PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

ON THE THEOREM OF HELLEY CONCERNING FINITE DIMENSIONAL SUBSPACES OF A DUAL SPACE

JAMES SHIREY

Let C and S denote Banach spaces for which $C \subset S^*$. Then (S, C) is said to have property $[P]$ if for any $s^{**} \in S^{**}$ there is an $s \in S$ such that $s^{**}(c) = c(s)$ for every $c \in C$. There is then a fixed $M > 0$ so that if $\varepsilon > 0$, s can always be chosen so that $||s|| \leqq M||s^{**}|| + \varepsilon$. If $M = 1$, then (S, C) is a 1-Helley pair.

A classical theorem of E. Helley states that (S, C) is a 1-Helley pair whenever C is finite dimensional. It is shown that such is the case whenever C is reflexive. As a partial converse, if (S, C) has property $[P]$ and if C is weak* closed, then C is reflexive.

It is also shown that if X and Y are closed subspaces of a Banach space, and if $X + Y$ is closed, then there is $M > 0$ so that for each $z \in X + Y$, there are $x \in X$ and $y \in Y$ for which $z = x + y$ and $||x|| \leqq M||z||$.

Introduction. Let C and S denote Banach spaces for which $C \subset S^*$. The pair (S, C) is said to have the property $[P]$ if for any $s^{**} \in S^{**}$ there is an $s \in S$ such that $s^{**}(c) = c(s)$ for every $c \in C$. That is, the canonical map $S \to C^*$ is surjective. Then the canonical map $S/C_\perp \to C^*$ is an isomorphism and it follows that there is some fixed $M > 0$ such that for any $\varepsilon > 0$, the $s \in S$ above may be chosen to satisfy $||s|| \leqq M||s^{**}|| + \varepsilon$. If M is the smallest constant for which this is true, then (S, C) is said to be an *M-Helley pair*. This terminology is suggested by the classical theorem of E. Helley (cf. [3], p. 103) to the effect that (S, C) is a 1-Helley pair whenever C is a finite dimensional subspace of S^*. It is shown in Theorem 1 below that such is the case whenever C is reflexive. As a partial converse, if (S, C) has property $[P]$, then C is reflexive provided that C is also weak* closed. If C is not weak* closed, then (S, C) is merely an M-Helley pair for some M.

As a consequence of Theorem 2, (S, C) is an M-Helley pair for some M if and only if $S + C^\perp = S^{**}$. There is then a constant K such that for any $s^{**} \in S^{**}$, there is an $s \in S$ and $y \in C^\perp$ for which $s^{**} = s + y$, $||s|| \leqq K||s^{**}||$, and the infimum of all such K is $1/M$. If $S + C^\perp$ is only a closed subspace of S^{**}, then the norm on $S/C_\perp$ induced by C is compatible with the usual equivalence class norm.

In either case, the closedness of $S + C^\perp$ is equivalent to a bounded decomposition property formalized as an "M-decomposition" below. This property is exemplified in the well-known theorem to the effect that if W is a Banach space and if X and Y are closed subspaces for which $X + Y = W$ and $X \cap Y = \{0\}$, then there is an M such that for any $w \in W$, there are unique $x \in X$ and $y \in Y$ for which $w = x + y$ and $\|x\| \leqq M\|w\|$. When there is no hypothesis on $X \cap Y$, the uniqueness of x and y is lost, and the classical proof of this theorem seems to break down irreparably. It is shown in Theorem 4 that the existence of an M for which $\|x\| \leqq M\|w\|$ is retained.

Definitions and notation. Let A, B, and C denote Banach spaces for which $C \subset A$ and $B \subset A^*$.

(B, C) is an *M-Helley pair* if (i) for any $a^* \in A^*$ and for every $\varepsilon > 0$ there is some $b \in B$ such that $a^*(c) = b(c)$ for every $c \in C$, (ii) $\|b\| \leqq M\|a^*\| + \varepsilon$ and (iii) M is the smallest constant for which this is true.

