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DEHN'S LEMMA AND HANDLE DECOMPOSITIONS OF SOME 4-MANIFOLDS

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We give two short proof of a weak version of the theorem of Laudenbach, Poenaru [3]. Also we show that an embedded $S^1 \times S^2$ in S^4 bounds a copy of $B^2 \times S^2$. Finally we establish that if W is a smooth 4-manifold with $\partial W = \#_n S^1 \times S^2$ and W is built from $\#_{n-1} B^2 \times S^2$ by attaching a 2-handle, then W is homeomorphic to $\#_n B^2 \times S^2$.

1. 4-Dimensional handlebodies. Let X, Y be the following smooth 4-manifolds:

$$X = \#_n B^3 \times S^1 \quad \text{and} \quad Y = \#_n B^2 \times S^2 .$$

In [3] it is proved that if $h\colon \partial X \to \partial Y$ is a diffeomorphism, then the smooth closed 4-manifold $X \bigcup_h Y$ which is obtained by gluing along h, is diffeomorphic to S^4.

We begin with two brief proofs, one using the Dehn's lemma in [5] and the other employing unknotting in codimension 3, of the following result:

THEOREM. *Let X, Y, h be as above. Then $X \bigcup_h Y$ is homeomorphic to S^4.*

Proof. (1) Let $\{x_i\} \times S^1$ be a circle in the boundary of the ith copy of $B^3 \times S^1$ in the connected sum $X = \#_n B^3 \times S^1$, for $1 \leqq i \leqq n$. Without loss of generality, all the loops $\{x_i\} \times S^1$ can be assumed to miss the cells which are used to construct X as a connected sum. By the Dehn's lemma in [5], it follows that all of the circles $h(\{x_i\} \times S^1)$ bound disjoint smooth embedded disks D_i in Y, for $1 \leqq i \leqq n$.

Let $N(D_i)$ denote a small tubular neighborhood of D_i in Y. Clearly $X \bigcup_h (N(D_1) \cup \cdots \cup N(D_n))$ is diffeomorphic to B^4, since $N(D_i)$ can be thought of as a 2-handle which geometrically cancels a 1-handle of X. On the other hand, let W denote the closure of $Y - N(D_1) - \cdots - N(D_n)$. Then $\partial W = S^3$ and W is contained in Y which can be embedded in S^4. By the topological Schoenflies theorem [1], W is homeomorphic to B^4. Consequently $X \bigcup_h Y$ is homeomorphic to $B^4 \cup B^4 = S^4$.

(2) By Van Kampen's theorem, $\pi_1(X \bigcup_h Y) = \{1\}$. Let Z be a bouquet of n circles which is embedded in X and is a deformation retract of X. By isotopic unknotting in codimension 3, Z is con-

tained in the interior of a PL 4-cell B in $X \bigcup_h Y$. Therefore, by an isotopy we can shrink X down towards Z until X is included in $\operatorname{int} B$. Exactly as in (1), by the topological Schoenflies theorem we obtain that $X \bigcup_h Y - \operatorname{int} B$ is homeomorphic to B^4 and so the result follows.

REMARK. Note that if the PL or smooth 4-dimensional Schoenflies theorem was known, then these arguments would establish that $X \bigcup_h Y$ is PL isomorphic or diffeomorphic to S^4.

2. Embeddings of $S^1 \times S^2$ in S^4. The following result was first proved by I. Aitchison (unpublished). We present a simplification of his method, which again uses the Dehn's lemma in [5].

THEOREM. *Let $h: S^1 \times S^2 \to S^4$ be a smooth embedding. Then h extends to a topological embedding of $B^2 \times S^2$ in S^4.*

Proof. Let V, W be the closures of the components of $S^4 - h(S^1 \times S^2)$ (by Alexander duality there are two such components). By the Mayer-Vietoris sequence, without loss of generality the inclusion $h(S^1 \times S^2) \to V$ induces an isomorphism $H_1(h(S^1 \times S_2)) \to H_1(V)$ and $H_1(W) = 0$.

