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RESOLUTION OF SIGN AMBIGUITIES IN JACOBI AND JACOBSTHAL SUMS

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Let p be a prime $\equiv 1 \pmod{16}$. We obtain extensions of known congruences involving parameters of bioctic Jacobi sums $\pmod{p}$. These extensions are used to give an elementary proof of an important congruence of Hasse relating parameters of quartic and octic Jacobi sums $\pmod{p}$. This proof leads directly to an elementary resolution of sign ambiguities of parameters of certain quartic, octic, and bioctic Jacobi and Jacobsthal sums. E. Lehmer's work on ambiguities in quartic sums is thereby extended.

1. Introduction and notation. Throughout this paper, $p = 16f + 1$ is a prime with fixed primitive root g. Fix a character $\chi \pmod{p}$ of order 16 such that $\chi(g) = \beta$, where $\beta = e^{2\pi i/16}$. Let $m = \operatorname{ind}_g 2$, so that $2 \equiv g^m \pmod{p}$.

For characters λ, $\psi \pmod{p}$, define the Jacobi sums

$$J(\lambda, \psi) = \sum_{n \pmod{p}} \lambda(n)\psi(1-n)$$

and

$$K(\lambda) = \lambda(4)J(\lambda, \lambda) .$$

For $\alpha \not\equiv 0 \pmod{p}$, define the Jacobsthal sums

$$\varphi_n(\alpha) = \sum_{\nu \pmod{p}} \left(\frac{\nu}{p}\right)\left(\frac{\nu^n + \alpha}{p}\right),$$

where the factors in the summands are Legendre symbols.

It is well known [8, Lemma 2] that if the character λ has order $2n$, then

$$K(\lambda) = J(\lambda, \lambda^n) .$$

Simple consequences of this formula are the following (see [1, Theorems 2.5 and 2.7]):

$$\varphi_n(\alpha) = \lambda(-1) \sum_{j=0}^{n-1} \lambda^{n+1+2j}(\alpha)K(\lambda^{2j+1}) , \tag{1}$$

and

$$K(\lambda) = \lambda(-1)J(\lambda, \lambda^{n-1}) ,$$

where the character λ has order $2n$.

Using this formula for $K(\lambda)$ and [11, eqs. (3.4), (3.6), (3.10), (3.13)], we see that

$$K(\chi^4) = -x + 2iy , \tag{2}$$

$$K(\chi^2) = -a + ib\sqrt{2} , \tag{3}$$

$$K(\chi) = c_0 + c_2\sqrt{2} + ic_1\sqrt{2-\sqrt{2}} + ic_3\sqrt{2+\sqrt{2}} , \tag{4}$$

and

$$J(\chi, \chi^2) = \sum_{i=0}^{7} d_i\beta^i . \tag{5}$$

The integers x and $|y|$ are uniquely determined by the conditions

$$p = x^2 + 4y^2, \quad x \equiv 1 \pmod 4) , \tag{6}$$

and the integers a and $|b|$ are uniquely determined by the conditions

$$p = a^2 + 2b^2, \quad a \equiv 1 \pmod 4 . \tag{7}$$

By [2, Theorem 3.5], c_0, $|c_2|$, and $\{|c_1|, |c_3|\}$ are uniquely determined by the conditions

$$\begin{gathered} p = c_0^2 + 2c_1^2 + 2c_2^2 + 2c_3^2, \quad c_0 \equiv -1 \pmod 8 , \\ 2c_0c_2 = c_1^2 - c_3^2 - 2c_1c_3 . \end{gathered} \tag{8}$$

See also [4, p. 338] and [9]. No simple criteria in terms of m are known to determine *in general* the signs of the ambiguous parameters in (2), (3), (4).

