PACIFIC JOURNAL OF MATHEMATICS
Vol. 81, No. 2, 1979

H-CLOSED AND COUNTABLY COMPACT EXTENSIONS

JOHN W. CARLSON

Nearness structures that are generated by countably compact T_1 strict extensions or H-closed extensions are characterized. For a Hausdorff topological space a compatible nearness structure is given for which the completion is the Fomin H-closed extension. A collection of compatible nearness structures for a given Hausdorff space is isolated; and it is shown that there exists a one-to-one correspondence between this collection and the collection of all strict H-closed extensions of the given space, up to the obvious equivalence.

This paper is concerned with the study of H-closed and countably compact extensions using nearness structures. A nearness structure ξ on a set X is said to be generated by an extension Y if ξ is precisely those collections $\mathscr{A}$ of subsets of X whose closures, when taken in Y, meet. Bentley and Herrlich [2], have studied this problem in general and have characterized those nearness structures that are generated by T_1 extensions that are compact, Hausdorff, regular, Lindelöf, paracompact, or realcompact. This paper characterizes those nearness structures that are generated by a strict countably compact T_1 extension or an H-closed extension.

Porter and Votaw [9], show that in general a given Hausdorff topological space X may have $|\mathscr{P}^3(X)|$ H-closed extensions and that it is impossible to generate all of them using proximities, in the same way one can generate all compact Hausdorff extensions of a completely regular space using proximities. Bently and Herrlich [2], have shown that one can generate all T_1 strict extensions of a space X by considering all compatible concrete nearness structures on X. In this paper it is shown that all strict H-closed extensions of a given Hausdorff space, up to the usual equivalence, are in one-to-one correspondence with the collection of all open B-complete concrete Hausdorff nearness structures compatible with the given topology. For a given Hausdorff topological space X, a compatible nearness structure ξ is obtained such that (X^*, ξ^*), the completion of (X, ξ), is homeomorphic to the Fomin H-closed extension of X. The relation of (X^*, ξ^*) to the Katětov H-closed extension is noted.

2. **Preliminaries.** Let X be a set; then $\mathscr{P}^n(X)$ will denote the power set of $\mathscr{P}^{n-1}(X)$ for each natural number n and $\mathscr{P}^0(X) = X$. Let ξ be a subset of $\mathscr{P}^2(X)$ and $\mathscr{A}$ and $\mathscr{B}$ subsets of $\mathscr{P}(X)$. Let A and B be subsets of X. Then the following notation is used.

(1) "$\mathscr{A}$ is *near*" or $\xi\mathscr{A}$ means $A \in \xi$; and "$\mathscr{A}$ is *far*" or $\bar{\xi}\mathscr{A}$ means $\mathscr{A} \notin \xi$.

(2) $A\xi B$ means $\{A, B\} \in \xi$.

(3) $\mathrm{cl}_\xi A = \{x \in X: \{\{x\}, A\} \in \xi\}$.

(4) $\mathscr{A} \vee \mathscr{B} = \{A \cup B: A \in \mathscr{A}, B \in \mathscr{B}\}$.

(5) $\mathscr{A}$ *corefines* $\mathscr{B}$ means that for each $A \in \mathscr{A}$ there exists a $B \in \mathscr{B}$ such that $B \subset A$.

DEFINITION 2.1. Let X be a set and $\xi \subset \mathscr{P}^2(X)$. Then (X, ξ) is called a *nearness space* provided:

(N1) $\cap \mathscr{A} \neq \phi$ implies $\mathscr{A} \in \xi$.

(N2) If $\mathscr{A} \in \xi$ and for each $B \in \mathscr{B}$ there exists $A \in \mathscr{A}$ with $A \subset \mathrm{cl}_\xi B$, then $\mathscr{B} \in \xi$.

(N3) If $\mathscr{A} \notin \xi$ and $\mathscr{B} \notin \xi$ then $A \vee \mathscr{B} \notin \xi$.