Let G denote the group which is the pushout of the homomorphisms $\pi_1(h(S^1 \times S^2)) \to \pi_1(V)$ and $\pi_1(h(S^1 \times S^2)) \to \pi_1(W)$. By Van Kampen's theorem, $G = \{1\}$. On the other hand there is a homomorphism of G onto $\pi_1(W)$ induced by the epimorphism $\pi_1(V) \to H_1(V) \cong H_1(h(S^1 \times S^2)) \cong \pi_1(h(S^1 \times S^2))$. Consequently $\pi_1(W) = \{1\}$ follows.

Now we can apply the Dehn's lemma in [5] to obtain that $h(S^1 \times *)$ bounds a smooth embedded disk D in W. Let $N(D)$ be a small tubular neighborhood of D in W. Then the closure of $W - N(D)$ is a topological 4-cell, by the topological Schoenflies theorem [1]. Therefore W is homeomorphic to $B^2 \times S^2$ and h extends to a topological embedding of $B^2 \times S^2$ as desired.

REMARK. This result is analogous to the classical theorem of Alexander that any smooth embedded $S^1 \times S^1$ in S^3 bounds a smooth solid torus $B^2 \times S^1$.

3. Handle decompositions and slice links. In [2], Kirby, Melvin proved that if a smooth 4-manifold M has boundary $S^1 \times S^2$ and is constructed by attaching a 2-handle to B^4 along a curve C with the 0-framing, then M is homeomorphic to $B^2 \times S^2$ and C is a slice knot. We prove the following generalization of their result:

THEOREM. *Let W be a smooth 4-manifold which is obtained by adding n 2-handles to B^4 along the curves $C_1, \cdots, C_n$. The 2-handles induce a framing of the link $C_1 \cup \cdots \cup C_n$. Assume that framed surgery on the sublink $C_1 \cup \cdots \cup C_i$ in S^3 yields $\#_i S^1 \times S^2$, for all i with $1 \leqq i \leqq n$. Then W is homeomorphic to $\#_n B^2 \times S^2$ and $C_1 \cup \cdots \cup C_n$ is a slice link.*

COROLLARY. *Let W be a smooth 4-manifold such that ∂W is diffeomorphic to $\#_n S^1 \times S^2$ and W is built by attaching a 2-handle to $\#_{n-1} B^2 \times S^2$. Then W is homeomorphic to $\#_n B^2 \times S^2$.*

Proof of theorem. By the assumption that surgery on the link $C_1 \cup \cdots \cup C_n$ gives $\#_n S^1 \times S^2$, it immediately follows that ∂W is diffeomorphic to $\#_n S^1 \times S^2$. If the handle decomposition of W is turned upside down, then W is constructed by attaching n 2-handles to $(\#_n S^1 \times S^2) \times I$ along some curves $C'_1 \times \{1\}, C'_2 \times \{1\}, \cdots, C'_n \times \{1\}$ and then adding a 4-handle. We will assume that the 2-handle glued along $C'_i \times \{1\}$ is *dual* to the 2-handle added along C_i to B^4.

Let W_i or W'_i denote the 4-manifold which is obtained by adjoining i 2-handles to B^4 or $(\#_n S^1 \times S^2) \times I$ respectively along the curves $C_1, \cdots, C_i$ or $C'_{n-i+1} \times \{1\}, \cdots, C'_n \times \{1\}$ respectively. Then ∂W_i is diffeomorphic to $\#_i S^1 \times S^2$, since surgery on $C_1 \cup \cdots \cup C_i$ gives $\#_i S^1 \times S^2$. Also $W - \operatorname{int} W'_i$ is diffeomorphic to W_{n-i} and therefore W'_i is a cobordism between $\#_n S^1 \times S^2$ and $\#_{n-i} S^1 \times S^2$. Note that W'_i can also be constructed by adding $n-i$ 2-handles to $(\#_{n-i} S^1 \times S^2) \times I$.