In § 2, we extend the congruence in (8) by characterizing $c_0 \pmod{16}$. This result is then used to give an elementary proof of the congruence

$$y \equiv 2b + m \pmod{16} . \tag{9}$$

Hasse [5, p. 232] gave a proof of (9) using deep results from class field theory. Also in § 2, we characterize $c_2 \pmod 8$, thus extending the known congruence $c_2 \equiv m \pmod 4$ [2, Theorem 3.6]. The congruences proved in § 2 are used in § 3, for certain m, to resolve sign ambiguities in the quartic, octic, and bioctic Jacobi sums in (2), (3), and (4), and in the corresponding Jacobsthal sums $\varphi_2(\alpha)$, $\varphi_4(\alpha)$, and $\varphi_8(\alpha)$.

The cyclotomic number (i, j) of order 16 is defined to be the number of integers $n \pmod p$ for which both n/g^i and $(1+n)/g^j$

are 16th power residues. In [3], the numbers 256 (i, j) are expressed as linear combinations of the parameters in (2) – (5). We will frequently use these formulas. The idea to apply the cyclotomic numbers of order 16 to resolve sign ambiguities originated in [7, p. 110].

2. Congruences for parameters of Jacobi sums. We record and then justify the following congruences:

$$y \equiv -m \pmod 8; \tag{10}$$

$$c_2 \equiv m \equiv b \pmod 4); \tag{11}$$

$$a \equiv x - 4b \pmod{32}, \text{ when } 4 \mid m; \tag{12}$$

$$a \equiv \begin{cases} (3-p)/2 \pmod{16}, & \text{if } 4 \mid m \\ (p-23)/2 \pmod{32}, & \text{if } 2 \| m; \end{cases} \tag{13}$$

$$x \equiv \begin{cases} (5-3p)/2 \pmod{64}, & \text{if } 8 \mid m \\ (-3p-59)/2 \pmod{64}, & \text{if } 4 \| m \\ (53-3p)/2 \pmod{64}, & \text{if } 2 \| m. \end{cases} \tag{14}$$

When $2 \| m$, (10) follows from [7, p. 108, eq. (32)], and when $4 \mid m$, (10) is a special case of [1, Theorem 3.17]. The first congruence in (11) follows from [2, Theorem 3.6] and the second follows from [1, Theorem 3.15]. By (6), (7), and (10), $x^2 \equiv a^2 + 2b^2 \pmod{64}$ when $4 \mid m$. Thus (12) follows, with the aid of (11). Finally, (13) and (14) are special cases of [1, Theorem 3.14] and [1, Theorem 3.16], respectively.

Theorem 1 below extends (14).

THEOREM 1. *We have*

$$x \equiv \begin{cases} (p+1)/2 \pmod{128} & \textit{if } 2 \mid f,\ 8 \mid m \\ (p-63)/2 \pmod{128} & \textit{if } 2 \nmid f,\ 8 \mid m \\ (9p-71)/2 \pmod{256} & \textit{if } 2 \mid f,\ 4 \| m \\ (p-127)/2 \pmod{256} & \textit{if } 2 \nmid f,\ 4 \| m \\ -8y + (p+145)/2 \pmod{128}, & \textit{if } 2 \mid f,\ 2 \| m \\ 8y + (p+17)/2 \pmod{128}, & \textit{if } 2 \nmid f,\ 2 \| m. \end{cases}$$

Proof. The result can be deduced using (6) and (10). For example, in the case $2 \mid f$, $4 \| m$, we have $4y^2 \equiv 64 \pmod{512}$ by (10). Hence by (6) $x^2 \equiv p - 64 \pmod{512}$ and the result follows.

In the next theorem, we characterize $c_0 \pmod{16}$, thus extending (8). Recall from (11) that $b \equiv m \pmod 4$.