If S is a Banach space, $A = S^*$, $B = S$ and the conditions (i) through (iii) obtain, then (S, C) is an M-Helley pair where S is identified with its canonical image in S^{**}.

(B, C) is an *M-semi-normed pair* if for any $\beta \in B/B \cap C^\perp$, $\sup\{b(c) : \in C, \|c\| = 1 \text{ and } b \in \beta\} \geqq M\|\beta\|$ and M is the largest constant for which this is true. Loosely stated, this means that the norm on $B/B \cap C^\perp$ induced by C is equivalent to the usual equivalence class norm.

Let X and Y denote closed subspaces of a Banach space Z. Then $[X, Y]$ is an *M-decomposition* in Z if there is a constant K such that for any $z \in X + Y$ there is an $x \in X$ and a $y \in Y$ for which $z = x + y$ and $\|x\| \leqq K\|z\|$, and M is the infimum of all such K.

Let X and Y denote Banach spaces for which $X \subset Y^*$, and suppose that X is closed in the weak* topology. Then X^{w^*} denotes the subspace of X^* consisting of those linear functionals on X which are continuous in the relative weak* topology.

Let X and Y denote Banach spaces and let $T: X \to Y$ denote a norm decreasing linear map. T is an *M-isomorphism* if M is the largest constant for which $M\|x\| \leqq \|Tx\| \leqq \|x\|$ holds for every $x \in X$.

All other notation, in particular the use of $\perp$, is consistent with that of [1].

Main results.

THEOREM 1. *Let S and C denote Banach spaces for which $C \subset S^*$.*

(a) *If C is reflexive, then (S, C) is a 1-Helley pair.*

(b) *Suppose that for any* $s^{**} \in S^{**}$ *there is an* $s \in S$ *for which* $s^{**}(c) = c(s)$ *for all* $c \in C$ *(property* $[P]$ *above).*

If C *is weak* closed, then* C *is reflexive and* (S, C) *is a* 1*-Helley pair. Otherwise,* (S, C) *is an* M*-Helley pair for some* M.

THEOREM 2. *Let* A, B *and* C *denote Banach spaces for which* $C \subset A$ *and* $B \subset A^*$. *The following statements are equivalent.*

(a) *The canonical map* $B/B \cap C^{\perp} \rightarrow C^*$ *is an* M*-isomorphic embedding* (resp. *an* M*-isomorphism*).

(b) (B, C) *is an* M*-semi-normed* (resp. *a* $1/M$*-Helley*) *pair.*

(c) $[B, C^{\perp}]$ *is a* $1/M$*-decomposition in* (resp. *of*) A^*.

COROLLARY 3. *The following statements are equivalent.*

(a) *The canonical map* $B \rightarrow C^*$ *has closed range* (resp. *is onto*).

(b) *For some* M, (B, C) *is an* M*-semi-normed pair* (resp. *a* $1/M$*-Helley pair*).

(c) $B + C^{\perp}$ *is closed in* A^* (resp. $B + C^{\perp} = A^*$).

THEOREM 4. *Let* X *and* Y *denote closed subspaces of a Banach space* Z.

(a) $X + Y$ *is closed if and only if* $[X, Y]$ *is an* M*-decomposition in* Z *for some* M.

(b) *If* $X + Y$ *is not closed, then there exists some* $w \in \mathrm{cl}\,(X + Y)$ *such that if* $\{x_n\} \subset X$, $\{y_n\} \subset Y$ *and* $\mathrm{Lim}\,(x_n + y_n) = w$, *then* $\mathrm{Lim}\,\|x_n\| = \mathrm{Lim}\,\|y_n\| = \infty$.

Proofs of main results.

