PACIFIC JOURNAL OF MATHEMATICS
Vol. 81, No. 2, 1979

VECTOR FIELDS AND EQUIVARIANT BUNDLES

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We introduce a new method which gives an easy computation of the Chern classes of V-equivariant bundles on the zero set of the holomorphic vector field V. By using this method we obtain the theorem of Riemann-Roch and Hirzebruch for V-equivariant bundles from the holomorphic Lefschetz fixed point formula in case V has arbitrary isolated zeros.

Introduction. For a given holomorphic vector field V on a complex manifold X, a holomorphic vector bundle $E \to X$ is said to be V-equivariant if there exists a $\boldsymbol{C}$-module homomorphism $\hat{V}: \underline{O}_X(E) \to \underline{O}_X(E)$ such that $\hat{V}(fs) = V(f)s + f\hat{V}(s)$ where f(resp. s) is a local section of $X \times \boldsymbol{C}$(resp. E). The importance of V-equivariant bundles comes from the fact that Chern numbers of such bundles are determined on the zeros of V.

In §1, we give cohomological and geometrical obstructions for a holomorphic vector bundle E to be V-equivariant.

In §2, the holomorphic Lefschetz fixed point formula for the arbitrary isolated fixed points is stated. A generalization of a theorem of N. R. O'Brian [6] is given.

Finally we show how the theorem of Riemann-Roch and Hirzebruch can be obtained naturally from the holomorphic Lefschetz fixed point formula via holomorphic vector fields.

This paper is a portion of the author's Ph. D. thesis. The thesis was written under the direction of Professors James B. Carrell and Larry Roberts at the University of British Columbia. The author would like to take this opportunity to thank Professors Carrell and Roberts for their help and guidance.

1. Equivariant vector bundles. Let X be a complex manifold, and let V be a holomorphic vector field on X. A holomorphic vector bundle $E \to X$ is said to be V-equivariant, if there exists a $\boldsymbol{C}$-module homomorphism $\hat{V}: \underline{O}_X(E) \to \underline{O}_X(E)$, ($\underline{O}_X(E)$ is the sheaf of holomorphic sections of E), such that $\hat{V}(fs) = V(f)s + f\hat{V}(s)$ where f(resp. s) is a local section of $X \times \boldsymbol{C}$(resp. E). Such a $\hat{V}$ is called a V-derivation on E.

If $\hat{V}$ is a V-derivation on E, then it induces naturally a V-derivation $\hat{V}$ on $\underline{O}_{P(E)}$ where $P(E) \to X$ is the bundle of projective spaces associated to $E \to X$ and $\underline{O}_{P(E)}$ is the sheaf of holomorphic functions on $P(E)$. Conversely it is not hard to see that any V-

derivations of $\underline{O}_{P(E)}$ is obtained from a V-derivation on E.

Let $\sigma: \boldsymbol{C} \times X \to X$ be a 1-parameter group of automorphisms of a complex manifold X, i.e., σ is a holomorphic map and the map $\boldsymbol{C} \to \mathrm{Aut}(X) t \to \sigma_t$ is a group homomorphism where $\sigma_t(x) = \sigma(t, x)$ and $\mathrm{Aut}(X)$ is the group of all holomorphic diffeomorphisms of X.

DEFINITION 1.1. A holomorphic vector bundle $E \to X$ is said to be σ-equivariant, if

(i) there exists a 1-parameter group of automorphisms $\hat{\sigma}: \boldsymbol{C} \times E \to E$ of E such that $\hat{\sigma}$ commutes with σ, i.e., $\hat{\sigma}_t(x, v) = (\sigma_t(x), \hat{\sigma}_t(x)v)$ for every $t \in \boldsymbol{C}$ and $(x, v) \in E$.

(ii) for $t \in \boldsymbol{C}$ and $x \in X$, the map $\hat{\sigma}_t(x): E_x \to E_{\sigma_t(x)}$ is $\boldsymbol{C}$-linear. We call such a $\hat{\sigma}$, a σ-equivariant 1-parameter group of automorphisms of E. Such a $\hat{\sigma}$ induces naturally a 1-parameter group of automorphisms of $P(E)$ commuting with σ, and conversely any such 1-parameter group of automorphisms of $P(E)$ is obtained from a σ-equivariant 1-parameter family $\hat{\sigma}$ of E, because $\mathrm{Aut}(P^n) \cong \mathrm{PGL}_{n+1} = \mathrm{GL}_{n+1}/\boldsymbol{C}^* \cdot I_{n+1}$.

Let V be a holomorphic vector field on a compact complex manifold X induced from the 1-parameter group of automorphisms $\sigma: \boldsymbol{C} \times X \to X$ of X, i.e.,

$$V(f) = \frac{d}{dt}(\sigma_t^*(f))\Big|_{t=0} = \frac{d}{dt}(f \circ \sigma_t)\Big|_{t=0} .$$

Then we have the following.

