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A COMMON FIXED POINT THEOREM FOR NESTED SPACES

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Let X be an arcwise connected Hausdorff space in which the union of any nest of arcs is contained in an arc. Let $f, g: X \to X$ be commuting functions (not necessarily continuous), which satisfy (1) $f(A)$ and $g(A)$ are arcwise connected for each arc $A \subset X$, and (2) $f^{-1}(x)$ and $g^{-1}(x)$ are arcwise connected for each $x \in X$. The principal result of this paper is:

THEOREM. *The functions f and g have a common fixed point.*

A space satisfying the conditions on X is called a *nested space*. Functions which satisfy condition (1) are called *arc preserving* and those satisfying condition (2) are called *strongly monotone*. In [3] Harris showed that continuous functions are arc preserving. Thus we have:

COROLLARY. *Two commuting, continuous strongly monotone selfmaps of a nested space have a common fixed point.*

In this context an arc is a continuum with exactly two noncutpoints. If X is metrizable, then this coincides with the classical definition of an arc as the homeomorphic image of the closed unit interval.

Before proceeding to the proof of the main result we give an example which shows that an arc preserving, strongly monotone function is not necessarily continuous and then give a few historical remarks.

EXAMPLE. Let $X_0 = \{(x, 0): 0 \leqq x \leqq 2\}$ and $X_n = \{(1/n, y): 0 \leqq y \leqq 1\}$ for $n \geqq 1$. Next set $X_{-1} = \{(2, y): 0 \leqq y \leqq 2\}$, $X_{-2} = \{(x, 2): 0 \leqq x \leqq 2\}$ and $X_{-3} = \{(0, y): 1 \leqq y \leqq 2\}$. Then set $X = \cup \{X_k: k \geqq -3\}$. Define $f: X \to X$ by $f(z) = (2, 0)$ if $z \in \bigcup_{i=1}^{3} U_{-i}$ and $f(z) = z$ otherwise. We see that f is arc preserving and strongly monotone but f is not continuous.

In 1967 W. J. Gray [1] proved that an abelian semigroup of continuous, monotone functions on an hereditarily unicoherent, hereditarily decomposable continuum into itself had a common fixed point. Further, in 1975 Gray and Smith [2] proved an extension of this result for hereditarily unicoherent, arcwise connected continua. In this note we prove a common fixed point theorem for commuting functions on a nested space.

The notion of a nested space was used by G.S. Young [6] in 1946. In [6] Young showed that nested spaces have the fixed point property for continuous selfmaps. This theorem was subsequently extended to multifunctions by Smithson [5] and further extended by Muenzenberger and Smithson in [4].

REMARK. It is clear from the definition that a nested space is acyclic. Thus if $x, y \in X$ where X is a nested space, we denote the unique arc in X with endpoints x, y by $[x, y]$. In the sequel X will denote a nested space, and the functions f, g satisfy conditions (1) and (2).

We define a partial order on X as follows: Let $e \in X$. Then $x \leqq y$ if and only if $x \in [e, y]$. The proof that $\leqq$ is a partial order is routine and is omitted. In the remainder of the paper we assume that X has this partial order.

LEMMA 1. *The partial order $\leqq$ satisfies the following:*

(i) *If $x < y$, then there is a z such that $x < z < y$.*

(ii) *If $C \subset X$ is totally ordered and nonempty, then* $\sup C$ *exists in X.*

(iii) *For each $x, y \in X$,* $\inf \{x, y\} = x \varLambda y$ *exists.*

Proof. For (i) let $z \in [x, y] - \{x, y\}$. Then $z \in [x, y] \subset [e, y]$ since $x \in [e, y]$ and thus $x < z < y$. For (ii) note that $\{[e, c]: c \in C\}$ is a nested collection of arcs in X and so C is contained in an arc $[e, a]$. Let $c_0 = \sup C$ in $[e, a]$. If $C \subset [e, b]$, then $C \subset [e, a] \cap [e, b]$ which is an arc and so $c_0 \leqq b$. Thus $c_0 = \sup C$ in X. For (iii), let $A = [e, x] \cap [e, y]$. Then A is an arc $[e, a]$ and $a = x \varLambda y$.

REMARK. We could also show that each nonempty subset of X has an infimum in X and that for each $x \in X$, there is a maximal element $m \in X$ with $x \leqq m$.

If x, y are not comparable, $x \varLambda y$ is a cutpoint of the arc $[x, y]$ and thus $[x, y] = [x \varLambda y, x] \cup [x \varLambda y, y]$.

