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TWO THEOREMS ON THE PRIME DIVISORS OF ZERO IN COMPLETIONS OF LOCAL DOMAINS

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The first theorem concerns the number of minimal prime ideals of a given depth (=dimension) in the completion of a finite integral extension domain of a semi-local domain. The second theorem characterizes local domains that have a depth one prime divisor of zero in their completion as those local domains whose maximal ideal M is a prime divisor (=associated prime) of all nonzero ideals contained in large powers of M.

For arbitrary local (Noetherian) domains R it is of some interest and importance to know as much as possible about the prime divisors of zero in the completion R^* of R. One reason (among several) for this is that knowledge about these prime ideals is necessary to solve the catenary chain conjectures. (For example, the Chain Conjecture, open since 1956, holds if all minimal prime ideals in R^* have the same depth whenever the integral closure of R is quasi-local.) In this paper we add to the existing knowledge in this area by proving two new theorems about such prime ideals.

For the first of these, it is well known that if the integral closure of R has k maximal ideals, then the completion R^* of R has at least k minimal prime ideals. (For example, see the proof of (33.10) in [3].) It is also known [8, (2.14.1) $\Rightarrow$ (2.14.2)] that if there exists a maximal chain of prime ideals of length n in some integral extension domain of R, then some minimal prime ideal in R^* has depth n. Our first theorem, (4), shows that these two results combine in a somewhat unexpected manner for finite integral extension domains of R. The second theorem, (9), characterizes local domains R such that there exists a depth one prime divisor of zero in R^*. (11) gives a similar characterization of a subclass of the local domains whose completions have depth one minimal prime ideals.

To help simplify the statement and proof of the first theorem, a few preliminaries are needed. We thus begin by fixing two notational conventions.

(1) NOTATION. We shall denote by A' the integral closure of a ring A in its total quotient ring, and by R^* the completion (in the natural topology) of a semi-local ring R.

The following three definitions are also needed.

(2) DEFINITIONS. For an integral domain A, $c(A) = \{m$; there exists a maximal chain of prime ideals of length m in $A\}$, and $uc(A) = \{n$; there exists a maximal chain of prime ideals of length n in some integral extension domain of $A\}$. If $P \in \operatorname{Spec} A$, then we let $c(P)$ and $uc(P)$ denote the sets $c(A_P)$ and $uc(A_P)$. Also, the class of quasi-local domains Q such that $uc(Q) = c(Q)$ will be denoted by $\mathscr{C}$.

It is clear that $c(A) \subseteq uc(A)$, and [3, Example 2, 203–205] in the case $m = 0$ shows that this containment may be proper even when A is a local domain, so there are local domains that are not in $\mathscr{C}$. It is an open conjecture (the Upper Conjecture) that the only local domains that are not in $\mathscr{C}$ are all similar to the cited example; that is, those whose altitude (=Krull dimension) is >1 and whose integral closure has a height one maximal ideal. (See [9, (4.10.3)].)

In (3) we list the known results concerning $uc(A)$ and $\mathscr{C}$ that will be needed below.

(3) REMARKS. (3.1) [8, (2.14.1) $\Leftrightarrow$ (2.14.2) $\Leftrightarrow$ (2.14.6)]. If (R, M) is a local domain and X is an indeterminate, then $uc(R) = \{\text{depth } z;$ z is a minimal prime ideal in $R^*\} = \{n - 1;\ n \in uc(R[X]_{(M,X)})\}$.

(3.2) [9, (4.1)] Every local domain of the form $R[X]_{(M,X)}$ is in $\mathscr{C}$, where (R, M) is an arbitrary local domain.

(3.3) [9, (4.8.2)] If $R \in \mathscr{C}$ and R' is quasi-local, then, for each integral extension domain S of R and for each maximal ideal N in S, $S_N \in \mathscr{C}$ and $uc(N) = uc(R)$.

The following theorem is the first of the main results in this paper.

