PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

THE COMPUTATION OF THE GENERALIZED SPECTRUM OF CERTAIN TOEPLITZ OPERATORS

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In an earlier memoir, "Charting the operator terrain," a new generalized spectrum for a bounded operator T on a separable Hilbert space, was defined as follows: Let $C^*(T)$ denote the C^*-algebra generated by T and the identity operator. We say another operator S is weakly contained in T if there exists a *-representation φ of $C^*(T)$ which maps the identity into an identity operator and $\varphi(T)=S$. The "spectrum" of T, denoted $\hat{T}$, is defined to be the space of unitary equivalence classes of irreducible operators weakly contained in T. In this paper this spectrum is explicitly computed for certain specific Toeplitz operators.

The purpose of the memoir [3] was to establish this "spectrum" as a natural generalization of the ordinary (scalar) spectrum of an operator. From the point of view of the theoretical structure the argument is quite convincing. Thus the spectrum is always non-empty and admits a (in general non-Hausdorff) topology relative to which it is compact. If T is normal, $\hat{T}$ may be identified with the ordinary spectrum. Further for a large class of well behaved (smooth) operators one obtains a theory analogous to the ordinary spectral multiplicity theory for normal operators. Thus to each (smooth) operator T one may associate a σ-finite measure class μ on $\hat{T}$ and a multiplicity function f defined on the measures absolutely continuous with respect to μ. These three invariants, $\hat{T}$, μ, f, then determine the operator up to unitary equivalence. This theory reduces to the ordinary spectral multiplicity theory when the operator is normal.

The difficulty with the theory is more practical than theoretical. Since the triplet $(\hat{T}, \mu, f)$ is a complete set of unitary invariants, the complexity of a nonnormal operator is mirrored in the complexity of the spectrum $\hat{T}$. Indeed $\hat{T}$ is a complete algebraic invariant for T in the sense that if S is another operator, then there is a C^*-algebra isomorphism φ of $C^*(T)$ onto $C^*(S)$ such that $\varphi(T) = S$, if and only if $\hat{T} = \hat{S}$. While we feel at home with the ordinary spectrum as a subset of the complex plane, we are somewhat intimidated by this space of equivalence classes of irreducible operators. The purpose of this paper is to make this generalized spectrum a bit less imposing by describing it concretely for certain nonnormal Toeplitz operators. We hope these examples will encourage others to attempt to compute this "spectrum" for their own favorite operators.

On the basis of the definition one might suspect that the computation of the "spectrum" of an operator T is equivalent to the computation of the spectrum of the C^*-algebra $C^*(T)$. While these are intimately related we shall see in these examples that the computation of the spectrum of the operator involves different considerations. (In the rest of the paper we shall be presumptuous enough to use the term "spectrum of an operator" to refer to the generalized spectrum defined above.)

We shall examine Toeplitz operators of the form $T = \alpha I + \beta S + \gamma S^*$, where I is the identity operator, S is the unilateral shift and α, β and γ are complex numbers. Then T is normal if and only if $|\beta| = |\gamma|$. If $|\beta| \neq |\gamma|$, then T is irreducible and generates the same C^*-algebra as S, i.e., $C^*(T) = C^*(S)$. Thus for the operators T considered in this paper the spectrum of the C^*-algebra $C^*(T)$ will be fixed, even though the spectra of the operators T will vary considerably.

We first state all the facts about these operators that we establish in this paper in one place. Thus the reader who is not interested in reading the somewhat pedestrian proofs of these assertions (which forms the second part of this paper) can stop at the end of the following theorem.

THEOREM. *Consider the Toeplitz operators of the form* $T = \alpha I + \beta S + \gamma S^*$ *where* $\alpha, \beta, \gamma \in \boldsymbol{C}$ *and* S *denotes the unilateral shift and* I *denotes the identity operator.*

(1) T *is normal if and only if* $|\gamma| = |\beta|$.

(2) *If* $|\gamma| \neq |\beta|$ *then* T *is irreducible and generates* $C^*(S)$.

(3) *We next examine the unitary equivalence problem for the irredubible operators* (i.e., *where* $|\gamma| \neq |\beta|$).