(N4) $\phi \in \mathscr{A}$ implies $\mathscr{A} \notin \xi$.

A nearness space is called a N1-*space* provided:

(N5) $\{x\}\xi\{y\}$ implies $x = y$.

Given a nearness space (X, ξ), the operator cl_ξ is a closure operator on X. Hence there exists a topoplogy associated with each nearness space in a natural way. This topology is denoted by $t(\xi)$. This topology is R_0. (Recall that a topology is R_0 provided $x \in \mathrm{cl}_X\{y\}$ implies $y \in \mathrm{cl}_X\{x\}$.) Conversely, given any R_0 topological space (X, t) there exists a compatible nearness structure ξ_0 given by

$$\xi_0 = \{\mathscr{A} \subset \mathscr{P}(X): \cap\, \mathrm{cl}_X \mathscr{A} \neq \phi\} .$$

To say that a nearness structure ξ is *compatible* with a topology t on a set X means that $t = t(\xi)$. A *near filter* on X is a filter that belongs to ξ.

DEFINITION 2.2. Let (X, ξ) be a nearness space.

(1) (X, ξ) is called *topological* provided $\mathscr{A} \in \xi$ implies $\cap\, \mathrm{cl}_X \mathscr{A} \neq \phi$.

(2) (X, ξ) is called *contigual* provided $\mathscr{A} \notin \xi$ implies that there exists a finite $\mathscr{B} \subset \mathscr{A}$ and $\mathscr{B} \notin \xi$.

(3) (X, ξ) is called *totally bounded* provided $\mathscr{A} \subset \mathscr{P}(X)$ and $\mathscr{A}$ has the finite intersection property implies $\mathscr{A} \in \xi$.

(4) (X, ξ) is called *B-complete* if each near ultrafilter converges.

(5) (X, ξ) is called *open totally bounded* provided every collection of $t(\xi)$-open subsets of X with the finite intersection property is near.

(6) (X, ξ) is called *open B-complete* provided every open near ultrafilter converges.

(7) (X, ξ) is called *countably bounded* provided each $\mathscr{A} \subset \mathscr{P}(X)$ with the countable intersection property is near.

(8) (X, ξ) is called *countably totally bounded* provided every countable $\mathscr{A} \subset \mathscr{P}(X)$ with the finite intersection property is near.

The following results (Carlson [3], [4]) are stated here for the convenience of the reader.

THEOREM A. *Let (X, ξ) be a nearness space.*

(1) *The underlying topology is compact if and only if ξ is B-complete and totally bounded.*

(2) *The underlying topology is Lindelöf if and only if ξ is countably bounded and every near filter with the countable intersection property clusters.*

(3) *The underlying topology is countably compact if and only if ξ is countably totally bounded and the closure of every near filter has the countable intersection property.*

(4) *If the underlying topology is Hausdorff then it is H-closed if and only if ξ is open totally bounded and open B-complete.*

DEFINITION 2.3. An *extension* of a topological space (X, t) is a dense embedding $e: (X, t) \to (Y, s)$ where (Y, s) is a topological space. It is called a *strict extension* if $\{\mathrm{cl}_Y e(A): A \subset X\}$ is a base for the closed sets in Y.

We will assume that the embeddings $e: X \to Y$ are injections and thus not distinguish between A and $e(A)$ for $A \subset X$.

A nearness structure ξ on X is said to be induced by a strict extension Y provided:

(1) $e: X \to Y$ is a strict extension, and

(2) $\xi = \{\mathscr{A} \subset \mathscr{P}(X): \ \cap \mathrm{cl}_Y \mathscr{A} \neq \phi\}$.

In a nearness space (X, ξ), a nonempty collection of subsets of X is called an *X-cluster* if it is maximal in ξ with respect to inclusion. The nearness space is called *complete* if every X-cluster has a nonempty adherence.