THEOREM 1.1. *Let $E \to X$ be a holomorphic vector bundle on X. Then the following are equivalent.*

(i) *E is σ-equivariant.*

(ii) *E is V-equivariant.*

(iii) *There exists a hermitian metric h on E, such that $i_V(\theta) = \bar{\partial}(L)$ for some differentiable section L of* $\mathrm{Hom}(E, E)$ *over X, where θ is the canonical curvature matrix associated to h and i_V is the contraction operator.*

Proof. (i) $\Rightarrow$ (ii). If E is σ-equivariant, then we have a σ-equivariant 1-parameter family $\hat{\sigma}$ of E. Now for each $t \in \boldsymbol{C}$ we have a $\boldsymbol{C}$-module homomorphism $\hat{\sigma}_t^*: \underline{O}_X(E) \to (\sigma_t)_*(\underline{O}_X(E))$, given by $\hat{\sigma}_t^*(s) = \hat{\sigma}_t^{-1} \circ s \circ \sigma_t$, where s is a local section of E. Now we define $\hat{V}(s) = (d/dt)(\hat{\sigma}_t^*(s))|_{t=0}$. Then $\hat{V}$ is a V-derivation on E, because

$$\hat{\sigma}_t^*(fs) = \sigma_t^*(f)\hat{\sigma}_t^*(s) \quad \text{and} \quad V(f) = \frac{d}{dt}(\sigma_t^*(f))|_{t=0}$$

where f(resp. s) is a local section of $X \times C$(resp. E). Therefore E is V-equivariant. Such a $\hat{V}$ is called V-derivation induced from $\hat{\sigma}$.

(ii) $\Rightarrow$ (i). This is the existence of the solution of a linear E-valued differential equation, and it can be seen as follows. Let $\hat{V}$ be a V-derivation on E; then $\hat{V}$ induces naturally a V-derivation $\hat{V}$ on $\underline{O}_{P(E)}$, i.e., a holomorphic vector field $\tilde{\hat{V}}$ on $P(E)$. Let $\tilde{\hat{\sigma}}$ be the 1-parameter group of automorphisms of $P(E)$ generated by $\tilde{\hat{V}}$ which exists because $P(E)$ is compact. Now $\tilde{\hat{\sigma}}$ commutes with σ because $\hat{V}$ is a V-derivation. Therefore, there exists a σ-equivariant 1-parameter family $\hat{\sigma}$ of E inducing $\tilde{\hat{\sigma}}$ on $P(E)$, hence E is σ-equivariant. In fact $\hat{V}$ is the V-derivation of E induced from $\hat{\sigma}$.

Since (ii) $\Leftrightarrow$ (iii) by a theorem in [4] we have the claim.

Examples of V-equivariant bundles:

(i) TX, the holomorphic tangent bundle of X, is V-equivariant for any holomorphic vector field V on X.

The Lie derivation $[V, \cdot]$ in the direction of V is a V-derivation on TX. Hence TX is V-equivariant. In fact, if X is compact and σ is the 1-parameter group of automorphisms of X generated by V, then $[V, \cdot]$ is the V-derivation induced from $\hat{\sigma}_t = d\sigma_t$, where $d\sigma_t$ is the differential of σ_t.

(ii) The Universal bundle and the Universal quotient bundle on Grassmannian manifold $G_{k,n}$ are V-equivariant for any holomorphic vector field V on $G_{k,n}$.

This follows from the fact that any 1-parameter group of automorphisms of $G_{k,n}$ is induced from an element $A \in gl_n$ and such an action can be lifted naturally to the Universal bundle and the Universal quotient bundle. See [1] for more details.

(iii) If $H^1(X, \underline{O}_X) = 0$, then any line bundle L on X is V-equivariant for any holomorphic vector field V on X. This follows from the condition (iii) in the theorem via $\mathrm{Hom}\,(L, L) \cong \underline{O}_X$.

REMARK. The theorem above is very usefull to compute the Chern classes of V-equivariant bundles on the set of zeros of V. Some computations have been done in [1] by using above method.

2. Riemann-Roch-Hirzebruch theorem for equivariant bundles. Let $E \to X$ be a holomorphic vector bundle on a compact complex manifold X. Then a holomorphic geometric endomorphism of E consists a pair (f, ϕ) where $f: X \to X$ is a holomorphic map and $\phi: f^*E \to E$ is a holomorphic bundle homomorphism. Such a geometric endomorphism (f, ϕ) of E induces functorially a $\boldsymbol{C}$-linear endomorphism $H^k(f, \phi)$ of $H^k(X, \underline{O}_X(E))$ for each k as it is defined in [7].