(a) *If* $\beta = 0$ *then* $\alpha I + \gamma S^* \cong \alpha I + |\gamma| S^*$.

(b) *If* $\beta \neq 0$ *then* $\alpha I + \beta S + \gamma S^* \cong \alpha I + |\beta| S + (|\beta| \gamma/\bar{\beta}) S^*$.

(c) *Thus we may assume that these special Toeplitz operators all have a special form, up to unitary equivalence, namely we assume* $\beta \geqq 0$ *and if* $\beta = 0$ *then* $\gamma \geqq 0$. *Under this condition we find that these irreducible operators represent distinct unitary equivalence classes,* i.e., $\alpha_1 I + \beta_1 S + \gamma_1 S^* \cong \alpha_2 I + \beta_2 S + \gamma_2 S^*$ *if and only if* $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ *and* $\gamma_1 = \gamma_2$.

(4) *The operator spectrum* $\hat{T}$ *of each irreducible operator* T *in this class* ($|\beta| \neq |\gamma|$) *is of the form* $\hat{T} = \{T\} \cup \hat{T}_1$ *where* $\{T\}$ *denotes the unitary equivalence class of* T *itself and* $\hat{T}_1$ *is a subset of the complex plane consisting of the one-dimensional operators*

weakly contained in T. Note $\hat{T}_1$ is also called the normal spectrum. $\hat{T}_1$ is always an ellipse and every nondegenerate ellipse in the plane arises as the normal spectrum of one of these operators. Note that two distinct (not unitarily equivalent) operators have distinct spectra but may have the same normal spectrum. For example S and S^* have the same normal spectrum (the unit circle). Similarly it follows from our next stated result that $S + 2S^*$, $S-2S^*$ and $2S+S^*$ all have the same normal spectrum, namely the ellipse

$$\left(\frac{x}{3}\right)^2 + y^2 = 1 .$$

(5) *Consider the irreducible operator $T = \beta S + \gamma S^*$ where $|\gamma| \neq \beta$, $\beta > 0$. Then the normal spectrum of $\hat{T}$ is an ellipse centered at the origin. If $\gamma = a + bi$, and if we rotate our x, y axis of the complex plane counterclockwise by an angle ν where $\cot 2\nu = a/b$ (if $b = 0$ do not rotate at all) then the equation of the ellipse relative to the rotated axis is*

$$\left(\frac{x}{\beta + |\gamma|}\right)^2 + \left(\frac{y}{\beta - |\gamma|}\right)^2 = 1 .$$

(Note: If $\gamma = re^{i\theta}$, $r > 0$ and $0 \leqq \theta \leqq 2\pi$, then $\nu = \theta/2$.) Of course the normal spectrum of an operator of the form $T = \alpha I + \beta S + \gamma S^*$ is an ellipse centered at α and is obtained by translating by α the ellipse given above (as the normal spectrum of $\beta S + \gamma S^*$).

(6) *There is a natural topology on $\hat{T}$ which is non-Hausdorff and easily described. The relative topology on the normal spectrum of T is just the ordinary (Hausdorff) topology as a subset of the complex plane. However, the closure of the singleton $\{T\}$ in $\hat{T}$ is all of $\hat{T}$.*

(7) *It is interesting to note how the operator spectrum collapses to the ordinary spectrum as β approaches $|\gamma|$. For example if $\beta>0$ and $\gamma = 1$ (and $\alpha = 0$) then as $\beta \to 1$ the ellipse*

$$\left(\frac{x}{1 + \beta}\right)^2 + \left(\frac{y}{1 - \beta}\right)^2 = 1$$

collapses to the line segment $[-2, +2]$, which is just the ordinary spectrum of the Hermitian operator $S + S^*$. ($\beta = \gamma = 1$, $\alpha = 0$.) Similarly the ordinary spectrum of the normal operators ($\beta = |\gamma|$) will also be a line segment in the complex plane.

