ON THE EXISTENCE OF CAPILLARY FREE SURFACES IN THE ABSENCE OF GRAVITY

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If we were to put water into a glass cylindrical container of circular cross section and transport it to outer space, the surface of the water would be uniquely determined as a lower hemisphere. Concus and Finn [1] have shown that if the circular cross section is replaced by a square section then no such surface can exist as a graph of a function. The question then arises, for what kind of cross section can we expect the existence of a surface in the form of a graph?

Mathematically, this question can be formulated as follows: Let Ω be the cross section of the cylinder and u be a capillary free surface defined over Ω. By the least action principle of physics, u would minimize the energy functional

$$(1.1) \qquad E[u] = \sigma \iint_\Omega \sqrt{1 + |\nabla u|^2}\, dxdy - \sigma\lambda \int_\Sigma u\, ds$$

subject to the volume constraint

$$(1.2) \qquad \iint_\Omega u\, dxdy = \text{constant}$$

where Σ is the boundary of Ω, ds is the arc length measure on Σ and ∇u is the gradient of u. The physical interpretation of (1.1) is as follows: The first term gives the potential energy in the free surfaces; the constant σ is referred to as the surface tension. The second term gives the wetting energy due to the boundary adhesion. The dimensionless constant λ satisfies $|\lambda| \leqq 1$ and depends on the material of the wall and the fluid. For glass and water, λ is close to 1, and for glass and mercury, λ is negative. We mention that the case $\lambda > 1$ corresponds physically to a situation in which adhesion dominate, so that the fluid would spread out along the walls and no equilibrium surface would exist. This phenomenon is observed, e.g., with liquid helium and glass.

The equilibrium condition $\delta E[u] = 0$ under the constraint (1.2) is expressed by the Euler equations:

$$(1.3) \qquad \text{div } Tu = H \quad \text{in} \quad \Omega$$

$$(1.4) \qquad Tu \cdot \nu = \lambda \quad \text{on} \quad \Sigma$$

where $Tu\langle u_x/\sqrt{1 + |\nabla u|^2},\, u_y/\sqrt{1 + |\nabla u|^2}\rangle$, H is a constant which is

twice the mean curvature of the capillary free surface, and ν is the unit outer normal of Σ.

If we introduce the angle γ by letting $\gamma = \cos \gamma$, then (1.3) and (1.4) imply that u is a surface of constant mean curvature which makes constant contact angle γ with the wall; the angle γ is measured inside the fluid, as shown in Figure 1.

It should be noted here that the constant H connot be prescribed; it is implicity determined by the geometry of Ω and the contact angle γ, as follows by applying the divergence theorem to (1.3) and using (1.4):

$$(1.5) \qquad\qquad H = \frac{\Sigma}{\Omega} \cos \gamma \ .$$

We use here the symbols Ω and Σ to denote both a set and its measure.   We may assume $0 \leqq \gamma \leqq \pi/2$; for further physical and geometrical background information, see [8].

The following necessary condition for the existence of solution of (1.3) and (1.4) was first observed by Concus and Finn [3]:

Let Ω^* be any subdomain in Ω bounded by a simple curve Γ and a subboundary Σ^* of Σ, as shown in Figure 2.   Integrate (1.3) over Ω^*, using (1.4) and the divergence theorem we have

$$\begin{aligned}
H\Omega^* &= \iint_{\Omega^*} \operatorname{div} Tu \, dxdy \\
&= \int_{\Sigma^*+\Gamma} Tu \cdot \nu \, ds \\
&= \int_{\Sigma^*} Tu \cdot \nu + \int_{\Gamma} Tu \cdot \nu \, ds \\
&= \Sigma^* \cos \Gamma + \int_{\Gamma} Tu \cdot \nu \, ds
\end{aligned}$$

substituing (1.5) and using the fact $|Tu| \leqq 1$, we obtain:

$$-\Gamma \leqq \frac{\Omega^*}{\Omega} \Sigma \cos \Gamma - \Sigma^* \cos \Gamma \leqq \Gamma$$

or equivalently

$$(1.6) \qquad\qquad \left| \frac{\Sigma^*}{\Sigma} - \frac{\Omega^*}{\Omega} \right| \cos \gamma \leqq \frac{\Gamma}{\Sigma}$$

for any Γ.

For a solution of (1.3) and (1.4) to exist, it is necessary that (1.5) hold for every choice of Γ.   As was shown in [3], an arbitrary Σ can be changed by an arbitrary small deformation into a boundary for which (1.6) will fail for some Γ.

Condition (1.6) was later shown by Giusti [13] to be not only necessary, but (almost) sufficient. For most purpose, Giusti required

$$(1.7) \qquad \left| \frac{\Sigma^*}{\Sigma} - \frac{\Omega^*}{\Omega} \right| \cos \gamma < (1 - \varepsilon) \frac{\Gamma}{\Sigma}$$

for some $\varepsilon > 0$ and any Γ.

Obviously (1.7) is implied by

$$(1.8) \qquad \left| \frac{\Sigma}{\Sigma} - \frac{\Omega}{\Omega} \right| \leqq \frac{\Gamma}{\Sigma}$$

for every contact angle $0 < \gamma \leqq \pi/2$. Thus if we could characterize those domains for which (1.8) holds, then (1.3) and (1.4) will have solution on them for any prescribed contact angle $\gamma > 0$.

The difficulty of using (1.7) and (1.8) is that one must verify them for every possible Γ. It is shown in [15] that if Ω is convex, then (1.8) follows from the following curvature condition on Σ:

$$K(p) \leqq \frac{\Sigma}{\Omega}$$

for every $p \in \Sigma$, where $K(p)$ denotes the curvature of Σ at p. For nonconvex domains there appears to be no analogous criterion, c.f. the examples in Finn and Giusti [11].