Define the sets $L(x)$ and $M(x)$ by: $L(x) = \{y \in X; y \leqq x\}$ and $M(x) = \{y: x \leqq y\}$. Then, since $L(x) = [e, x]$, $L(x)$ is totally ordered. Also $M(x)$ is arcwise connected. We have:

LEMMA 2. *If A is arcwise connected, if $A \cap M(x) \neq \phi \neq A \cap (X - M(x))$, then $x \in A$.*

Proof. Let $y_1 \in A \cap M(x)$ and $y_2 \in A \cap (X - M(x))$. Then $y_1 \varLambda y_2 \notin$

$M(x)$, but $y_1 \varLambda y_2 \in L(y_1)$. Thus $x \in [y_1 \varLambda y_2, y_1] \subset [y_1, y_2] \subset A$.

LEMMA 3. *If $a < b$ and if $[a, b]$ contains a fixed point of f, then $x_0 = \inf\{x \in [a, b]: f(x) = x\}$ is a fixed point of f. Hence, $[a, b]$ contains a smallest (in $[a, b]$) fixed point of f.*

Proof. Let $x_1 \in [a, b]$ be a fixed point of f. Then if $f(x_0) \neq x_0$, $x_0 < x_1$ and we may assume that $f(x_0) \nleqq x_1$. Let $z_0 = f(x_0)\varLambda x_1$. Since f is arc preserving, there is a $z_1 \in [x_0, x_1]$ such that $f(z_1) = z_0$. Note $z_1 \neq x_0$. Now let x_2 be a fixed point of f such that $x_0 < x_2 < z_1$. Since $x_2 < x_1$, $f(x_0)\varLambda x_2 \leqq f(x_0)\varLambda x_1 \leqq f(x_0)$. But x_2 and $f(x_0)$ are elements of $f[x_0, x_2]$ and thus so are $f(x_0)\varLambda x_2$ and $f(x_0)$. Hence, $z_0 \in f[x_0, x_2]$. This implies that $f^{-1}(z_0) \cap M(x_2) \neq \varnothing$ and $f^{-1}(z_0) \cap (X - M(x_2)) \neq \varnothing$ and thus $x_2 \in f^{-1}(z_0)$ since f is strongly monotone. This is a contradiction to $f(x_2) = x_2$ and $x_2 < z_0$.

Next we need another definition.

DEFINITION. Let $a \in X$. The branch at a containing $x_1 \in M(a) - a$ is the set $B = \{x: a < x\varLambda x_1\}$.

Thus if B_1, B_2 are two different branches at a and if $x_i \in B_i$, $i = 1, 2$, then $a = x_1 \varLambda x_2$ and $a \in [x_1, x_2]$.

Before proving the main result we need two more lemmas.

LEMMA 4. *If $A \subset X$ is a nonempty totally ordered set such that $x \leqq f(x)$ for $x \in A$, then, $x_0 \leqq f(x_0)$ where $x_0 = \sup A$.*

Proof. Suppose $f(x_0) \notin M(x_0)$. Let $b = f(x_0)\varLambda x_0$ and let $b \leqq c \leqq x_0$. Since $x_0 = \sup A$, there is an x_1, $c < x_1 < x_0$, such that $x_1 \leqq f(x_1)$. Then $f[x_1, x_0]$ meets $M(c)$ and $X - M(c)$ and hence, contains c. Let $z_1 \in [x_1, x_0]$ with $f(z_1) = c$. Next let x_2 be in A with $z_1 < x_2 < x_0$. Then $f[x_2, x_0]$ meets both $M(c)$ and $X - M(c)$ and also contains c. But then $f^{-1}(c)$ meets $M(x_2)$ and $X - M(x_2)$ and therefore contains x_2. This contradicts the choice of x_2. Thus $x_0 \leqq f(x_0)$.

LEMMA 5. *Let $a < f(a)$. (i) If B is the branch at a containing $f(a)$, then B contains a fixed point of f. (ii) If $f(f(a)) \notin M(f(a))$, then $X - M(f(a))$ contains a fixed point of f.*

Proof. For part (i) first let $a < x < f(a)$. Then if $f(x) \notin M(x)$, $f[a, x]$ contains x. Hence, there is an x, $a < x < f(a)$, with $x \leqq f(x)$. Now let C be a maximal totally ordered set containing x such that $c \leqq f(c)$ for all $c \in C$. Let $x_0 = \sup C$. Then $f(x_0) \in M(x_0)$ follows from Lemma 4. Note that x_0 is in the branch at a containing $f(a)$.

For $x_0 < x_1 < f(x_0)$, $f(x_1) \notin M(x_1)$. Thus $x_0 \in f[x_0, x_1]$. Suppose $f(z_1) = x_1$ where $x_0 < z_1 < x_1$. But then $z_1 \in C$ which contradicts the definition of x_0 and C. Thus $f(x_0) = x_0$.

