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APPROXIMATION PROPERTIES OF POLYNOMIALS WITH BOUNDED INTEGER COEFFICIENTS

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For every fixed positive integea N, let $\mathscr{P}_N$ denote the set of all polynomials $p(x)=\sum a_i x^i$ where a_i is an integer, $|a_i|\leqq N$. For a fixed real number t set $\mathscr{P}_N(t)=\{p(t): p\in\mathscr{P}_N\}$.

THEOREM 1. **Suppose $1<t<N+1$ and t is not a root of map of the polynomials from $\mathscr{P}_N$. Then $\mathscr{P}_N(t)$ is dense in $\boldsymbol{R}$.**

THEOREM 2. **If t is an S-number then $\mathscr{P}_N(t)$ is discrete for every N.**

1. For every fixed positive integer N, let $\mathscr{P}_N$ denote the set of all polynomials $p(x)$ with integer coefficients, $p=\sum a_i x^i$, such that $|a_i|\leqq N$. For a fixed real number t set

$$\mathscr{P}_N(t)=\{p(t): p\in\mathscr{P}_N\} .$$

It was shown in [1] that if $N=1$, t is a number such that $1<t<2$ and t is not a root of any of the polynomials from $\mathscr{P}_1$ then the set $\mathscr{P}_1(t)$ is dense in the real line. (It is fairly easy to see that if $t\notin(1, N+1)$, $t>0$ than $\mathscr{P}_N(t)$ cannot be dense in $\boldsymbol{R}$). At the same time it was shown that $\mathscr{P}_1(1/2(1+\sqrt{5}))$ is discrete. As far as we know this is the only known example of N and $t\in(1, N+1)$ such that $\mathscr{P}_N(t)$ is discrete. In this paper we prove two extentions of these results. The first is a straightforward generalization of [1]:

THEOREM 1. *Suppose $1<t<N+1$ and t is not a root of any of the polynomials from $\mathscr{P}_N$. Then $\mathscr{P}_N(t)$ is dense in $\boldsymbol{R}$.*

The second result is more intriguing and has a curious connection with what is known as $P-V$ numbers or S-numbers ($P-V$ numbers for Pisot-Vijayaragharan, see [2] for details).

DEFINITION. A number $t>1$ is called a $P-V$ number if it is an algebraic integer and all of its conjugates have absolute value strictly less than 1.

THEOREM 2. *If t is a $P-V$ number then $\mathscr{P}_N(t)$ is discrete for every N.*

It follows, for instance, that $\mathscr{P}_N(1/2(1+\sqrt{5}))$ is discrete for all N, not just $N=1$.

Let $||s||$ denote the distance from s to the nearest integer. A number θ is said to have property (P) if for some $\lambda > 1$, $||\lambda\theta^n|| \to 0$. It is known that every $P - V$ number has property (P). A conjecture is raised in [2] as to whether the converse is true: Is every number with property (P) a $P - V$ number? It is known that every algebraic number with property (P) is a $P - V$ number. Thus the conjecture would be settled if one could show that for every number t having property (P), the set $\mathscr{P}_N(t)$ is discrete.

The proof of Theorem 1 is essentially no different from the proof given in [1] for $N = 1$. We proceed with the proof of Theorem 2 now.

LEMMA 1. *Suppose $t > 1$ and 0 is an accumulation point of $\mathscr{P}_N(t)$. Let k, m be any positive integers. There exists polynomial p of the form $p(x) = x^{m_1}f(x)$, $f \in \mathscr{P}_N$, $m_1 > m$ such that*

$$t^{-k-1} \leqq p(t) < t^{-k} .$$

Proof. Let $r(x)$ be a polynomial in $\mathscr{P}_N$ such that

$$0 < r(t) < t^{-k-m_1} .$$

Let m_1 be the smallest integer such that

$$t^{-k-m_1-1} < r(t) .$$

Then $m_1 > m$ and $r(t) < t^{-k-m_1}$. Thus

$$t^{-k-1} < t^{m_1}r(t) \leqq t^{-k} .$$

LEMMA 2. *Suppose $t > 1$ and 0 is an accumulation point of $\mathscr{P}_N(t)$. Then $\mathscr{P}_N(t)$ is dense in $\boldsymbol{R}$.*

Proof. Let $u > 0$ and $\eta > 0$ be fixed. Let k be so large that the interval $[t^{-k-1}, t^{-k}]$ has length less than η. There is a sequence of polynomials $p_1, p_2, \cdots$, having no common terms $\alpha_j x^j$ such that

$$t^{-k-1} < p_n(t) \leqq t^{-k} .$$

This follows by applying Lemma 1 with fixed k and making m_1 larger and larger. If

$$q_m(t) = p_1(t) + \cdots + p_m(t)$$

then $q_m(t) > mt^{-k}$, so $q_m(t) \to \infty$. Hence for some m, $q_m(t)$ will be inside the interval $[u - \eta \quad u - \eta]$. Since u and η were arbitrary, the result follows.

Proof of Theorem 2. It is enough to show that $\mathscr{P}_N(t)$ is not dense for any $N = 1, 2, \cdots$. Indeed, suppose this is done and assume that $\mathscr{P}_{N_0}(t)$ is not discrete. Then clearly $\mathscr{P}_{2N_0}(t)$ has 0 as an accumulation point and by Lemma 2 is dense. To show $\mathscr{P}_N(t)$ is not dense for any N we argue as follows. Let

$$t = t_1, t_2, \cdots, t_p$$

be all the roots of the irreducible monic polynomial of t, and let

$$\sigma = \max\{|t_2|, |t_3|, \cdots, |t_p|\}$$

so that $0 < \sigma < 1$. For any k

$$t^k + t_2^k + \cdots + t_p^k$$

is an integer, hence

$$|t^k\text{-integer}| \leqq |t_2|^k + \cdots + |t_p|^k \leqq (p-1)\sigma^k .$$

Let $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_j x^j$ be a polynomial in $\mathscr{P}_N$. Then

$$t^k p(t) = \sum_{n=0}^{j} \alpha_n t^{k+n}$$

so

$$\begin{aligned} |t^k p(t) - \text{integer}| &\leqq \sum_{n=0}^{j} |\alpha_n| (p-1)\sigma^{k+n} \\ &\leqq N(p-1)\frac{\sigma^k}{1-\sigma} . \end{aligned}$$

Choose k so large that the right hand side is less than 1/3. Then

$$\left| p(t) - \frac{\text{integer}}{t^k} \right| < \frac{1}{3}\frac{1}{t^k}$$

or, if the integer is odd

$$\left| p(t) - 1/2\frac{\text{integer}}{t^k} \right| \geqq \frac{1}{6}\frac{1}{t^k}$$

for any $p \in \mathscr{P}_N$.

References

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