(4) THEOREM. *Let (R, M) be a local domain such that R' is quasi-local, let S be a finite integral extension domain of R, and let N be a maximal ideal in S. Then the following statements hold*:

(4.1) *There exists a depth n minimal prime ideal in $(S_N)^*$ if and only if there exists a depth n minimal prime ideal in R^*.*

(4.2) *If $n \in uc(R)$ and if there are k maximal ideals in S' that lie over N, then there are at least k minimal prime ideals of depth n in $(S_N)^*$.*

(4.3) *If $n \in uc(R)$ and if there are exactly m maximal ideals in S', then there are at least m minimal prime ideals of depth n in S^*.*

Proof. (4.1). By (3.1), there exists a depth n minimal prime ideal in $(S_N)^*$ if and only if $n \in uc(N)$, and there exists a depth n minimal prime ideal in R^* if and only if $n \in uc(R)$. Also, if $n \in uc(N)$, then clearly $n \in uc(S)$, and so $n \in uc(R)$, since S is integral over R.

Thus it remains to show that if $n \in uc(R)$, then $n \in uc(N)$.

For this, let $D = R[X]_{(M,X)}$ and $C = S[X]_{(R[X]-(M,X))}$. Then $D \in \mathscr{C}$, by (3.2), and C is integral over D. Also, D' is quasi-local, by the hypothesis on R, so by (3.3) $uc(P) = uc(D)$ for each maximal ideal P in C. Now $(N, X)C$ is a maximal ideal in C, since N is maximal in S, and $C_{(N,X)C} \cong S_N[X]_{(NS_N,X)}$. Therefore, by (3.1), if $n \in uc(R)$, then $n + 1 \in uc(D) = uc((N, X)C)$, so $n \in uc(N)$.

(4.2): Since there are only finitely many maximal ideals in S', there exists a finite integral extension domain $B \subseteq S'$ of S such that B and S' have the same number of maximal ideals. Therefore there exist k maximal ideals in B that lie over N. Also, if $n \in uc(R)$, then for each maximal ideal Q in B there exists a minimal prime ideal of depth n in $(B_Q)^*$, by (3.1) and (4.1). Now $(B_Q)^* \cong B^*_{QB^*}$, so there exists a minimal prime ideal w in B^* such that $w \subseteq QB^*$ and height $QB^*/w = n$. Then B^*/w is a complete domain, so B^*/w is local, and so QB^* is the only maximal ideal in B^* that contains w. Therefore depth $w =$ height $QB^*/w = n$.

Now B^* and S^* have the same total quotient ring T (since B is a finite S-algebra and $S \subseteq B \subseteq S'$), so there exists a one-to-one correspondence between the minimal prime ideals z in S^* and the minimal prime ideals w in B^* given by $w = zT \cap B^*$ and $z = w \cap S^*$, and then depth $z =$ depth w (since B^*/w is integral over S^*/z). Therefore, since there exist k maximal ideals in B^* that lie over NS^*, it follows from the preceding paragraph that there are at least k minimal prime ideals of depth n in S^* that are contained in NS^*. Then, as in the preceding paragraph, if z is such a minimal prime ideal, then height $NS^*/z =$ depth $z = n$. Hence, since $(S_N)^* \cong S^*_{NS^*}$, it follows that there are at least k minimal prime ideals of depth n in $(S_N)^*$.

(4.3) follows from (4.2), since $S^* \cong \bigoplus (S_{N_i})^*$, where the N_i are the maximal ideals in S.

Concerning (4), it is an open problem if there exists $n \in uc(R)$ such that $n <$ altitude R. If such n exist, then, as noted in the introduction, the Chain Conjecture does not hold.

(5) Remarks. With the notation of (4), the following statements hold:

(5.1) R is quasi-unmixed if and only if some S_N is quasi-unmixed if and only if all S_N are quasi-unmixed.

(5.2) If there exists a minimal prime ideal of depth d in $(S_N)^*$, then there exist at least k minimal prime ideals of depth d in $(S_N)^*$ and there exist at least m minimal prime ideals of depth d in S^*.

(5.3) There exists a local integral extension domain $S_0 \subseteqq S$ of R such that $(S_0)' = S'$ and $(S_0)^*$ has at least m minimal prime ideals of depth n, for each $n \in uc(R)$.

Proof. (5.1) is clear by (4.1).

(5.2). If there exists a minimal prime ideal of depth d in $(S_N)^*$, then $d \in uc(R)$, by (4.1) and (3.1), so the conclusion follows from (4.2) and (4.3).