Let $\{C\}$ denote the homotopy class of a loop C relative to some base point and let $\langle * \rangle$ denote the normal closure of the set of elements $*$ in some group. By Van Kampen's theorem applied to the two handle decompositions of W'_i, we conclude that

$$\pi_1(W'_i) \cong \pi_1(\#_n S^1 \times S^2)/\langle \{C'_{n-i+1}\}, \cdots, \{C'_n\} \rangle$$

and $\pi_1(W'_i)$ has rank $\leqq n - i$. Consider the case when $i = 1$. By a classical theorem of Whitehead (see Exercise 20 on p. 283 of [4]) and by Corollary 5.14.2 on p. 354 of [4], it follows that $\pi_1(W'_1)$ is free and $\{C'_n\}$ is primitive, i.e., is contained in a free basis of the free group $\pi_1(\#_n S^1 \times S^2)$.

Next, $\pi_1(W'_2)$ has a presentation consisting of a set of free generators of $\pi_1(W'_1) \cong \pi_1(\#_n S^1 \times S^2)/\langle \{C'_n\} \rangle$ and the one relation $\{C'_{n-1}\}$. Hence by the results on p. 283 and p. 354 of [4] again, $\pi_1(W'_2)$ is free and $\{C'_{n-1}\}$ is primitive. Therefore we obtain that $\{\{C'_{n-1}\}, \{C'_n\}\}$ is contained in a free basis for $\pi_1(\#_n S^1 \times S^2)$. Continuing on with this argument, we conclude that $\{\{C'_1\}, \cdots, \{C'_n\}\}$ is a free basis of $\pi_1(\#_n S^1 \times S^2)$. So by Lemma 2 of [3], there is a diffeomorphism h: $\#_n S^1 \times S^2 \to \#_n S^1 \times S^2$ such that $h(S^1 \times \{x_i\})$ is homotopic to C'_i for all i, $1 \leqq i \leqq n$, where $S^1 \times \{x_i\}$ is contained in the ith copy of $S^1 \times S^2$ used to form $\#_n S^1 \times S^2$ and is disjoint from the 3-cells employed for the connected sum.

Let M be the smooth 4-manifold with $\partial M = S^3$ which is built by adding n 3-handles and 4-handles to W'_n, using the component $(\#_n S^1 \times S^2) \times \{0\}$ of $\partial W'_n$. The 3-handles can be attached along the 2-spheres $h(\{y_i\} \times S^2) \times \{0\}$, for $1 \leqq i \leqq n$, where $\{y_i\} \times S^2$ is in the ith copy of $S^1 \times S^2$ used to obtain $\#_n S^1 \times S^2$ and $\{y_i\} \times S^2$ misses the 3-cells utilized for the connected sum. Turning the 3- and 4-handles of M upside down, we find that M can be constructed with a 0-handle, n 1-handles and n 2-handles. Note that each 2-handle of M algebraically cancels one of the 1-handles, since C'_i is homotopic to $h(S^1 \times \{x_i\})$.

The Mazur trick can now be applied. $M \times I$ is a 5-manifold composed of a 0-handle, n 1-handles and n 2-handles. By the Whitney trick, the 2-handles geometrically cancel the 1-handles. Consequently $M \times I$ is diffeomorphic to B^5 and $2M = \partial(M \times I)$ is diffeomorphic to S^4. By the topological Schoenflies theorem [1], M is homeomorphic to B^4.

Let N denote the smooth closed 4-manifold which is obtained by gluing a 4-cell to M along $\partial M = S^3$. Then N is homeomorphic to S^4. Since $N = W \cup \#_n B^3 \times S^1$ it follows that W is homeomorphic to $\#_n B^2 \times S^2$, either by isotopic unknotting in codimension 3 or by using the Dehn's lemma in [5] plus the topological Schoenflies theorem as in §2. This proves the first part of the theorem. Finally, exactly the same argument as in [2] applies to show that $C_1 \cup \cdots \cup C_n$ is a slice link.

Proof of corollary. If W satisfies the conditions of the corollary, then W can be constructed by adding n 2-handles to B^4 along the curves $C_1, \cdots, C_n$ where $C_1 \cup \cdots \cup C_{n-1}$ is a trivial link of $n-1$ components in S^3. Hence W satisfies the hypotheses of the theorem and so W is homeomorphic to $\#_n B^2 \times S^2$.

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