset of S with $S - A$ nonvoid. Choosing b in $S - A$, compute a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leqq \xi < b$ or $b < \xi \leqq a$. Then $\xi \in S$, and so $\xi \in A \cap (S - A)^-$. Thus S is C-connected.

THEOREM 2. *A necessary and sufficient condition that a located subset S of R be 0-connected is that $S \supset]a, b[$ whenever a, b are points of S and $a < b$.*

If S is 0-connected, and a, b are points of S with $a < b$, and $x \in]a, b[$, we apply the 0-connectivity condition to $A \equiv S \cap]-\infty, x[$, to obtain ξ in S with $d(\xi, y) > 0$ for each y in A. As ξ is clearly equal to x, we have $x \in S$, as required.

Conversely, suppose the stated condition holds, and let A be an open, located subset of S with $S - A$ nonvoid. Choosing b in $S - A$, compute a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leqq \xi < b$ or $b < \xi \leqq a$. As $\xi \in A$ entails $A \cap (S - A)$ nonvoid, and A is open, we see that $d(\xi, x) > 0$ for each x in A. In particular, either $a < \xi < b$ or $b < \xi < a$; so that $\xi \in S$, and S is 0-connected.

PROPOSITION 1. *Let a, b, c be real numbers with $a < b < c$. Then $[a, b] \cup]b, c]$ is connected.*

Let A be an open, closed, located (and therefore totally bounded) subset of $S \equiv [a, b] \cup]b, c]$. We first prove that, if $A \cap [a, b]$ is nonvoid, then $A \supset [a, b]$. Indeed, given x_0 in $A \cap [a, b]$ and x in $[a, b]$, we have either $x_0 \leqq x - r$ or $x + r \leqq x_0$. Without loss of generality, we may assume the former. Letting

$$B \equiv A \cap [a, x] = A \cap [a, x[,$$

we see that B is open and closed in $[a, b]$. On the other hand, if $0 < \varepsilon < r$ and $\{x_1, \cdots, x_\nu\}$ is an ε-net of A, we may assume that $x_1, \cdots, x_s$ belong to $A \cap [a, x - r]$, and that $x_{s+1}, \cdots, x_\nu$ belong to $A \cap [x + r, c]$. It is now easy to show that $\{x_1, \cdots, x_s\}$ is an ε-net of B; whence B is totally bounded, and therefore located in $[a, b]$. By connectivity of $[a, b]$, we now have $B = [a, b]$; whence we obtain the contradiction $x \in A$. Thus $r = 0$, $x \in \bar{A} \cap [a, b]$, and so $A \supset [a, b]$.

In a similar manner, we can show that if $A \cap]b, c]$ is nonvoid, then $A \supset]b, c]$. Given ξ in A, we now see that either $\xi \in [a, b]$, in which case $[a, b] \subset A$ and therefore (as A is open in S)$A \cap]b, c]$ is nonvoid; or $\xi \in]b, c]$, when $]b, c] \subset A$, and therefore (as A is closed in S) $b \in A$. In either case, we have $A \supset [a, b] \cup]b, c]$, and therefore $A = S$. Thus S is connected.

THEOREM 3. *The proposition,*

a located, 0-connected subset of R is C-connected ,

is essentially nonconstructive.

Consider the located subset $S \equiv \{0\} \cup]0, 1]$ of R. It follows from Theorem 2 that S is 0-connected. On the other hand, by Theorem 1, the C-connectivity of S would entail the proposition

$$\forall x \in [0, 1](x > 0 \vee x = 0) ,$$

which is known to be essentially nonconstructive.

THEOREM 4. *The proposition,*

$$a \text{ located, connected subset of } R \text{ is } 0\text{-connected} ,$$

is essentially nonconstructive.

Consider the located subset $S \equiv [-1, 0] \cup]0, 1]$ of R. By Proposition 1, S is connected. However, the 0-connectivity of S would entail the proposition

$$\forall x \in [-1, 1](x > 0 \vee x \leqq 0) ,$$

which is known to be essentially nonconstructive.

2. A subset U of the metric space E is *colocated* (in E) if it is the metric complement of a located set. U is then an open subset of E. Colocated sets, like located sets (although to a lesser degree), are easier to handle than general subsets of E. It therefore seems reasonable to investigate what happens when we formulate alternative connectivity properties in terms of colocated sets.

When we do so, we find that the natural analogue of C-connectivity is just a condition of disconnectedness. That of 0-connectivity is given by the property,

> if U is a nonvoid, colocated subset of E, then exists ξ in $\bar{U}$ such that $d(\xi, x) > 0$ for each x in U,

a property easily shown to be equivalent to that of C-connectivity. Finally, there is no direct analogue of connectivity, although a natural property (readily seen to be equivalent to that of connectivity) is that any open, closed, colocated subset of E is empty.

Of greater interest is the following property, analogous to that of M-connectivity (defined in [4], and there shown to be equivalent to 0-connectivity):

(*) if U, V are nonvoid, disjoint subsets of E with U colocated and V open, then there exists ξ in E such that $d(\xi, x) > 0$ for each x in $U \cup V$.