(5.3). Let $S_0 = R + J$, where J is the Jacobson radical of S. Then S_0 is a finite integral extension domain of R (since S is and $R \subseteqq S_0 \subseteqq S$), and J is the only maximal ideal in S_0 and is the conductor of S_0 in S. Therefore S_0 is a local domain and $(S_0)' = S'$, so the conclusion follows from (4.3) (with S_0 in place of S).

(4) can be used to say something about the number of minimal prime ideals in the completion of finite integral extension domains of arbitrary semi-local domains, as will now be shown.

(6) COROLLARY. *Let S be a finite integral extension domain of a semi-local domain R. Let $M_1, \cdots, M_g$ be the maximal ideals in R', assume that there are k_i maximal ideals in S' that lie over M_i (for $i = 1, \cdots, g$) and that $n \in uc(M_i)$ if and only if $i = 1, \cdots, h \leqq g$. Then there are at least $k_1 + \cdots + k_h$ minimal prime ideals of depth n in S^*.*

Proof. There exist finite integral extension domains A of R and B of S such that $A \subseteqq R'$, $B \subseteqq S'$, R' and A have the same number of maximal ideals, and S' and B have the same number of maximal ideals; and it can be assumed that $A \subseteqq B$. Therefore, $L_i = A_{M_i \cap A}$ is a local domain such that $(L_i)'$ is quasi-local and there exist k_i maximal ideals in B that lie over $P_i = M_i \cap A$. Also, by the Going Up Theorem, $n \in uc(L_i)$ if and only if $i \leqq h$. Hence, by (4.3), for $i = 1, \cdots, h$ there are at least k_i minimal prime ideals of depth n in $(B_{A-P_i})^*$. Now $(B_{A-P_i})^*$ is isomorphic to a direct summand (with k_i maximal ideals) of B^*, so the k_i minimal prime ideals of depth n in $(B_{A-P_i})^*$ correspond to k_i minimal prime ideals of depth n in B^*. Thus it follows that there are at least $k_1 + \cdots + k_h$ minimal prime ideals of depth n in B^*. Therefore the conclusion follows as in the second paragraph of the proof of (4.2), since S^* and B^* have the same total quotient ring.

Before considering the second of the main theorems, we give two small applications of (6). The second of these uses a number of new results from the recent paper [10].

(7) *With the notation of* (6), *if* $Z = \operatorname{Rad} S^*$, *then* $(S^*/Z)'$ *has at least* $k_1 + \cdots + k_h$ *maximal ideals of height* n. *In particular, if* S *is analytically unramified, then there exist at least* $k_1 + \cdots + k_h$ *maximal ideals of height* n *in* $S^{*\prime}$. For, $(S^*/Z)' \cong \bigoplus \{(S^*/z)'; z$ is a minimal prime ideal in $S^*\}$, so since each S^*/z is a complete local domain and since at least $k_1 + \cdots + k_h$ of the z's have depth n, by (6), it follows that $(S^*/Z)'$ has at least $k_1 + \cdots + k_h$ maximal ideals of height n. And if S is analytically unramified, then $Z = (0)$, so the second statement follows from the first.