(8) *As an application of the general theory [3] it is interesting to note how all operators weakly equivalent to one of these special (nonnormal irreducible) Toeplitz operators can be specified by the general theory. Thus, except for multiplicity, they are*

determinant by a σ-finite measure class on $\hat{T}$ whose support is all of $\hat{T}$. This means that $\{T\}$ must have positive measure, i.e, *must appear as a direct summand. The measure restricted to the normal spectrum $\hat{T}_1$ then gives rise to a normal operator. We have just proved the theorem: Every operator weakly equivalent to T is of the form*

$$ⓝ T \oplus N$$

where ⓝT denotes the direct sum of n copies of T $(1 \leqq n \leqq \aleph_0)$ and N is any normal operator whose spectrum is contained in the ellipse $\hat{T}_1$. (This is a generalization of a result of Coburn [1], [2] *which asserts every operator weakly equivalent to the unilateral shift S is of the form*

$$ⓝ S \oplus U$$

where U is a unitary operator. Of course unitary operators can be characterized as normal operators whose spectra are contained in the unit circle—the normal spectrum of S.)

Proof of assertion 1. A direct computation shows that $T^*T = TT^*$ if and only if $|\beta|^2 I + |\gamma|^2 SS^* = |\beta|^2 SS^* + |\gamma|^2 I$. But clearly such an equation can hold if and only if $|\gamma| = |\beta|$. (For example, apply the equation to the unit vector in l^2 with one as its first coordinate.)

Proof of assertion 2. We may assume, without loss of generality, that $\alpha = 0$. If $\beta = 0$ the result is obvious as S^* generates $C^*(S)$. By multiplying T by β^{-1} we may assume $\beta = 1$. Then $\bar{\gamma}T - T^* = (|\gamma|^2 - 1)S^* \in C^*(T)$. Since $|\gamma|^2 \neq |\beta|^2 = 1$ we have $S^* \in C^*(T)$ and thus $C^*(T) = C^*(S)$. Note that any closed subspace which reduces T also reduces $C^*(T)$ and hence S. Since S is irreducible, T must be also.

Proof of assertion 3.

(a) If $\beta = 0$ then $\gamma S^* \cong |\gamma| S^*$ by Lemma 5.7 of [3]. (Cf. [1], [2] and Theorem 1, page 15 of [4].)

(b) If $\beta \neq 0$ then there exists a unitary map U such that $U(\beta S)U^* = |\beta| S$ by Lemma 5.7 of [3]. Thus

$$U(\gamma S^*)U^* = \gamma(USU^*)^* = (\gamma |\beta| / \bar{\beta}) S^* .$$

Note: In the proofs to follow we shall use e_n to denote the unit vector of l_2 which has 1 as its nth coordinate and zero elsewhere. Part (c) of assertion 3 is proven by the following lemma and proposition.

LEMMA. *If* $S \cong \alpha I + \beta S + \gamma S^*$ *then* $\alpha = 0$, $\gamma = 0$ *and* $|\beta| = 1$.

Proof. Since S is an isometry the operator $\alpha I + \beta S + \gamma S^*$ (henceforth denoted T) is also. Applying T to the unit vectors e_1 and e_2 implies $\gamma = 0$. Further since S^*S is the identity operator it follows that T^*T is well or

$$(\alpha\bar{\alpha} + \beta\bar{\beta})I + \bar{\alpha}\beta S + \bar{\beta}\alpha S^* = I .$$

Applying this to the unit vector e_1 implies $\bar{\alpha}\beta = 0$. But $\beta = 0$ leads to the contradiction $S \cong \alpha I$. Thus $\alpha = 0$ and $\beta\bar{\beta} = 1$.

PROPOSITION. *Suppose* $\alpha_1 I + \beta_1 S + \gamma_1 S^* \cong \alpha_2 I + \beta_2 S + \gamma_2 S^*$ *and suppose further that* $\beta_1 \geqq 0$, $\beta_2 \geqq 0$ *and* $\gamma_1 \geqq 0$ *if* $\beta_1 = 0$ *and* $\gamma_2 \geqq 0$ *if* $\beta_2 = 0$, *and* $|\gamma_1| \neq \beta_1$, $|\gamma_2| \neq \beta_2$. *Then* $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$.