Herrlich's completion of a nearness space was presented in [7]. A brief description of it appears in [2] which we provide here for the convenience of the reader. Let (X, ξ) be a nearness space and let Y be the set of all X-clusters $\mathscr{A}$ with empty adherence. Set $X^* = X \cup Y$. For each $A \subset X$, define $\mathrm{cl}A = \{y \in Y: A \in y\} \cup \mathrm{cl}_\xi A$. A nearness structure ξ^* is defined on X^* as follows: $\mathscr{B} \in \xi^*$ provided $\mathscr{A} = \{A \subset X$: there exists $B = \mathscr{B}$ with $B \subset \mathrm{cl}A\} \in \xi$. (X^*, ξ^*) is a complete nearness space with $\mathrm{cl}_{\xi^*} X = X^*$. Also, for $A \subset Y$, $\mathrm{cl}_{\xi^*} A = \mathrm{cl}A$.

The following important theorem is due to Herrlich and Bentley [2].

THEOREM B. *For any nearness space (X, ξ) the following conditions are equivalent*:

(1) *ξ is a nearness structure induced on X by a strict extension.*

(2) *The completion (X^*, ξ^*) of (X, ξ) is topological.*

(3) *Every nonempty X-near collection is contained in some X-cluster.*

A nearness space satisfying the above equivalent conditions is called concrete.

3. Countably compact extensions. The basic problem in this section is to determine which nearness structures are generated by countably compact strict extensions.

DEFINITION 3.1. A nearness space (X, ξ) is said to satisfy condition CC provided: $\bigcup_{k \in N} \mathscr{A}_k \notin \xi$ implies there exists a finite set $F \subset N$ such that $\bigcup_{k \in F} \mathscr{A}_k \notin \xi$.

THEOREM 3.1. *Let (Y, t) be an R_0 topological space and $X \subset Y$ such that Y is a strict extension of X. Set $\xi = \{\mathscr{A} \subset \mathscr{P}(X): \bigcap \mathrm{cl}_Y \mathscr{A} \neq \phi\}$. Then Y is countably compact if and only if (X, ξ) satisfies condition CC.*

Proof. Suppose that Y is countably compact. Let $\mathscr{A}_n \subset \mathscr{P}(X)$ for each $n \in N$ and suppose $\mathscr{A} = \bigcup \{\mathscr{A}_n: n \in N\} \notin \xi$. Set $S_k = \bigcap \{\mathrm{cl}_Y A: A \in \mathscr{A}_k\}$ and $\mathscr{S} = \{S_k: k \in N\}$. Now $\bigcap \{\mathrm{cl}_Y S_k: k \in N\} = \bigcap \mathrm{cl}_Y \mathscr{A} = \phi$. Therefore $\mathscr{S} \notin \xi_t = \{\mathscr{A} \subset \mathscr{P}(Y): \bigcap \mathrm{cl}_Y \mathscr{A} \neq \phi\}$, the compatible topological nearness structure for (Y, t). By Theorem A, since Y is countably compact, ξ_t is countably totally bounded. Hence there exists a finite subcollection of S, say $S_{k_1}, \cdots, S_{k_n}$, with $\bigcap \{S_{k_i}: 1 \leqq i \leqq n\} = \phi$. Thus $\bigcap \{\bigcap \mathrm{cl}_Y A: A \in \mathscr{A}_{k_i}\}: 1 \leqq i \leqq n\} = \phi$. Therefore $\mathscr{A}_{k_1} \cup \mathscr{A}_{k_2} \cup \cdots \cup \mathscr{A}_{k_n} \notin \xi$. Hence ξ satisfies condition CC.