(8) *With the notation of* (6), *let* Q *be an open ideal in* S, *let* r *be a large integer, and let* I *be an ideal in* S *such that* $Q^r \subseteqq I \subseteqq (Q^r)_a =$ *the integral closure of* Q^r *in* S. *Then there exist at least* $k_1 + \cdots + k_h$ *minimal prime ideals of depth* n *in the form ring* (=*associated graded ring*) $\mathscr{F}(S, I)$ *of* S *with respect to* I. To prove this, note first that it is known [11, Theorem 2.1] that $\mathscr{F}(S, I) \cong \mathscr{R}/t^{-1}\mathscr{R}$, where $\mathscr{R} = \mathscr{R}(S, I) = S[tI, t^{-1}]$ (t is an indeterminate) is the Rees ring of S with respect to I. (The restriction to local rings in [11] is not essential.) Therefore it suffices to prove that $t^{-1}\mathscr{R}$ has at least $k_1 + \cdots + k_h$ minimal prime divisors of depth n. For this, it may be assumed that $I = (Q^r)_a$, by [10, (7.7)]. Let $\mathscr{R}^0 = \mathscr{R}(S^*, IS^*)$, let z be a minimal prime ideal in S^*, and let $z^* = zT[t, t^{-1}] \cap \mathscr{R}^{0\prime}$, where T is the total quotient ring of S^*. Then [7, Corollary 2.23] says there exists a height one prime ideal $p^{0\prime}$ in $\mathscr{R}^{0\prime}$ such that $(z^*, t^{-1})\mathscr{R}^{0\prime} \subseteqq p^{0\prime}$. Next, since r is large, [10, (7.1)] shows that z^* is the only minimal prime ideal in $\mathscr{R}^{0\prime}$ that is contained in $p^{0\prime}$. Also, by [10, (7.3)] and since r is large, there exists a one-to-one correspondence between the minimal prime divisors $p^{0\prime}$ of $t^{-1}\mathscr{R}^{0\prime}$, p^0 of $t^{-1}\mathscr{R}^0$, and p of $t^{-1}\mathscr{R}$, and then depth $p =$ depth $p^0 =$ depth $p^{0\prime} =$ depth z, by [10, (7.3) and (5.2.1)]. Therefore, since S^* has at least $k_1 + \cdots + k_h$ minimal prime ideals of depth n, by (6), it follows that $t^{-1}\mathscr{R}$ has at least $k_1 + \cdots + k_h$ minimal prime divisors of depth n, and so $\mathscr{F}(S, I)$ has at least $k_1 + \cdots + k_h$ minimal prime ideals of depth n.

We now consider the second theorem on prime divisors of zero in R^*. Concerning (9), it is shown in [1] that every power of the maximal ideal in an analytically irreducible local domain R contains a prime ideal of height = altitude $R - 1$. (9) characterizes local domains that fail about as completely as possible to have this latter property.

(9) THEOREM. *The following statements are equivalent for a*

local domain (R, M) *such that altitude* $R \geqq 1$:

(9.1) *There exists a depth one prime divisor of zero in* R^*.

(9.2) *There exist* M*-primary ideals* Q *in* R *such that every nonzero ideal contained in* Q *has* M *as a prime divisor.*

Proof. $(9.1) \Rightarrow (9.2)$. Let $z_1, \cdots, z_g$ be the prime divisors of zero in R^* with depth $z_1 = 1$ and let $(0) = \bigcap_1^g q_i$ be a primary decomposition of zero in R^*. Let x be a nonzero element in $\bigcap_2^g q_i$. Then, for all $P^* \in \operatorname{Spec} R^* - \{z_1, M^*\}$, $x \in \operatorname{Ker}(R^* \to R^*_{P^*})$, so x is in every P^* primary ideal. Hence if I is a nonzero ideal in R such that $I: M = I$, then a primary decomposition of IR^* shows that $x \in IR^*$, since $IR^*: M^* = IR^*$ and $IR^* \not\subseteqq z_1$. Since $x \neq 0$, there exist M-primary ideals Q in R such that $x \notin QR^*$, so $x \notin IR^*$, for all ideals $I \subseteqq Q$. Thus it follows that M is a prime divisor of all nonzero ideals $I \subseteqq Q$.

$(9.2) \Rightarrow (9.1)$. Assume (9.1) does not hold and let $z_1, \cdots, z_k$ be the maximal prime divisors of zero in R^*, so depth $z_i > 1 (i = 1, \cdots, k)$. Assume it is known that there exist $P_1, \cdots, P_k \in \operatorname{Spec} R^* - \{M^*\}$ such that $z_i \subset P_i$ and such that there are no containment relations among the $p_i = P_i \cap R$ (so there are no containment relations among the P_i). Let $S = R - \bigcup_1^k p_i$, let $S^* = R^* - \bigcup_1^k P_i$, and for $n \geqq 1$ let $I_n = (\bigcap_1^k p_i^n) R_S \cap R$ and let $I_n^* = (\bigcap_1^k P_i^n) R^*_{S^*} \cap R^*$. Then $I_n \subseteqq I_n^* \cap R$ and I_n (resp., I_n^*) is the intersection of the k primary ideals $p_i^{(n)} = p_i^n R_{p_i} \cap R$ (resp., $P_i^{(n)} = P_i^n R^*_{P_i} \cap R^*$). Also, $I_n^* \supseteqq I_{n+1}^*$ and $\bigcap I_n^* = (0)$, since $\bigcap_n (\bigcap_1^k P_i^n) R^*_{S^*} = (0)$ and $\operatorname{Ker}(R^* \to R^*_{S^*}) = (0)$. Therefore for each $m \geqq 1$ there exists $n(m)$ such that $I^*_{n(m)} \subseteqq M^{*m}$, by [3, (30.1)]. Hence $I_{n(m)} \subseteqq I^*_{n(m)} \cap R \subseteqq M^{*m} \cap R = M^m$ and M is not a prime divisor of $I_{n(m)}$, so (9.2) does not hold. Therefore it remains to show the existence of the P_i.