Proof. Consider first the case where $\gamma_1 = 0$. Then $\beta_1 S \cong (\alpha_2 - \alpha_1)I + \beta_2 S + \gamma_2 S^*$. If $\beta_1 = 0$ then the operator on the right is the zero operator. Applying it to the unit vector e_1 we easily conclude that $\alpha_1 = \alpha_2$, $\beta_2 = 0$ and $\gamma_2 = 0$. Thus assuming $\beta_1 \neq 0$ we have

$$S \cong \beta_1^{-1}(\alpha_2 - \alpha_1)I + \beta_1^{-1}\beta_2 S + \beta_1^{-1}\gamma_2 S^* .$$

By the lemma we have $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_2 = 0$. Thus we may assume without loss of generality that $\gamma_1 \neq 0$.

Similarly we may assume $\beta_1 \neq 0$. Indeed if $\beta_1 = 0$ we may reduce the situation to the case just considered by taking adjoints, i.e.,

$$\bar{\alpha}_1 I + \bar{\gamma}_1 S \cong \bar{\alpha}_2 I + \bar{\beta}_2 S^* + \bar{\gamma}_2 S .$$

We now consider the general case where $\alpha_1 I + \beta_1 S + \gamma_1 S^* \cong {}_U \alpha_2 I + \beta_2 S + \gamma_2 S^*$ where $\beta_1 > 0$ and $\gamma_1 \neq 0$. Then

$$(1) \qquad \beta_1 S + \gamma_1 S^* \cong_U (\alpha_2 - \alpha_1)I + \beta_2 S + \gamma_2 S^*.$$

Taking adjoints gives

$$(2) \qquad \beta_1 S^* + \bar{\gamma}_1 S^* \cong_U \overline{(\alpha_2 - \alpha_1)}I + \beta_2 S^* + \bar{\gamma}_2 S.$$

(Throughout this argument the subscript U is to emphasize that each of these equivalences are effected by the same unitary operator.) We next multiply equivalence (1) by $\beta_1\gamma_1^{-1}$ to obtain

$$(3) \qquad \beta_1^2\gamma^{-1}S + \beta_1 S^* \cong_U \beta_1\gamma_1^{-1}(\alpha_2 - \alpha_1)I + \beta_1\beta_2\gamma_1^{-1}S + \beta_1\gamma_1^{-1}\gamma_2 S^* .$$

Subtracting (2) from (3) gives

$$(\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1)S \cong_U [\beta_1\gamma_1^{-1}(\alpha_2 - \alpha_1) - \overline{(\alpha_2 - \alpha_1)}]I + (\beta_1\gamma_1^{-1}\gamma_2 - \beta_2)S^* + (\beta_1\beta_2\gamma_1^{-1} - \bar{\gamma}_2)S .$$

Note that $\beta_1 \neq |\gamma_1|$ implies that $\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1 \neq 0$. Hence

$$S \cong_U \frac{[\beta_1\gamma_1^{-1}(\alpha_2 - \alpha_1) - \overline{(\alpha_2 - \alpha_1)}]}{(\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1)} I + \frac{(\beta_1\gamma_1^{-1}\gamma_2 - \beta_2)}{(\beta_2^2\gamma_1^{-1} - \bar{\gamma}_1)} S^* + \frac{(\beta_1\beta_2\gamma_1^{-1} - \bar{\gamma}_2)}{(\beta_2^2\gamma_1^{-1} - \bar{\gamma}_1)} S . \tag{4}$$

It follows from our lemma that

$$\beta_1\gamma_1^{-1}(\alpha_2 - \alpha_1) - \overline{(\alpha_2 - \alpha_1)} = 0 \tag{5}$$

$$\beta_1\gamma_1^{-1}\gamma_2 - \beta_2 = 0 \tag{6}$$

$$|\beta_1\beta_2\gamma_1^{-1} - \bar{\gamma}_2| = |\beta_2^2\gamma_1^{-1} - \bar{\gamma}_1| . \tag{7}$$

From (5) note that if $\alpha_1 \neq \alpha_2$ then $\beta_1\gamma_1^{-1} = \overline{\alpha_2 - \alpha_1}/\alpha_2 - \alpha_1$ has modulus 1 or $\beta_1 = |\gamma_1|$, a contradiction to our hypotheses. Thus $\alpha_1 = \alpha_2$.