Now suppose (X, ξ) satisfies condition CC. It suffices to show that ξ_t is countably totally bounded. Let $\mathscr{S} = \{S_i: i \in N\}$ be a countably collection of subsets of Y. Let $\mathscr{T} = \{T_i: T_i = \mathrm{cl}_Y S_i\}$. Suppose $\mathscr{S} \notin \xi_t$, then $\mathscr{T} \notin \xi_t$. Since Y is a strict extension of X there exists a collection $\mathscr{A}_i$ of subsets of X for each $i \in N$ such that $T_i = \bigcap \{\mathrm{cl}_Y A: A \in \mathscr{A}_i\}$. Now $\mathscr{A} = \bigcup \{\mathscr{A}_i: i \in N\} \in \xi$ if and only if $\bigcap \mathrm{cl}_Y \mathscr{A} \neq \phi$. But $\mathscr{T} \notin \xi_t$ and thus $\bigcap \mathrm{cl}_Y \mathscr{A} = \bigcap \{\bigcap \{\mathrm{cl}_Y A: A \in \mathscr{A}_i\}: i \in N\} = \bigcap \{T_k: k \in N\} = \phi$. Therefore $\mathscr{A} = \bigcup \mathscr{A}_k \notin \xi$ and since ξ satisfies condition CC, there exists finitely many $\mathscr{A}_{k's}$, say $\mathscr{A}_{k_1}, \cdots, \mathscr{A}_{k_n}$, such that $\bigcap_{i=1}^n \mathscr{A}_{k_i} \notin \xi$. Thus $\bigcap_{i=1}^n \mathrm{cl}_Y \mathscr{A}_{k_i} = \phi$; that is $\bigcap_{i=1}^n T_i = \phi$ and consequently $\bigcap_{i=1}^n S_i = \phi$. Thus ξ_t is countably totally bounded. ξ_t is topological and thus the closure of each near filter has the countable intersection property. Hence by Theorem A, (Y, ξ_t) is countably compact.

The following theorem characterizes the T_1 nearness structures that are generated by a countably compact strict T_1 extension.

THEOREM 3.2. *Let (X, ξ) be a T_1 nearness space. The following are equivalent*:

(1) *ξ is a nearness structure induced on X by a countably compact strict T_1 extension.*

(2) *The completion (X^*, ξ^*) of (X, ξ) is topological and countably compact.*

(3) *(X, ξ) satisfies condition CC and each $\mathscr{A} \in \xi$ is contained in a X-cluster.*

Proof. (1) implies (3). (X, ξ) satisfies condition CC by Theorem 3.1. That each $A \in \xi$ is contained in a X-cluster follows from Theorem B. (3) implies (2). By Theorem B, (X^*, ξ^*) is topological and X^* is a strict extension of X. By Theorem 3.1, (X^*, ξ^*) is countably compact. (2) implies (1). This follows from Theorem B.

4. H-closed extension.

THEOREM 4.1. *Let (Y, t) be a Hausdorff topological space with dense subspace X. Define ξ by $\xi = \{\mathscr{A} \subset \mathscr{P}(X): \cap \operatorname{cl}_Y \mathscr{A} \neq \phi\}$. Then the following are equivalent*:

(1) *(Y, t) is H-closed.*

(2) *(X, ξ) is open totally bounded.*

Proof. (2) implies (1). Let $\mathscr{O} = \{O_\alpha: \alpha \in I)$ be an open cover of Y such that $\bar{\mathscr{O}} = \{\operatorname{cl}_Y O_\alpha: \alpha \in I\}$ has no finite subcover. Then $\mathscr{S} = \{Y - \operatorname{cl}_Y O_\alpha: \alpha \in I\}$ is a collection of open sets in Y with the finite intersection property. Now $\mathscr{S}' = \{X \cap (Y - \operatorname{cl}_Y O_\alpha): \alpha \in I\}$ is a collection of open sets in X with the finite intersection property since $\operatorname{cl}_Y X = Y$. Since ξ is open totalley bounded, $\mathscr{S}' \in \xi$. Therefore $\mathscr{S}' \in \xi_t$; that is $\cap \{\operatorname{cl}_Y(X \cap (Y - \operatorname{cl}_Y O_\alpha)): \alpha \in I\} \supset \{x\}$, for some $x \in Y$. There exists $\alpha' \in I$ such that $t \in O_{\alpha'}$. But $x \in \operatorname{cl}_Y(X \cap (Y - \operatorname{cl}_Y O_{\alpha'}))$. Therefore $O_{\alpha'} \cap (X \cap (Y - \operatorname{cl}_Y O_{\alpha'})) \neq \phi$; that is $O_{\alpha'} \cap (Y - \operatorname{cl}_Y O_{\alpha'}) \neq \phi$ which is impossible. Hence $\bar{\mathscr{O}}$ must have a finite subcover and thus (Y, t) is H-closed.