LEMMA 1. *Let* S *and* C *denote Banach spaces for which* $C \subset S^*$. *If* C *is reflexive, the* C *is closed in the weak* topology.*

Proof. Let U denote any strongly closed, convex and bounded subset of C. Since C is reflexive, U is $\sigma(C^*, C)$ compact. An application of the Hahn-Banach theorem shows that U is also $\sigma(S^{**}, S^*)$ compact. Then U is compact in the coarser $\sigma(S, S^*)$ topology. Thus, every such U is weak* closed and it follows (cf. p. 141 of [4]) that C is weak* closed.

Proof of Theorem 1.

(a) By [1], p. 25, Lemma 1, $(S/C_{\perp})^*$ is canonically isometric to $C_{\perp}^{\perp}$, and since Lemma 1 above implies that $C = C_{\perp}^{\perp}$, it follows that $(S/C_{\perp})^{**}$ is canonically isometrically isometric to C^*. Since C is reflexive, so is $S/C_{\perp}$, and it follows that $S/C_{\perp}$ is canonically isometric to C^*. It is easily shown that this equivalent to the statement that (S, C) is a 1-Helley pair.

(b) Let V denote the unit ball of C. Since V is strongly closed and convex, V is weak* closed relative to C. If C is weak* closed, then V is weak* compact.

Choose $y \in C^*$. There is an $s^{**} \in S^{**}$ for which $s^{**}(c) = y(c)$ for all $c \in C$. From the hypothesis there is then an $s \in S$ for which $y(c) = c(s)$ for all $c \in C$. From this and the weak* compactness of V, it follows that y attains its maximum value on V. Since this holds for every $y \in C^*$, it follows from the well-known theorem of R. C. James [2] that C is reflexive.

Without the assumption that C is weak* closed, it is still evident from the hypothesis that the canonical map $S \to C^*$ is surjective. The canonical map $S/C_\perp \to C^*$ is then an M-isomorphism for some M, and this is easily seen to imply that (S, C) is an M-Helley pair (see Theorem 2).

LEMMA 2. *Let A, B, and C denote Banach spaces for which $C \subset A$ and $B \subset A^*$. The following statements are equivalent.*

(a) *The canonical mapping $B/B \cap C^\perp \to C^*$ is a $1/M$-isomorphic embedding* (resp. *a $1/M$-isomorphism*).

(b) *$[B, C^\perp]$ is an M-decomposition in A^** (resp. *of A^**).

(c) *The projection π of $B + C^\perp/B \cap C^\perp$ onto $B/B \cap C^\perp$ along $C^\perp/B \cap C^\perp$ is of norm M* (resp. *and $B + C^\perp = A^*$*).

Proof. The proof is an immediate consequence of the following three statements. Proof of the parenthesized portions is omitted.

(I) If $B/B \cap C^\perp \to C^*$ is a $1/M$-isomorphic embedding, then $[B, C^\perp]$ is a K-decomposition in A^* for some $K \leqq M$.

(II) If $[B, C^\perp]$ is a K-decomposition in A^*, then $||\pi|| \leqq K$.

(III) $B/B \cap C^\perp \to C^*$ is a $1/||\pi||$-isomorphic embedding.

Proof of (I). Let $w = b + g$ where $b \in B$ and $g \in C^\perp$, let β denote the equivalence class of b, let y denote the linear map on C induced by w and choose $M_0 > M$. By hypothesis, $M_0^{-1}||\beta|| < ||y||$, and it is obvious that $||y|| < ||w||$. There is then an x in β such that $||x|| \leqq M_0||w||$, and it is clear that $w - x \in C^\perp$. Since this holds for every $M_0 > M$, it follows that $[B, C]$ is a K-decomposition in A^* for some $K \leqq M$.

Proof of (II). Choose $\xi \in B + C^\perp/B \cap C^\perp$ and $w \in \xi$. By hypothesis, there are $x \in B$ and $y \in C^\perp$ for which $w = x + y$ and if $K_0 > K$, for which $||x|| \leqq K_0||w||$. If $[x]$ denotes the class of x, it then follows that $||[x]|| \leqq K_0||\xi||$. Since this holds for all $K_0 > K$, we conclude that $||\pi|| \leqq K$.