(By "disjoint" here, we mean that $d(u, v) > 0$ whenever $u \in U$ and $v \in V$.) We have

$$C\text{-connectivity} \Longrightarrow (*) \Longrightarrow 0\text{-connectivity} .$$

To see this, suppose first that E is C-connected, and let A be a located subset of E such that $U \equiv E - A = E - \bar{A}$ is nonvoid. Then there exists ξ in $\bar{A} \cap \bar{U}$. As U is open, $d(\xi, x) > 0$ for each x in U. If also V is a nonvoid open subset of E such that U, V are disjoint, then $\xi \in V$ entails $V \cap U$ nonvoid; whence, as V is open, $d(\xi, x) > 0$ for each x in V. Thus E satisfies (*).

On the other hand, if E satisfies (*) and A is an open, located subset of E with $U \equiv E - A$ nonvoid, then there exists ξ in E such that $d(\xi, x) > 0$ for each x in $U \cup A$. Were $\operatorname{dist}(\xi, A) > 0$, we would have the contradiction $\xi \in U$; hence $\operatorname{dist}(\xi, A) = 0$, $\xi \in \bar{A}$, and so E is 0-connected.

On the real line, we can say more:

THEOREM 5. *A necessary and sufficient condition that a located subset E of R satisfy* (*) *is that E be* 0-*connected.*

Let E be 0-connected. Let U be a nonvoid, colocated subset of E, V a nonvoid, open subset of E such that U, V are disjoint, and choose u in U, v in V. We may assume that $u < v$. By the lemma in [5], the set

$$B \equiv \{x \in [u, v]: [u, x] \subset U\}$$

is totally bounded. Let $\xi \equiv \sup B$. Then (as $U \cup V$ is open in E) it is clear that $|\xi - x| > 0$ for each x in $U \cup V$. Thus $u < \xi < v$, and so, by Theorem 2, $\xi \in E$. Hence E satisfies (*). Reference to the remarks preceding this theorem completes the proof.

THEOREM 6. *Let E be either an open ball in a Banach space, or a complete, convex subset of a normed space. Then E satisfies* (*).

Let A be a located subset of E, with $U \equiv E - A$ nonvoid. Using the argument of the proof of 2.1 of [3], we can construct a point ξ of $E \cap \bar{A} \cap \bar{U}$. It is easy to see that, if V is a nonvoid, open subset of E such that U, V are disjoint, then $\|\xi - x\| > 0$ for each x in $U \cup V$.

Theorems 5 and 6 support the (classically true) conjecture that 0-connectivity and (*) are equivalent properties of a metric space.

3. An immediate consequence of Theorems 1 and 3 is that the proposition,

> if S is a located, 0-connected subset of R, and a, b are points of S with $a < b$, then $[a, b] \subset S$,

is essentially nonconstructive. This, and Theorem 1 itself, extends the work of Mandelker [6] in response to the first of two questions with which we ended [3]. On the other hand, Proposition 1 enables us to progress towards an answer to the second of these questions, which we shall consider in a form slightly different to that found in [3]:

> *if f is a uniformly continuous mapping of* $[0, 1]$ *into* R, *what connectivity properties obtain for* $f([0, 1])$?

LEMMA 2. *Let K be a compact, connected metric space, $f: K \to R$ a uniformly continuous mapping, and a, b points of $f(K)$ with $a \leqq b$. Then $f(K) \cap [a, b]$ is dense in $[a, b]$.*

Let $y \in [a, b]$, and suppose that $0 < r \equiv \operatorname{dist}(y, f(K))$. Then

$$a \leqq y - r < y < y + r \leqq b .$$

Compute α in $]0, r[$ so that

$$A \equiv f^{-1}(]-\infty, y - \alpha]) = f^{-1}(]-\infty, y[)$$

is compact [1, Ch. 4, Thm. 8]. Then A is an open, closed and located subset of K. Hence $A = K$, and so $y \in A$ — a contradiction. Thus $\operatorname{dist}(y, f(K)) = 0$.

THEOREM 7. *The proposition,*

> *a uniformly continuous mapping $f: [0, 1] \longrightarrow R$ has 0-connected range ,*

is essentially nonconstructive.

Let $\alpha \in [-1, 1]$, and define a uniformly continuous mapping $f: [0, 1] \to R$ so that $f(0) = -1$, $f(1/3) = f(2/3) = \alpha$, $f(1) = 1$, and f is linear in each of the intervals $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$. Let $S \equiv f([0, 1])$ and $A \equiv [-1, 0[\cap S$. Then A is open in S. As S is dense in $[-1, 1]$ (by Lemma 2), A is dense in $[-1, 0[$, and therefore totally bounded. Hence A is located in S. Also, $\operatorname{dist}(1, A) > 0$, and $1 \in S$. Suppose that S is 0-connected. Then there exists ξ in $\bar{A} \cap S$ with $|\xi - x| > 0$ for each x in A. It is clear that $\xi = 0$; whence $0 \in S$, and we can compute z in $[0, 1]$ with $f(z) = 0$. Either $1/3 < z$ or $z < 2/3$. In the former case, we have $\alpha = f(1/3) \leqq f(z) = 0$; in the latter, $\alpha = f(2/3) \geqq f(z) = 0$. Thus we see that the proposition in question entails

$$\forall \alpha \in [-1, 1] \quad (\alpha \geqq 0 \vee \alpha \leqq 0) ,$$

a proposition known to be essentially nonconstructive.

We have yet to answer the final question of [3] in its original form:

> *if f is a uniformly continuous mapping of an interval I in R into a metric space, is $f(I)$ connected?*

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