(1) implies (2). Suppose (Y, t) is H-closed. Let $\mathscr{O} = \{O_\alpha: \alpha \in I\}$ be a collection of open sets in X with the finite intersection property. Suppose $\cap \operatorname{cl}_Y \mathscr{O} = \phi$. Set $\mathscr{S} = \{Y - \operatorname{cl}_Y O_\alpha: \alpha \in I\}$. Then $\mathscr{S}$ is an open cover of Y. Since Y is H-closed there exists $\alpha_1, \cdots, \alpha_n \in I$ such that $Y = \cup \{\operatorname{cl}_Y(Y - \operatorname{cl}_Y O_{\alpha_i}): 1 \leqq i \leqq n\}$. Now there exists $x \in \cap \{O_{\alpha_i}: 1 \leqq i \leqq n\}$ with $x \in X$ since $\mathscr{O}$ has the finite intersection property. Also there exists i_0 with $1 \leqq i_0 \leqq n$, such that

$x \in \mathrm{cl}_Y(Y - \mathrm{cl}_Y O_{\alpha i_0})$. Now there exists Q, open in Y, such that $Q \cap X = O_{\alpha i_0}$. Then $x \in Q$ and since $x \in \mathrm{cl}_Y(Y - \mathrm{cl}_Y O_{\alpha i_0})$ we have that $P = Q \cap (Y - \mathrm{cl}_Y O_{\alpha i_0}) \neq \phi$ and P is open in Y. Since $\bar{X} = Y$, $P \cap X \neq \phi$; that is, $X \cap (Q \cap (Y - \mathrm{cl}_Y O_{\alpha i_0})) \neq \phi$, and $O_{\alpha i_0} \cap (Y - \mathrm{cl}_Y O_{\alpha i_0}) \neq \phi$ since $O_{\alpha i_0} = X \cap Q$. But this is impossible and thus $\cap\, \mathrm{cl}_Y \mathscr{O} \neq \phi$. Hence (X, ξ) is open totally bounded.

DEFINITION 4.1. Let (X, ξ) be a nearness space. A collection $\mathscr{A} \subset \mathscr{P}(X)$ is called *micrometric* provided the collection of all subsets of X which meet every member of $\mathscr{A}$ is a near collection. (X, ξ) is called *Hausdorff* if for each micrometric $\mathscr{A} \in \xi$ then $\mathscr{B} = \{B \subset X: \mathscr{A} \cup \{B\} \in \xi\} \in \xi$.

Theorems C and D stated below are found in Bentley and Herrlich [2]. They will be needed in the proof of our next theorem.

THEOREM C. (1) *A topological nearness space is Hausdorff in the nearness space sense if and only if it is Hausdorff in the topological space sense.*

(2) *Every nearness subspace of a Hausdorff nearness space is Hausdorff.*

(3) *The completion of a Hausdorff nearness space is Hausdorff.*

THEOREM D. *Let (X, ξ) be a nearness space. The following are equivalent*:

(1) *ξ is a nearness structure induced on X by a Hausdorff extension.*

(2) *ξ is a nearness structure induced on X by a strict Hausdorff extension.*

(3) *(X, ξ) is concrete and Hausdorff.*

In Theorem 4.1 we did not require that Y be a strict extension of X as we did in Theorem 3.1. Theorem D, however indicates that in the presence of the Hausdorff condition the nearness structure induced will be concrete.