THEOREM 2. *We have*

$$c_0 \equiv \begin{cases} -1 \pmod{16}, & \text{if } (8 \mid m,\ 8 \mid b) \text{ or } (4 \| m,\ 4 \| b) \text{ or } (2 \nmid f,\ 2 \| m) \\ \ \ 7 \pmod{16}, & \text{if } (8 \mid m,\ 4 \| b) \text{ or } (4 \| m,\ 8 \mid b) \text{ or } (2 \mid f,\ 2 \| m) . \end{cases}$$

Proof. We shall use the formulas for 256 (i, j) found in [3]. First suppose that $8 \mid m$. We must show that

$$c_0 \equiv 2b - 1 \pmod{16} . \tag{15}$$

Assume that $2 \mid f$. Then

$$2p + 2 + 4x + 8a + 16c_0 = 256\{(1, 2) + (3, 6)\} \equiv 0 \pmod{256} .$$

By (12), $8a$ can be replaced by $8x - 32b$ above. Then (15) follows with use of (14). Now assume that $2 \nmid f$. Then

$$\begin{aligned} 2p - 30 + 4x - 64y + 8a - 16c_0 = 256\{&(3, 0) + (2, 2) \\ &+ (1, 1) - (2, 0)\} \equiv 0 \pmod{256} . \end{aligned}$$

By (12), $8a$ can be replaced by $8x - 32b$ above. Then (15) follows, with use of (10) and (14).

Next suppose that $4 \| m$. We must show that

$$c_0 \equiv 2b + 7 \pmod{16} . \tag{16}$$

Assume that $2 \mid f$. Then

$$\begin{aligned} 2p + 2 + 4x + 8a + 64y + 16c_0 = 256\{&(3, 11) + (1, 8) \\ &+ (2, 8) - (2, 10)\} \equiv 0 \pmod{256} . \end{aligned}$$

By (12), $8a$ may be replaced by $8x - 32b$ above. Then (16) follows, with use of (10) and (14). Now assume that $2 \nmid f$. Then

$$2p + 2 + 4x - 24a - 16c_0 = 256\{(3, 5) + (2, 1)\} \equiv 0 \pmod{256} .$$

By (12), $24a$ may be replaced by $24x - 96b$ above. Then (16) follows, with use of (14).

Finally, suppose that $2 \| m$. Then $2 \| c_2$ by (11). Since by (8), $c_0 \equiv -1 \pmod 8$ and

$$16f + 1 = p \equiv c_0^2 + 2c_2^2 + 2(c_1^2 + c_3^2) \pmod{32} ,$$

the result follows.

We now give an elementary proof of (9), the congruence $y \equiv 2b + m \pmod{16}$. Our proof will use the useful and easily proved fact [6, p. 426] that the cyclotomic number $(0, j)$ is odd if and only if $j \equiv m \pmod{16}$.

THEOREM 3. *We have* $y \equiv 2b + m \pmod{16}$.

Proof. Whiteman [11, p. 411] has given an elementary proof in the case $8 \mid m$. It remains to consider the cases $2 \| m$ and $4 \| m$. There is no loss of generality in assuming that $m \equiv 2$ or $m \equiv 4 \pmod{16}$.

Case 1. $m \equiv 2 \pmod{16}$, $2 \mid f$.

Since $(0, 2)$ is odd,

$$\begin{aligned} 4(p+1) - 8x + 64b - 32y = 256\{(0, 2) - (0, 6) + 2(4, 8) \\ + 2(4, 10)\} \equiv 256 \pmod{512} . \end{aligned}$$

By (14), $8x$ can be replaced by $212 - 12p$ above, and the result follows.

Case 2. $m \equiv 2 \pmod{16}$, $2 \nmid f$.

Since $(0, 2)$ is odd,

$$\begin{aligned} 8x + 28 - 4p - 32y - 32a - 64b = 256\{(0, 2) - (0, 6) \\ - 2(4, 2) - 2(4, 0)\} \equiv 256 \pmod{512} . \end{aligned}$$

Using (13) and (14), we deduce that $y + 2b \equiv 10 \pmod{16}$. Since $2 \| b$ by (11), it follows that $y \equiv 2b + 2 \pmod{16}$.

Case 3. $m \equiv 4 \pmod{16}$, $2 \mid f$.

Since $(0, 4)$ is odd,

$$32c_0 + 32y - 14x - p - 17 = 256\{(0, 4) - 2(2, 10)\} \equiv 256 \pmod{512} .$$

By Theorem 1, $2x \equiv 9p - 71 \pmod{512}$, and by Theorem 2, $c_0 \equiv 2b + 7 \pmod{16}$. We thus deduce that $y \equiv 2b + 4 \pmod{16}$.