Proof of (III). Let T denote the indicated map, choose $\xi \in B/B \cap C^\perp$ with $\|\xi\| = 1$, choose $x \in \xi$, and let x_0 denote the canonical image of x in C^*. Since $A^*/C^\perp$ is canonically isometric to C^* (cf. [1], p. 26, Lemma 2), $\|x_0\| = \inf\{\|x - y\| : y \in C^\perp\}$. The left side of this equation has the same value for any $x \in \xi$ and it follows that

$$\begin{aligned}\|T\xi\| &= \inf\{\|x - y\| : x \in \xi,\ y \in C^\perp\} \\ &= \inf\{\|\xi - \tau\| : \tau \in C^\perp/B \cap C^\perp\}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\inf\{\|T\xi\| : \xi \in B/B \cap C^\perp,\ \|\xi\| = 1\} \\ &\quad = \inf\{\|\xi - \tau\| : \xi \in B/B \cap C^\perp,\ \tau \in C^\perp/B \cap C^\perp,\ \|\xi\| = 1\}\,.\end{aligned}$$

It is easily shown that the right side of the above equation is $1/\|\pi\|$, and this completes the proof.

Proof of Theorem 2. The equivalence of (a) and (c) is a consequence of Lemma 2, and the equivalence of the non-parenthesized portions of (a) and (b) is a direct consequence of the definitions. It only remains to show that (B, C) is a $1/M$-Helley pair if and only if the canonical map $T: B/B \cap C^\perp \to C^*$ is an M-isomorphism. This is a direct consequence of the following statements.

(i) If (B, C) is a $1/M$-Helley pair, then T is a K-isomorphism for some $K \geqq M$.

(ii) If T is a K-isomorphism, then (B, C) is a K_0-Helley pair for some $K_0 < 1/K$.

Proof of (i). Choose $y \in C^*$, let $(\cdot)/C$ denote the canonical map of A^* onto C^*, and let $\varepsilon > 0$. From the canonical isometry of $A^*/C^\perp$ with C^* (loc. cit.), it follows that there is an $a^* \in A^*$ for which $a^*/C = y$ and $\|a^*\| \leqq \|y\| + M\varepsilon/2$. By hypothesis, there is $b \in B$ for which $b/C = a^*/C$ and $\|b\| \leqq M^{-1}\|a^*\| + \varepsilon/2$. Hence, $\|b\| \leqq M^{-1}\|y\| + \varepsilon$. Let β denote the class of b in $B/B \cap C^\perp$. Then $T\beta = y$, from which we conclude that T is surjective, and the last inequality implies that $M\|\beta\| \leqq \|T\beta\|$.

Proof of (ii). Choose $a^* \in A^*$. From the hypothesis, there is some $\beta \in B/B \cap C^\perp$ for which $T\beta = a^*/C$ and $\|\beta\| \leqq K^{-1}\|a^*\|$. Then, for any $\varepsilon > 0$, there is $b \in \beta$ for which $\|b\| \leqq K^{-1}\|a^*\| + \varepsilon$ and $b/C = a^*/C$. The desired conclusion follows from this.

Proof of Corollary 3. Parts (a) and (b) of Corollary 3 are respectively equivalent to (a) and (b) of Theorem 2 where the value of M is unspecified. To obtain the remaining equivalence, note that by

Lemma 2, $[B, C^\perp]$ is a $1/M$-decomposition in A^* for some M if and only if $B/B \cap C^\perp$ and $C^\perp/B \cap C^\perp$ are complementary subspaces of $B + C^\perp/B \cap C^\perp$, and this is the case if and only if $B + C^\perp$ is a complete (hence, closed) subspace of A^*.