THEOREM 4.2. *Let (X, ξ) be a Hausdorff nearness space. The following are equivalent.*

(1) *ξ is a nearness structure induced on X by a strict H-closed extension.*

(2) *ξ is a nearness structure induced on X by an H-closed extension.*

(3) *The completion X^* of X is H-closed and topological.*

(4) *ξ is open totally bounded, concrete and Hausdorff.*

Proof. (1) implies (2) is clear and (3) implies (1) follows from

Theorem B. (2) implies (3). By Theorem 4.1, ξ must be open totally bounded. Now the H-closed extension is Hausdorff and thus by Theorem D, ξ is both concrete and Hausdorff. By Theorems B and C, X^* is Hausdorff and topological, and by Theorem 4.1 X^* is H-closed. (3) implies (4). ξ is open totally bounded by Theorem 4.1, and concrete by Theorem B, and Hausdorff by Theorem C. (4) implies (3). By Theorems B and C, X^* is topological and Hausdorff by Theorem 4.1, X^* is H-closed.

5. Katětov and Fomin H-closed extensions. Let (Y, t) be a topological space and $\mathrm{cl}_Y X = Y$. $t(X)$ will denote the subspace topology on X. For each $y \in Y$, set $\mathscr{O}y = \{O \cap X : y \in O \in t\}$. Then $\{\mathscr{O}y : y \in Y\}$ is called the filter trace of Y on X.

Let t(strict) be the topology on Y generated by the base $\{O^* : O \in t(X)\}$ where $O^* = \{y \in Y : O \in \mathscr{O}y\}$. Let t(simple) be the topology on Y generated by the base $\{O \cup \{y\} : O \in \mathscr{O}y, y \in Y\}$. Then t(strict) and t(simple) are such that Y with either of these topologies is an extension of $(X, t(X))$, called a strict extension, or simple extension of X, respectively. Note that

$$t(\text{strict}) \leqq t \leqq t(\text{simple}) .$$

Moreover; a topology s on Y with the same filter trace as t, forms an extension of $(X, t(X))$ if and only if it satisfies the above inequality. (See Banaschewski [1]).

Let (X, t) be a Hausdorff topological space. Let M be the collection of all free open ultrafilters on X. Set $Y = X \cup M$. Let κX be the set Y with the topology generated by the base $\{U : U \in t\} \cup \{\{\mathscr{M}\} \cup U : \mathscr{M} \in M \text{ and } U \in \mathscr{M}\}$. Then κX is called the *Katětov H-closed extension* of (X, t), [9]. Let σX be the set Y with the topology generated by the base $\{U^* : U \in t\}$ where $U^* = U \cup \{\mathscr{M} \in M : U \in \mathscr{M}\}$. Then σX is called the *Fomin H-closed extension* of (X, t), [6].

Note that κX and σX have the same underlying set and note that κX is a simple extension and σX is a strict extension. In the notation of Banaschewski, the underlying set Y in both cases corresponds to the collection of all open ultrafilters on X; those that converge correspond to the points of X, those that do not are the points in M.

THEOREM 5.1. *Let (X, t) be a Hausdorff topological space. Set $\xi_h = \{\mathscr{A} \subset \mathscr{P}(X) : \cap \bar{\mathscr{A}} \neq \phi$ or there exists a free open ultrafilter $\mathscr{M}$ on X such that*

$$A \cap O \neq \phi \quad \textit{for each } A \in \mathscr{A} \quad \textit{and} \quad O \in \mathscr{M}\} .$$

Then:

(1) ξ_h *is the nearness structure induced on* X *by the Katětov* H*-closed extension* κX.

(2) ξ_h *is the nearness structure induced on* X *by the Fomin* H*-closed extension* σX.

(3) ξ_h *is a compatible nearness structure on* (X, t) *that is concrete, Hausdorff and open totally bounded.*

(4) (X^*, ξ_h^*), *the completion of* (X, ξ_h), *is homeomorphic to the Fomin* H*-closed extension of* (X, t).