Let $z_1, \cdots, z_n$ be local holomorphic coordinates centered at the isolated fixed point $p \in X$ of $f: X \to X$. Then on a small neighborhood $N(p)$ of p in X, the only common zeros of the functions $z_i - f^i(z) i = 1, \cdots, n$ is $0 = (0, \cdots, 0)$, where $f^i(z) = z_i \circ f$ for $i = 1, \cdots, n$. Let

$$\nu_p(f, \phi, E) = \operatorname{Res}_p \left\{ \frac{\operatorname{trace} \phi(z) dz_1, \cdots, dz_n}{z_1 - f^1(z), \cdots, z_n - f^n(z)} \right\}$$

where $\phi(z)$ is the matrix representation of ϕ on $N(p)$ and Res_p is the Grothendieck residue symbol. If the fixed point set X^f of $f: X \to X$ is finite, then by [7], the Holomorphic Lefschetz fixed point formula can be stated as

$$L(f, \phi, E) := \sum_{k=0}^{n} (-1)^k \operatorname{trace} H^k(f, \phi) = \sum_{p \in X^f} \nu_p(f, \phi, E) .$$

Let V be a holomorphic vector field on a compact complex manifold X, induced from the 1-parameter group of automorphisms σ of X, and let $\hat{V}$ be a V-derivation induced from the σ-equivariant 1-parameter group of automorphisms $\hat{\sigma}$ of a holomorphic vector bundle $E \to X$. Then for each $t \in \boldsymbol{C}$, we have a bundle isomorphism $\hat{\sigma}_t^{-1}: E \to (\hat{\sigma}_t^{-1})^* E$. By taking the pull-back relative to $\sigma_t: X \to X$, we obtain a bundle isomorphism $\phi_t = (\hat{\sigma}_t^{-1})^*: \sigma_t^* E \to E$. Hence we have a natural geometric endomorphism (σ_t, ϕ_t) of E for each $t \in \boldsymbol{C}$.

If p is an isolated zero of V, then p is an isolated fixed point of σ_t at least for small values of $|t|$, hence $\nu_p(\sigma_t, \phi_t, E)$ makes sense at least for small values of $|t|$, say in a neighborhood W of $0 \in \boldsymbol{C}$. Now we will compute $\nu_p(\sigma_t, \phi_t, E)$ for $t \in W$.

Let $V|_{N(p)} = \sum_{i=1}^{n} a_i \partial / \partial z_i$ be the local expression of V on $N(p)$ where $z = (z_1, \cdots, z_n)$ are the local coordinates on $N(p)$ centered at p. Now consider the functions $v_i(z)$, $w_j(z)$ on $N(p)$ defined from the following identities:

$$\det (I_q + \lambda \hat{V}(z)) = \sum_{0}^{q} c_i(\hat{V}(z)) \lambda^i = \prod_{i=1}^{q} (1 + \lambda v_i(z))$$

$$\det (I_n + \lambda L(z)) = \sum_{0}^{n} c_i(L(z)) \lambda^i = \prod_{j=1}^{n} (1 + \lambda w_j(z))$$

where $L(z) = (a_i(z) / \partial z_j)$, q is the rant of E, $\hat{V}(z)$ is the matrix representation of $\hat{V}$ on $N(p)$, and I_k is the $k \times k$ identity matrix. We define for $(t, z) \in \boldsymbol{C} \times N(p)$, the Chern character of $\hat{V}(z)$ by $\operatorname{Ch}(E, z, t) = \sum_1^q \exp(t v_i(z))$, and the Todd class of $L(z)$ by

$$Td(z, t) = \prod_{1}^{n} \frac{t w_i(z)}{1\text{-}\exp(t w_i(z))} ,$$

where the functions on the right-hand side should be regarded as standing for the corresponding power series expansion. Both $\mathrm{Ch}(E, z, t)$ and $Td(z, t)$ are holomorphic functions on $W \times N(p)$, since

$$c_i(\hat{V}(z))(\text{resp. } c_j(L(z)))$$

is the elementary symetric functions in the $v_k(z)$(resp. $w_r(z)$) and $c_i(\hat{V}(z))$, $c_j(L(z))$ are holomorphic functions. Under the above assumptions, we have the following.