For this, let $0 \neq a \in M$ and for $i = 1, \cdots, k$ let $P_{i,a}$ be a minimal prime divisor of $(z_i, a)R^*$. Let $\mathscr{P}_i = \{P_{i,a} \cap R; 0 \neq a \in M\}$. It will now be shown that for each i: (i) $\mathscr{P}_i$ is an infinite set such that $\cup \mathscr{P}_i = M$; and, (ii) the intersection of each infinite subset of $\mathscr{P}_i$ is zero. Namely, depth $P_{i,a} \cap R \geqq$ depth $P_{i,a} \geqq 1$, so each $\mathscr{P}_i$ is an infinite set such that $\cup \mathscr{P}_i = M$ (since a is an arbitrary nonzero element in M), so (i) holds. Also, (ii) holds since the intersection of each infinite subset of $\{P_{i,a}; 0 \neq a \in M\}$ is z_i.

Now assume $1 \leqq h < k$ and $p_1 \in \mathscr{P}_1, \cdots, p_h \in \mathscr{P}_h$ have been chosen such that are no containment relations among $p_1, \cdots, p_h$. Then by (i) (and since M is not the union of finitely many prime ideals) there exist infinitely many $p \in \mathscr{P}_{h+1}$ such that $p \not\subseteqq \bigcup_1^h p_i$, and by (ii) $\bigcap_1^h p_i$ is not contained in at least one of these p, so there exists $p_{h+1} \in \mathscr{P}_{h+1}$ such that there are no containment relations among

$p_1, \cdots, p_{h+1}$. From this the existence of $P_1, \cdots, P_k \in \operatorname{Spec} R^* - \{M^*\}$ such that $z_i \subset P_i$ and such that there are no containment relations among the $P_i \cap R$ readily follows.

(10) REMARKS. (10.1) It is clear by (9) that if there exists an ideal Q in R as in (9.2), then for all finite local integral extension domains $(L, P) \subseteqq R'$ of R there exists a P-primary ideal Q' such that P is a prime divisor of every nonzero ideal in L that is contained in Q'.

(10.2) It is interesting to note that the equivalent conditions in (9) imply that if $P \in \operatorname{Spec} R - \{0, M\}$ and i is a large integer, then P^i is not P-primary (since $P^i \subseteqq M^i \subseteqq Q$). It would be interesting to know if this is, in fact, equivalent to the equivalent statements in (9).

(10.3) It was shown in [6, (4.8) (5)] that if R is as in [3, Example 2, 203-205] in the case $m = 0$, then for all $i \geqq 2$ and for all $P \in \operatorname{Spec} R - \{0, M\}$, P^i is not P-primary.

(10.4) If R is as in (10.3), then M^2 is a suitable Q for (9.2).

Proof. (10.4). R' is a regular domain with exactly two maximal ideals, say P, Q, such that height $P = 1 <$ height $Q = r + 1$, so there exist exactly two minimal prime ideals in R'^*, say z^*, w^*, with $z^* \cap w^* = (0)$ and depth $z^* = 1 <$ depth $w^* = r + 1$. Then $w = w^* \cap R^*$ and $z = z^* \cap R^*$ are the prime divisors of zero in R^* and $z \cap w = (0)$. Also, $M = P \cap Q$, so $M^* = P^* \cap Q^*$. Therefore $w = w^* \cap M^* = w^* \cap (P^* \cap Q^*) = w^* \cap P^*$ is a minimal prime ideal in R^* and $w \nsubseteq M^{*2}$ (since $wR'^*_{P^*} = (w^* \cap P^*)R'^*_{P^*} = P^*R'^*_{P^*} \nsubseteq P^{*2}R'^*_{P^*} = M^{*2}R'^*_{P^*}$). So x in the proof of (9.1) $\Rightarrow$ (9.2) can be chosen in $w, \notin M^{*2}$.