(5) (X, t') *is homeomorphic to the Katětov* H*-closed extension of* (X, t), *where* t' *is the simple extension topology corresponding to the extension topology* $t(\xi_h^*)$.

Proof. (1) Let $\xi_{\kappa X}$ denote the nearness structure generated by the Katětov H-closed extension. Let $\mathscr{A} \in \xi_h$. Then either $\cap \mathrm{cl}_X \mathscr{A} \neq \phi$ and $\mathscr{A} \in \xi_{\kappa X}$, or $\cap \mathrm{cl}_X \mathscr{A} = \phi$ and there exists a free open ultrafilter $\mathscr{M}$ on X such that $A \cap O \neq \phi$ for each $O \in \mathscr{M}$ and $A \in \mathscr{A}$. Then $\mathscr{M} \in \mathrm{cl}_{\kappa X} A$ for each $A \in \mathscr{A}$ and thus $\cap \mathrm{cl}_{\kappa X} A \neq \phi$ and therefore $\mathscr{A} \in \xi_{\kappa X}$.

Now suppose $\mathscr{A} \in \xi_{\kappa X}$. Then $\cap \mathrm{cl}_{\kappa X} \mathscr{A} \supset \{p\}$. If $p \in X$ then $p \in \cap \mathrm{cl}_X \mathscr{A}$ and $\mathscr{A} \in \xi_h$. If $p \in \kappa X - X$ then $p = \mathscr{M}$ for some free open ultrafilter $\mathscr{M}$ on X. Now $\mathscr{M} \in \cap \mathrm{cl}_{\kappa X} \mathscr{A}$ implies that $\mathscr{M} \in \mathrm{cl}_{\kappa X} A$ for each $A \in \mathscr{A}$; that is $(\{\mathscr{M}\} \cup O) \cap A \neq \phi$ for each $O \in \mathscr{M}$. Thus $O \cap A \neq \phi$ for each $O \in \mathscr{M}$ and $A \in \mathscr{A}$. Hence $\mathscr{A} \in \xi_h$.

Proof of (2). Let σX denote the Fomin H-closed extension of (X, t) and $\xi_{\sigma X}$ the nearness structure induced by σX. Let $\mathscr{A} \in \xi_{\sigma X}$. Then $\cap \mathrm{cl}_{\sigma X} \mathscr{A} \supset \{p\}$. If $p \in X$ then $\cap \mathrm{cl}_X \mathscr{A} \neq \phi$ and $\mathscr{A} \in \xi_h$. If $p \in \sigma X - X$ then $p = \mathscr{M}$, a free open ultrafilter on X. Now $\mathscr{M} \in \mathrm{cl}_{\sigma X} A$ for each $A \in \mathscr{A}$. Let $O \in \mathscr{M}$; then $O^* = O \cup \{\mathscr{N} \in \sigma Y : O \in \mathscr{N}\}$ is an open set in σX. Now O^* is an open set containing $\mathscr{M}$ and thus $O^* \cap \mathscr{A} \neq \phi$, and consequently $O \cap A \neq \phi$ for each $A \in \mathscr{A}$. Since this holds for each $O \in \mathscr{M}$ it follows that $\mathscr{A} \in \xi_h$.

Let $\mathscr{A} \in \xi_h$. Either $\cap \mathrm{cl}_X \mathscr{A} \neq \phi$ and $\mathscr{A} \in \xi_{\sigma X}$ or there exists a free open ultrafilter $\mathscr{M}$ such that $O \cap A \neq \phi$ for each $O \in \mathscr{M}$, and $A \in \mathscr{A}$. Let O^* be a basic open set containing $\mathscr{M}$. Then $O \in \mathscr{M}$ and $O^* \cap A \neq \phi$ for each $A \in \mathscr{A}$. Thus $\mathscr{M} \in \cap \mathrm{cl}_{\sigma X} \mathscr{A}$ and $\mathscr{A} \in \xi_{\sigma X}$.

Proof of (3). This follows by either (1) or (2) and Theorem 4.1.