For statement (ii) set $X_0 = \{X - M(f(a))\} \cup \{f(a)\}$, and define $g: X_0 \to X_0$ by $g(x) = f(x)$ if $f(x) \notin M(f(a))$ and $g(x) = f(a)$ if $f(x) \in M(f(a))$. Since $X - M(f(a))$ is arcwise connected, X_0 is a nested space and g is arc preserving, and strongly monotone. Now let C be a maximal totally ordered set in X_0 which contains a such that $x \leqq f(x)$ for all $x \in C$. Then let $x_0 = \sup C$. (We are using $\leqq$ restricted to X_0.) From the hypothesis for (ii) $f(f(a)) \notin M(f(a))$ and thus $x_0 \neq f(a)$. Then by the same argument used in part (i), $f(x_0) = x_0$ and part (ii) follows.

Now we give the proof of the main theorem.

Proof. Let $A = \{x \in X: x \leqq f(x)$ and $x \leqq g(x)\}$ and let C be a maximal totally ordered subset of A. Set $a = \sup C$. Then by applying Lemma 4 to f and g we see that $a \in C$. The remainder of the proof is divided into a number of parts.

First suppose that $a < f(a) \Lambda g(a)$; without loss of generality we may suppose that for $a < x < b = f(a) \Lambda g(a)$ there is a z, $a < z < x$ with $f(z) \notin M(z)$. Thus let $a < z_1 < b_1 < b$ with $f(z_1) \notin M(z_1)$. Then $b_1 \in f[a, z_1]$. Say $a < y_1 < z_1$ and $f(y_1) = b_1$. But then there is a z_2, $a < z_2 < y_1$ with $f(z_2) \notin M(z_2)$. Thus $f[a, z_2]$ also contains b_1 which contradicts the assumption that $f^{-1}(b_1)$ is arcwise connected.

Next we assume that $a \neq f(a)$ and $a \neq g(a)$, and that $a = f(a) \Lambda g(a)$. For each $i = 1, 2$, let B_i be the branch at a containing $x_1 = f(a)$ and $x_2 = g(a)$ respectively. Since f and g commute $f(x_2) = g(x_1)$ and hence, either $g(x_1) \notin B_2$ or $f(x_2) \notin B_1$. Say $f(x_2) \notin B_1$. Then $a \in f[a, x_2]$. By Lemmas 3 and 4 there is a fixed point y_1 of f in B_1 such that the only f-fixed point in $[a, y_1]$ is y_1. Next note that $fg(y_1) = gf(y_1) = g(y_1)$ and so $g(y_1)$ is also a fixed point of f. Further, if $y_2 = g(y_1) \notin B_1$, then $a \in [y_2, y_1]$ and thus $a \in f[y_2, y_1]$. This leaves two possibilities: either $y_2 \in B_1$ or $y_2 \in B_2$ for otherwise $a \in f^{-1}(a)$ which is a contradiction. So suppose $y_2 \in B_1$ and let $b = y_1 \Lambda y_2$. Then if $f(b) \notin M(b)$, $b \in f[b, y_1] \cap f[b, y_2]$ and so $b \in f^{-1}(b)$ which contradicts the choice of y_1. Thus $f(b) \in M(b)$. Then $g(b) \notin M(b) - b$ follows from the maximality of a. Next set $y_3 = g(y_2)$. If $y_3 \in M(b)$ we have $b \in g[b, y_1] \cap g[b, y_2]$ which is a contradiction. Hence, $y_3 \notin M(b)$. Note $f(y_3) = y_3$. So we have $b \in f[y_3, b]$ and either $b \in f[b, y_1]$ or $b \in f[b, y_2]$ which is another contradiction.

We still have the subcase $y_2 \in B_2$ to consider. By Lemma 3 we may assume that f does not have another fixed point in $[a_1, y_2]$. Then set $y_3 = g(y_2)$. As above either $y_3 \in B_1$ or $y_3 \in B_2$. In either of these cases we can obtain the same contradiction as in the case where $y_2 \in B_1$. Hence, the argument for $f(a) \neq a \neq g(a)$ and $f(a) \Lambda g(a) = a$ is concluded.

Finally, assume $f(a) = a$ and $a < g(a) = y_1$. Since y_1 is a fixed point of f, $g(y_1) \notin M(y_1)$. Thus there is a g-fixed point $x_1 \in M(a) - M(y_1)$. Let $b = x_1 \Lambda y_1$. Then $a \leqq b < y_1$ and so let c be such that $b < c < y_1$. Now if $x_2 = f(x_1) \notin M(a)$, then $c \in g[x_2, a]$ and $c \in g[a, x_1]$ which is a contradition since $a \notin g^{-1}(c)$. Thus $x_2 \in M(a)$ and $x_2 \notin M(x_1)$. From Lemma 3 we may assume that x_1 is the only g-fixed point in $[a, x_1]$. Now let $d = x_1 \Lambda x_2$. As in the previous arguments, applied to g in this case, $f(d) \in M(d)$. Finally we set $x_3 = f(x_2)$ and the same arguments as above give the same contradiction. Thus we conclude that a is a fixed point of both f and g.

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