Since $\beta \neq 0$ we have from (6) that $\gamma_2 = \beta_1^{-1}\beta_2\gamma_1$. Placing this value of γ_2 in equation (7) gives

$$|\beta_1\beta_2\gamma_1^{-1} - \beta_1^{-1}\beta_2\bar{\gamma}_1| = |\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1| \tag{8}$$

or

$$(\beta_1^{-1}\beta_2)\,|\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1| = |\beta_1^2\gamma_1^{-1} - \bar{\gamma}_1| . \tag{9}$$

If $|\beta_2^2\gamma_1^{-1} - \bar{\gamma}_1| = 0$ then $\beta_2^2 = \gamma_1\bar{\gamma}_1$ contrary to the assumption that $\beta_2 \neq |\gamma_1|$. Hence $\beta_1^{-1}\beta_2 = 1$ or $\beta_1 = \beta_2$ and $\gamma_2 = \beta_1^{-1}\beta_2\gamma_1 = \gamma_1$.

Proof of assertion 4. As we have already seen, T and S generate the same C^*-algebra. The spectrum of S is known to be $\{S\} \cup \tau$ where τ denotes the unit circle in the complex plane. (Cf. Example 2.54, page 86 of [3].) Thus the irreducible representations of $C^*(T)$ consist (up to unitary equivalence) of the identity representation and the characters of the form π_λ, for $\lambda \in \tau$, where π_λ is determined by the requirement that $\pi_\lambda(S) = \lambda$. Hence

$$\hat{T} = \{T\} \cup \hat{T}_1$$

where

$$\hat{T}_1 = \{\pi_\lambda(T): \lambda \in \tau\} = \{\alpha + \beta\lambda + r\bar{\gamma}: \lambda \in \tau\} .$$

That this is the ellipse described in assertion 5 is an exercise in elementary coordinate geometry. Note that it follows from assertion 3 that $S+2S^*$, $S-2S^*$ and $2S+S^*$ are nonequivalent, yet they have the same normal spectrum, namely the ellipse

$$\left(\frac{x}{3}\right)^2 + y^2 = 1 \ .$$

Proof of assertion 6. We first must define the topology of $\hat{T}$ by the following procedure (cf. Definitions 2.52 and 2.53 of [3]). Let $\mathscr{S}$ be any subset of $\hat{T}$. We say an operator R is *weakly contained* in $\mathscr{S}$ if R is weakly contained in an operator which is the direct sum of concrete irreducible operators, one from each of the elements (i.e., unitary equivalence classes) of $\mathscr{S}$. Further an element $r \in \hat{T}$ is said to be *weakly contained* in $\mathscr{S}$, if any concrete operator R in the class r is weakly contained in $\mathscr{S}$. The *closure of* $\mathscr{S}$, denoted $\bar{\mathscr{S}}$, is defined to be the set of elements of $\hat{T}$ which are weakly contained in $\bar{\mathscr{S}}$. A nontrivial verification shows that this closure operation satisfies the Kuratowski closure axioms and hence defines a topology on $\hat{T}$.

Since the closure of $\{T\}$ is the collection of elements of $\hat{T}$ weakly contained in T it follows from the definition of $\hat{T}$ that the closure of $\{T\}$ is all of $\hat{T}$.

In general this topology on $\hat{T}_n$ (the subset of $\hat{T}$ consisting of the irreducible operators which act on a Hilbert space $\mathscr{H}_n$ of dimension n) is just the quotient topology obtained from the *-strong operator topology on the set of all irreducible operators of norm less than or equal $\|T\|$, acting on $\mathscr{H}_n$. In particular for $n=1$ this topology is just the ordinary topology of the complex plane.

We remark that these considerations do apply to other classes of Toeplitz operators. For example, for each positive integer n, the operator

$$\alpha I + \beta S^n + \gamma (S^*)^n$$

is a (reducible) Toeplitz operator with exactly the same (generalized) spectrum as the Toeplitz operator $\alpha I + \beta S + \gamma S^*$, whose spectrum has been computed above.

In conclusion we hope these elementary computations for a very special class of operators have established that, while the spectrum $\hat{T}$ introduced in [3] may be complicated, it is not completely intractable.

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Received May 12, 1978.

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