Proof of (4). By Bentley and Herrlich [2], any strict extension Y of X that generates the nearness structure ξ, must be equivalent to the strict extension (X^*, ξ^*). Now the Fomin H-closed extension is strict and it generates the nearness structure ξ_h on X. Therefore the Fomin H-closed extension σX is homeomorphic to (X^*, ξ_h^*).

Proof of (5). The Katětov H-closed extension can be obtained by taking σX, the Fomin H-closed extension, and replacing the topology on the Fomin extension with the corresponding simple extension topology, [10]. The result holds then by (4).

COROLLARY 5.2. *Let (X, t) be a Hausdorff topological space. If ξ_h has only finitely many X-clusters with empty adherence then:*

(1) $\kappa X = \sigma X$.

(2) *X has a finite cover of almost H-closed subspaces.*

(3) *X is locally H-closed.*

Proof. In Porter and Votaw [10], (1) and (2) are shown to be equivalent and imply (3). The hypothesis and Theorem 5.1 imply that $\sigma X - X$ is finite. Flachsmeyer [5] shows that this implies (1).

THEOREM 5.3. *Let (X, t) be a Hausdorff topological space. Then there exists a one-to-one correspondence between the collection of all strict H-closed extensions of X, up to the obvious equivalence, and the collection of all compatible nearness structures on X that are Hausdorff, concrete and open totally bounded.*

Proof. Let $\mathscr{E}$ denote the collection of all strict H-closed extensions of X, with equivalent extensions identified. Let $\mathscr{S} = \{\xi \subset \mathscr{P}^2(X): t = t(\xi), \xi$ is Hausdorff, concrete, and open totally bounded$\}$. Define a mapping $T: \mathscr{E} \to \mathscr{S}$ by $T(Y) = \xi_Y = \{\mathscr{A} \subset \mathscr{P}(X): \bigcap \text{cl}_Y \mathscr{A} \neq \phi\}$. Since Y is an extension of X it follows that $T(Y) = \xi_Y$ is compatible with (X, t). Since Y is H-closed it follows by Theorem 4.1 that $\xi_Y \in \mathscr{S}$. To see that T is one-to-one, let Y and Z belong to $\mathscr{E}$ and $\xi_Y = \xi_Z$. Since Y and Z are strict extensions of $(X, \xi_Y) = (X, \xi_Z)$ then Y and Z are both equivalent to (X^*, ξ_Y) and thus are equivalent. (See Bentley and Herrlich [2].) Now T is onto since $\xi \in \mathscr{S}$ implies (X^*, ξ^*) belongs to $\mathscr{E}$ by Theorem 4.2 and thus $T(X^*) = \xi$.

REFERENCES

1. B. Banaschewski, *Extensions of topological spaces*, Canad. Math. Bull., **7** (1964), 1-22.
2. H. L. Bentley and H. Herrlich, *Extensions of otpological spaces.*
3. J. W. Carlson, *B-Completeness in nearness spaces*, Gen. Top. Appl., **5** (1975), 263-268.
4. ———, *Topological properties in nearness spaces*, Gen. Top. Appl., to appear.
5. J. Flachsmeyer, *Zur theorie der H-abgeschlossen Erweiterungen*, Math. Z., **94** (1966), 349-381.
6. S. Fomin, *Extensions of topological spaces*, Ann. of Math., **44** (1943), 471-480.

7. H. Herrlich, *A concept of nearness*, Gen. Top. Appl., **5** (1974), 191-212.
8. ———, *Topological structures*, Mathematical Centre Tracts 52, Amsterdam, 1974.
9. K. M. Katětov, *Uber H-abgeschlossene and bikompaktc Raume*, Casopis Pest. Mat. Rys., **69** (1940), 36-49.
10. J. R. Porter and C. Votaw, *H-closed extensions I*, Gen. Top. Appl., to appear.

Received March 13, 1978 and in revised form September 7, 1978.

EMPORIA STATE UNIVERSITY
EMPORIA, KS 66801