THEOREM 2.1. *For* $t \in W - 0$,

$$\nu_p(\sigma_t, \phi_t, E) = \frac{1}{t^n} \operatorname{Res}_p \left\{ \frac{\mathrm{Ch}(E, z, t)\, Td(z, t) dz_1, \cdots, dz_n}{a_1(z), \cdots, a_n(z)} \right\}.$$

Theorem 2.1 generalizes a theorem of N. R. O'Brian in [6], where he actually proves this theorem for $\hat{\sigma}_s = (\bigwedge^p d\sigma_s^{-1})^t \colon \bigwedge^p (T^*X) \to \bigwedge^p (T^*X)$, t stands for the transpose. The same arguments given there can be applied with the help of above technics to prove Theorem 2.1, so we leave the details to the reader to check. We have been informed that the Theorem 2.1 has been obtained algebraically by N. R. O'Brian.

It is clear by the Theorem 2.1 and by the algorithm for the Grothendieck residue in [2] that $\nu_p(\sigma_t, \phi_t, E)$ is a meromorphic function of t in W with a pole at $t = 0$. $\nu_p(\sigma_t, \phi_t, E)$ has a unique analytic continuation as a meromorphic function on the whole complex plane. But $L(\sigma_t, \phi_t, E) = \sum_0^n (-1)^k$ trace $H^k(\sigma_t, \phi_t)$ is a holomorphic function of t in the complex plane. On the other hand, by the holomorphic Lefschetz fixed point formula $L(\sigma_t, \phi_t, E)$ equals to $\sum_{p \in Z} \nu_p(\sigma_t, \phi_t, E)$ for sufficiently small $|t|$, where Z is the zero set of V which is finite. Hence by the uniqueness of the analytic continuation the fixed point formula

$$L(\sigma_t, \phi_t, E) = \sum_{p \in Z} \nu_p(\sigma_t, \phi_t, E) \tag{1}$$

holds for all $t \in \boldsymbol{C}$. Therefore the singular parts of $\nu_p(\sigma_t, \phi_t, E)$ must cancel out as we sum over $p \in Z$, and the constant term must add up to the left member at $\mathrm{t} = 0$. But for $t = 0$, $\phi_t = \mathrm{id}$, hence the left member of (1) reduces to $\sum_0^n (-1)^k \dim \cdot H^k(X, \underline{O}_X(E)) \colon = \chi(\underline{O}_X(E))$. By Theorem 2.1. and the algorithm for the Grothendieck residue in [2], it is easy to see that the constant term of $\nu_p(\sigma_t, \phi_t, E)$ is equal to

$$\operatorname{Res}_p \left\{ \frac{\mathrm{Ch}(E, z) Td(X, z) dz_1, \cdots, dz_n}{a_1(z), \cdots, a_n(z)} \right\}$$

where $\mathrm{Ch}(E, z)\, Td(X, z) =$ the coefficient of t^n in the power series expansion of $\mathrm{Ch}(E, z, t)\,Td(z, t)$ around $t = 0$. Hence we get

$$\chi(\underline{O}_x(E)) = \sum_{p \in Z} \mathrm{Res}_p \left\{ \frac{\mathrm{Ch}(E, z)Td(X, z)dz_1, \cdots, dz_n}{a_1, \cdots, a_n} \right\} . \tag{2}$$

Now the right-hand side of (2) can be viewed as the value of the global residue on a section of $\underline{O}_X = \underline{O}_X / i_V(\Omega^1)$ where Ω^1 is the sheaf of holomorphic 1-forms. This operator, Res: $\Gamma(Z, \underline{O}_z) \to \boldsymbol{C}$, is defined as follows: Let W be a holomorphic function in a neighborhood $N(p)$ of $p \in Z$ representing the function $s \in \Gamma(Z, \underline{O}_z)$ at p. Then

$$\mathrm{Res}\,(s) = \sum_{p \in Z} \mathrm{Res}_p \left\{ \frac{W(z)dz_1, \cdots, dz_n}{a_1, \cdots, a_n} \right\}$$

is a well defined linear map. Moreover by [5] we have a commutative diagram

$$\begin{tikzcd} \Gamma(Z, \underline{O}_z) \arrow[rr, "\mathrm{Res}"] \arrow[dr, "m"'] & & \boldsymbol{C} \\ & H^n(X, \Omega^n) \arrow[ur] & \end{tikzcd} \qquad \left(\frac{1}{2\pi i}\right)^n \int_X .$$

If we apply (2) to this commutative diagram, by the explicit description of the edge map m: $\Gamma(Z, \underline{O}_Z) \to H^n(X, \Omega^n)$ given in [1] or in [5], we get the theorem of Riemann-Roch and Hirzebruch for V-equivariant holomorphic vector bundle E.

REMARK. Via this method, R. Bott has obtained in [3] the theorem of Riemann-Roch and Hirzebruch for the trivial line bundle. It is this beautiful work of R. Bott, who let us study this subject. It would be interesting to obtain the above theorem for the meromorphic vector field.

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Received December 18, 1977 and in revised form August 14, 1978. Partially supported by the National Research Council of Canada grant 67-8497.

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