(10.4) and (10.2) give a different proof of the result noted in (10.3).

It follows from the next result (see (12.2)) that if R' is a finite R-algebra (for example, if R is analytically unramified), then (9.2) characterizes when there exists a depth one *minimal* prime divisor of zero in R^*.

(11) PROPOSITION. *Let (R, M) be a local domain such that altitude $R \geqq 1$ and every M-primary ideal contains a nonzero integrally closed ideal. Then the following statements are equivalent:*

(11.1) *There exists a depth one minimal prime ideal in R^*.*

(11.2) *There exist M-primary ideals Q in R such that every nonzero ideal contained in Q has M as a prime divisor.*

Proof. (11.1) $\Rightarrow$ (11.2) by (9).

For the converse, let Q be such an ideal and let I be a nonzero integrally closed ideal contained in Q. Let $b \in I$, $b \neq 0$. Then $(bR)_a \subseteqq I \subseteqq Q$, so M is a prime divisor of $(bR)_a$. Now $(bR)_a = bR' \cap R$, so there exists a (height one) prime divisor p' of bR' that lies over M. Therefore p' is a height one maximal ideal, and so there exists a depth one minimal prime ideal in R^* by [4, Proposition 3.3].

This paper will be closed with the following three remarks related to (11).

(12) REMARKS. (12.1) There exists a local domain (R, M) such that altitude $R > 1$ and there exist M-primary ideals Q in R such that there does not exist a nonzero integrally closed ideal contained in Q. Therefore if $0 \neq b \in Q$, then $(b^n R)_a \not\subseteqq Q$, for all $n \geqq 1$.

(12.2) If (R, M) is a local domain and if there exists b in M such that $R_b \cap R'$ is a finite R-algebra (in particular, if R' is a finite R-algebra), then every M-primary ideal contains an integrally closed ideal.

(12.3) If (R, M) is a local domain, if there exists b in M such that $R_b \cap R^{(1)}$ is a finite R-algebra (where $R^{(1)} = \cap\{R_p;\ p \in \operatorname{Spec} R$ and height $p = 1\}$), and if there exists an ideal Q in R as in (11.2), then altitude $R = 1$.

Proof. (12.1). By (9) and (11) it suffices to show the existence of a local domain R such that R^* has a depth one prime divisor of zero but does not have a depth one minimal prime ideal. Such an example is given in [2, Proposition 3.3].

(12.2). Let Q be an M-primary ideal and let n such that $M^n \subseteqq Q$, so $b^n \in Q$. Now $\{r/b^k;\ r \in b^k R' \cap R\} = R_b \cap R'$ is a finite R-algebra, so there exists $k \geqq 1$ such that $b^k(R_b \cap R') \subseteqq R$. Therefore if $r \in (b^{k+i}R)_a = b^{k+i}R' \cap R$, then $r/b^{k+i} \in R' \cap R_b$, so $r \in b^i R$, hence $(b^{k+i}R)_a \subseteqq b^i R$ for all $i \geqq 1$. Thus $(b^{k+n}R)_a \subseteqq b^n R \subseteqq Q$.

(12.3). [5, Lemma 5.15(10)] shows that there exists k such that $b^{k+1}(R_b \cap R^{(1)}) \subseteqq bR$, if $R_b \cap R^{(1)}$ is a finite R-algebra, so $b^{k+n}(R_b \cap R^{(1)}) \subseteqq b^n R$. Also, $b^{k+n}(R_b \cap R^{(1)})$ is a finite intersection of height one primary ideals, by [5, Lemma 5.15(4)], and height one prime ideals in $R_b \cap R^{(1)}$ lie over height one prime ideals in R, by [5, Lemma 5.15(6)]. Therefore with n such that $b^n \in Q$, we have $b^{k+n}(R_b \cap R^{(1)}) \subseteqq Q$ and is a finite intersection of height one primary ideals, so altitude $R = 1$ by the property of Q.

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