Case 4. $m \equiv 4 \pmod{16}$, $2 \nmid f$.

Since $(0, 4)$ is odd,

$$\begin{aligned} 2p - 62 - 4x + 32y - 32a + 32c_0 = 256\{(0, 0) - (0, 4) \\ + 2(2, 0)\} \equiv 256 \pmod{512} . \end{aligned}$$

By Theorem 1, $2x \equiv p - 127 \pmod{512}$, and by Theorem 2, $c_0 \equiv 2b +$

$7 \pmod{16}$. By (13), $2a \equiv 3 - p \pmod{32}$. We thus deduce that $y \equiv 2b + 4 \pmod{16}$.

In the next theorem, we characterize $c_2 \pmod 8$, thus extending (11).

THEOREM 4. *We have*

$$c_2 \equiv \begin{cases} b \pmod 8, & \text{if } 8 \mid m,\ 2 \mid f \text{ or } 4 \| m,\ 2 \nmid f \\ (-1)^{f+(m-2)/4} b \pmod 8, & \text{if } 2 \| m \\ b + 4 \pmod 8, & \text{if } 4 \| m,\ 2 \mid f \text{ or } 8 \mid m,\ 2 \nmid f\,. \end{cases}$$

Proof. First suppose that $2 \| m$. It suffices to consider the case $m \equiv 2 \pmod 8$. When $2 \mid f$,

$$256\{(2, 4) - (4, 10)\} = 32(c_2 - b)\ ,$$

and so $c_2 \equiv b \pmod 8$. When $2 \nmid f$,

$$256\{(4, 2) - (2, 6)\} = 32(b + c_2)\ ,$$

and so $c_2 \equiv -b \pmod 8$.

Finally, suppose that $4 \mid m$. Cyclotomic numbers are not needed in this case. By (11), $4 \mid b$ and $4 \mid c_2$. By Theorem 2, $c_0 \equiv 2(b + m) - 1 \pmod{16}$, so $c_0^2 \equiv 4(b + m)(b + m - 1) + 1 \pmod{32}$. Hence, by (8),

$$p - 1 = 16f \equiv 4(b + m)(b + m - 1) + 2(c_1^2 + c_3^2) \pmod{32}\ .$$

Thus,

$$\begin{cases} 4 \mid c_1,\ 4 \mid c_3, \text{ if } 4f \equiv b + m \pmod 8 \\ 2 \| c_1,\ 2 \| c_3, \text{ if } 4f \not\equiv b + m \pmod 8\ . \end{cases}$$

Also by (8), $2c_0c_2 = c_1^2 - c_3^2 - 2c_1c_3$, so

$$\begin{cases} 8 \mid c_2, \text{ if } 4f \equiv b + m \pmod 8 \\ 4 \| c_2, \text{ if } 4f \not\equiv b + m \pmod 8\ . \end{cases}$$

We thus obtain the result.

Incidentally, Theorem 4 and (11) yield the following criterion for the octic character of 2 modulo primes $p = 16f + 1$:

2 is an octic residue $\pmod p$ iff $4 \mid b$ and $b \equiv c_2 + 4f \pmod 8$.

Of course, simpler criteria (in terms of y) are known [10].

3. Resolution of sign ambiguities. We begin with quartic

Jacobi and Jacobsthal sums in the case $4 \| m$, i.e., 2 is a quartic but not octic residue $(\bmod p)$. Without loss of generality, let $m \equiv 4 \pmod{16}$. Note that $4 | b$, by (11).