LEMMA 3. *Let W and S denote Banach spaces for which $S \subset W^*$. If S is weak* closed, then S^{w^*} is canonically isometric to $W/S_\perp$, and S^{w^*} is thereby a norm closed subspace of W^*.*

Proof. Since $W/S_\perp$ is canonically isometric to $S_\perp^\perp$ (loc. cit.), and since $S_\perp^\perp = S$ by hypothesis, it follows that $(W/S_\perp)^{**}$ is canonically isometric to S. The composition of this isometry with the canonical map $W/S_\perp \to (W/S_\perp)^{**}$ is the canonical map $T\colon W/S_\perp \to W^*$, and it follows that T is isometric onto its range. Since the weak* topology in W^* is locally convex, one may apply the Hahn-Banach theorem (cf. [4], p. 108, Theorem 2) to conclude that the range of T is S^{w^*}.

LEMMA 4. *Let X and Y denote closed subspaces of a Banach space Z. The following statements are equivalent.*

(1) *The canonical mapping $X/X \cap Y \to (Y^\perp)^*$ is a $1/M$-isomorphic embedding* (resp. *$1/M$-isomorphism*).

(2) *$[X, Y]$ is an M-decomposition in Z* (resp. *of Z*).

(3) *The projection of $X + Y/X \cap Y$ onto $X/X \cap Y$ along $Y/X \cap Y$ is of norm M* (resp. *and $X + Y = Z$*).

Proof. The statements (I), (II) and (III) that were used to prove Lemma 2 may be slightly modified and used to prove this lemma. The proofs of the corresponding statements are the same with the exception of statement (III). To prove the analogue of statement (III), note that $(Z/Y)^*$ is canonically isometric to $Y^\perp$. Hence, $(Z/Y)^{**}$ is canonically isometric to $(Y^\perp)^*$, and so the canonical map of Z/Y into $(Y^\perp)^*$ is norm preserving.

Choose $\xi \in X/X \cap Y$ with $||\xi|| = 1$, choose any $x \in \xi$ and let x_0 denote the canonical image of x in $(Y^\perp)^*$. From the above, $||x_0|| = \inf\{||x - y||\colon y \in Y\}$, and the rest of the proof follows the outline of Lemma 2.

Proof of Theorem 4. The proof of (a) is essentially the same as the proof of Corollary 3 with Lemma 4 playing the role of Lemma 2. As for (b), we prove the contrapositive. Let $W = \mathrm{cl}(X + Y)$, let $T\colon W \to (Y^\perp)^{w^*}$ denote the canonical surjection, and define

$$B_N = \{y \in (Y^\perp)^{w^*}\colon ||y|| \leqq 1, \text{ and there is } \{x_n\} \subset X \text{ such that } \lim ||Tx_n - y|| = 0 \text{ and } ||x_n|| \leqq N \text{ for all } n\}\,.$$

If V denotes the unit ball of $(Y^{\perp})^{w^*}$, then it is clear that B_N is a closed subset of V for each N. If $w \in W$ as described in (b) of Theorem 4 does not exist, then $\cup B_N = V$. Lemma 3 implies that $(Y^{\perp})^{w^*}$ is a Banach space and an application of the Baire category theorem then shows that B_N contains a neighborhood in $(Y^{\perp})^{w^*}$ for some N. It follows that T is an open map. Then $X/X \cap Y \rightarrow (Y^{\perp})^{w^*}$ is an isomorphic embedding and by an application of Lemma 4, $[X, Y]$ is an M-decomposition in W for some M. Then by (a) of this theorem, $X + Y$ is closed.

References

1. M. Day, *Normed Linear Spaces*, Springer-Verlag, Berlin, 1962.
2. R. C. James, *Reflexivity and the supremum of linear functionals*, Ann. of Math., **66** (1957), 159–169.
3. A. Wilansky, *Functional Analysis*, Blaisdell, New York, 1964.
4. K. Yosida, *Functional Analysis*, Academic Press, New York, 1965.

Received December 12, 1977 and in revised form January 10, 1979.

OHIO UNIVERSITY
ATHENS, OH 45701