THEOREM 5. *Let ψ be a character* $(\bmod p)$ *of order* 4 *chosen such that $\psi(g) = i$. Suppose that $m \equiv 4 \pmod{16}$. When $p \nmid \alpha$, define $r = \operatorname{ind}_g \alpha$. Then*

$$K(\psi) = -x + 2iy$$

and

$$\varphi_2(\alpha) = \begin{cases} -(-1)^{r/2} 2x, & \text{if } 2 | r \\ (-1)^{(r-1)/2} 4y, & \text{if } 2 \nmid r \end{cases}$$

where x and y are uniquely determined by the conditions

$$p = x^2 + 4y^2, \ x \equiv 1 \pmod 4, \text{ and } y \equiv \begin{cases} 4 \pmod{16}, & \text{if } 8 | b \\ -4 \pmod{16}, & \text{if } 4 \| b \,. \end{cases}$$

Proof. The evaluation of $K(\psi)$ follows from (2), (6), and Theorem 3. The evaluation of $\varphi_2(\alpha)$ then follows with use of (1).

For quartic sums $K(\psi)$ and $\varphi_2(\alpha)$ modulo primes $\equiv 1 \pmod 4$ for which 2 is not a quartic residue, the sign ambiguities have been resolved (in terms of m) by E. Lehmer [7], [8]. The signs remain undetermined for $K(\psi)$ and $\varphi_2(\alpha)$ modulo primes $\equiv 1 \pmod{16}$ for which 2 is an octic residue, and also modulo primes $\equiv 9 \pmod{16}$ for which 2 is a quartic residue.

An evaluation of $\varphi_6(\alpha)$ modulo primes $\equiv 1 \pmod{12}$, up to some undetermined signs, is given in [1, Theorem 4.8]. However, for the particular primes $\equiv 1 \pmod{12}$ for which 2 is not a quartic residue and the primes $\equiv 1 \pmod{48}$ for which 2 is a quartic but not octic residue, $\varphi_6(a)$ can be completely determined. This is because $K(\psi)$ is completely determined for these primes, and consequently so is $\varphi_6(\alpha)$ by [1, eqs. (4.3) and (4.5)].

We now consider octic and bioctic Jacobi and Jacobsthal sums in the case $2 \| m$. Without loss of generality, let $m \equiv 2 \pmod{16}$. The signs of y and b are simply determined, since $y \equiv -2 \pmod 8$ by (10) and $b \equiv -1 + y/2 \pmod 8$ by Theorem 3.

THEOREM 6. *Let λ be a character* $(\bmod p)$ *of order* 8 *chosen such that $\lambda(g) = e^{2\pi i/8}$. Suppose that $m \equiv 2 \pmod{16}$. When $p \nmid \alpha$, define $r = \operatorname{ind}_g \alpha$. Then*

$$K(\lambda) = -a + ib\sqrt{2}$$

and

$$\varphi_4(\alpha) = \begin{cases} -(-1)^{r/4}4a, & if\ 4\,|\,r \\ 0 & ,\ if\ 2\,||\,r \\ 4b & ,\ if\ r \equiv 1\ or\ r \equiv 3 \pmod 8 \\ -4b & ,\ if\ r \equiv 5\ or\ r \equiv 7 \pmod 8\ , \end{cases}$$

where a and b are uniquely determined by the conditions

$$p = a^2 + 2b^2,\ a \equiv 1 \pmod 4,$$
$$and\ b \equiv \begin{cases} 2 \pmod 8, & if\ y \equiv 6 \pmod{16} \\ -2 \pmod 8, & if\ y \equiv -2 \pmod{16}\ . \end{cases}$$

Proof. The evaluation of $K(\lambda)$ follows from (3), (7), and Theorem 3. The evaluation of $\varphi_4(\alpha)$ then follows with use of (1).

The octic sums $K(\lambda)$ and $\varphi_4(\alpha)$ that remain ambiguous are those modulo primes $\equiv 9 \pmod{16}$, and modulo primes $\equiv 1 \pmod{16}$ for which 2 is a quartic residue.

THEOREM 7. *Let χ be a character* $\pmod p$ *of order* 16 *chosen such that $\chi(g) = e^{2\pi i/16}$. Suppose that $m \equiv 2 \pmod{16}$. When $p \nmid \alpha$, define $r = \operatorname{ind}_g \alpha$. Then*

$$K(\chi) = c_0 + c_2\sqrt{2} + ic_1\sqrt{2 - \sqrt{2}} + ic_3\sqrt{2 + \sqrt{2}}$$

and

$$(-1)^f\varphi_8(\alpha) = \begin{cases} (-1)^{r/8}8c_0, & if\ 8\,|\,r \\ 0 & ,\ if\ 4\,||\,r \\ 8c_2\ or\ -8c_2\ according\ as\ r \equiv \pm 2\ or\ \pm 6 \pmod{16} & \\ 8c_1 & ,\ if\ r \equiv 1\ or\ r \equiv 7 \pmod{16} \\ -8c_1 & ,\ if\ r \equiv 9\ or\ r \equiv 15 \pmod{16} \\ 8c_3 & ,\ if\ r \equiv 3\ or\ r \equiv 5 \pmod{16} \\ -8c_3 & ,\ if\ r \equiv 11\ or\ r \equiv 13 \pmod{16}\ , \end{cases}$$

where c_0 and c_2 are uniquely determined by the conditions

$$p = c_0^2 + 2c_1^2 + 2c_2^2 + 2c_3^2,\ c_0 \equiv -1 \pmod 8,\ 2c_0c_2 = c_1^2 - c_3^2 - 2c_1c_3,\ and$$

$$(-1)^f c_2 \equiv \begin{cases} 2 \pmod 8, & if\ y \equiv 6 \pmod{16} \\ -2 \pmod 8, & if\ y \equiv -2 \pmod{16}\ . \end{cases} \tag{17}$$

Proof. The evaluation of $K(\chi)$ follows from (4), (8), and Theorems 3 and 4. The evaluation of $\varphi_8(\alpha)$ then follows with the aid of (1) (see [2, Theorem 3.9]).

Theorem 7 gives only a partial resolution of signs, because while c_0 and c_2 are determined, c_1 and c_3 are not. Theorem 8 below shows that c_1 and c_3 can also be determined in Theorem 7 if one makes the additional assumption that $2|f$, i.e., $p \equiv 1 \pmod{32}$. We assume without loss of generality that $m \equiv 2 \pmod{32}$.

THEOREM 8. *If in Theorem 7 the additional assumptions* $2|f$ *and* $m \equiv 2 \pmod{32}$ *are made, then the evaluations of* $K(\chi)$ *and* $\varphi_8(\alpha)$ *are valid with* $c_0, c_1, c_2,$ *and* c_3 *uniquely determined by the conditions in* (17) *together with the conditions*

$$\begin{cases} 4|c_1 \text{ and } 2\|c_3, \text{ if } y \equiv 6 \pmod{16} \\ 4|c_3 \text{ and } 2\|c_1, \text{ if } y \equiv -2 \pmod{16} \end{cases} \tag{18}$$

and

$$c_1 + c_3 \equiv (y + 2b - 2)/4 \pmod 8 . \tag{19}$$

Proof. The conditions in (18) are easily proved with use of (17). Hasse [5, p. 233] proved that when $p \equiv 1 \pmod{32}$,

$$y + 2b - 4(c_1 + c_3) \equiv m \pmod{32} . \tag{20}$$

Thus (19) follows from (20). To see that the signs of c_1 and c_3 are uniquely determined by (17), (18), and (19), note first that $c_1 + c_3 \equiv \pm 2 \pmod 8$ by (18), so that the sign of $c_1 + c_3$ is determined by (19). The result now follows because the sign of $2c_1c_3 = c_1^2 - c_3^2 - 2c_0c_2$ is determined by (17) and (18).

The bioctic sums $K(\chi)$ and $\varphi_8(\alpha) \pmod p$ that remain ambiguous are those for which $p \equiv 17 \pmod{32}$, and those for which $p \equiv 1 \pmod{32}$ with 2 a quartic residue $(\bmod\, p)$.

Note added in proof. Congruences (9) and (20) are the cases $n=4$ and $n = 5$ of a general congruence $(\bmod\, 2^n)$ conjectured by J. B. Muskat in 1971. The author has recently obtained an elementary proof of Muskat's conjecture.

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