Instead of thinking about the curvature conditions on Σ, Finn [9] gave the following vector field condition on Ω which is equivalent to (1.7):

THEOREM (*Finn*). *The condition* (1.7) *holds for arbitrary* $\Gamma \subset \Omega$ *whenever there is a vector field* $W(x, y)$ *in* Ω *satisfying*

$$\operatorname{div} W = \frac{\Sigma}{\Omega} \quad in \quad \Omega$$

$$w \cdot \nu = 1 \quad on \quad \Sigma$$

$$(1.9) \qquad \operatorname*{Sup}_{\Omega} |w| < \frac{1}{\cos \gamma} .$$

By using this theorem, Finn proved the existence of solution over triangles, tetrahedrons, parallelograms and some special polygons by constructing the vector field directly. He required $\alpha + \gamma \geqq \pi/2$, where 2α denotes the smallest angle of the vertices of the domain (In case $\alpha + \gamma < \pi/2$, Concus and Finn [1] have proved that no solutions exist.) For general domains a difficult point in connection with the use of Finn's theorem.

In this paper, we shall use variational method to reduce the class

of curves on which we need to test Concus-Finn-Giusti's condition
(1.7) to some special curves Γ, which have property that if (1.7)
holds for these special curves Γ, then it holds for all possible curves.
This enables us to state following existence theorem depending on
the geometric properties of Ω:

THEOREM 4.1. *If a circle of radius* $R = \Omega/\Sigma$ *can be rotated along*
Σ *in the interior of* Ω *such that no antipodes of the circle lie on* Σ,
then (1.8) *holds and hence a solution of* (1.3), (1.4) *exists in* Ω *for*
all $\gamma > 0$.

Actually, we prove that (1.8) holds strictly for every $\Omega^* \neq \varnothing$, Ω;
by a theorem of Giusti (Theorem 1, [15]), this implies that a solution
exists for $\gamma = 0$.

The sufficient condition of Theorem 4.1 is also necessary for exist-
ence of the solution for domains without a "neck" of radius $R = \Omega/\Sigma$.
(Here what we mean by a neck domain of radius R is a domain which
has two large components with narrow connection through which one
cannot pass through a circle of radius R.)   If such a "neck" appears,
the necessity can fail.   For example, a domain which is the union of
two disks of equal radius will fail to satisfy the condition of Theorem
4.1 if the aperture is sufficiently small; however, as we shall show
in § 5, the solution nevertheless exists.

2. **Isoperimetric inequalities.** The purpose of this section is
to provide some basic isoperimetric inequalities which will be used
to prove theorems in the later sections.

A well known form of the classical isoperimetric inequality can
be stated as follows:

Among all plane curves of prescribed arc length which pass
through the two given P, Q; the circular arc bounds the maximum
area between it and the line segment through P and Q.

This result still holds if the line segment is replaced by a plane
curve which does not have intersections with the circular arc of
prescribed arc length.

The following lemma is a generalization of the classical isoperi-
metric inequality in a bounded domain.

LEMMA 2.1. *Let* Ω *be a plane domain with piecewise smooth*
boundary Σ, *let* P *and* Q *be two points on* Σ *which divide* Σ *into*
two parts Σ_1, Σ_2 *and let* Γ *be a curve in* Ω *joining* P *and* Q *with*
prescribed arc length less then Σ_1 *and bounds with* Σ_2 *an area* $\Delta(\Gamma)$.
Then $\Delta(\Gamma)$ *achieves its maximum in one of the following cases;*

(i) *Γ is a circular* arc *if one can draw a circular* arc *in Ω with the prescribed arc length*

(ii) *Γ is a curve which coincides with some parts of Σ and circular arcs of equal radii. Furthermore, if the end points of the circular arcs are smooth points of Σ, then the circular arcs are tangent to Σ there except possibly for the initial points P and Q.* (Figure 3 a, b, c.)

Proof. (Uniqueness). If one can draw a circular arc of the prescribed arc length in Ω, $Δ(Γ)$ achieve its maximum by the classical isoperimetric inequality. Suppose that is not the case. Since fixing any two points of $Γ$ and deforming the subarc of $Γ$ between these two points into a circular arc always increases the enclosed area, $Γ$ will consist of some pieces of $Σ$ and some circular arc in $Ω$. Let C_i, C_j be two circular arc with radii r_i, r_j respectively. If $r_i ≠ r_j$ we can cut C_i and C_j a little bit by two chords l_i and l_j respectively with $l_i = l_j$ (Figure 4). Let S_i and S_j be the small regions cut out from C_i and C_j respectively. Moving S_i to l_j and S_j to l_i, one obtains a new curve $Γ'$ which bounds a region with the same area as $Γ$, but now the subarcs of $Γ'$ are not circular. Hence we can deform it into a circular arc and increase the area bounded. This proves that $r_i = r_j$.

Next, we want to prove that any circular C_i is tangent to $Σ$ if the end points of C_i are smooth points of $Σ$. Let E be an end point of C_i which is a smooth point of $Σ$ and an interior point of $Γ$. Let A and B be two points on $Γ$ in a neighborhood of E such that $A ∈ Σ$, $B ∈ C_i$, and let A be the origin of the coordinate system with AB as x-axis (Figure 5). By taking a small neighborhood we can assume that $Σ$ is expressed as $y = g(x)$ in a neighborhood $Σ_{AF}$. Now let l be the arc length of $Γ_{AB}$. We will use a variational method to show that $Γ_{AB}$ should be tangent to $Σ$ at E.

Let $(b, 0)$ and $(x_0, g(x_0))$ be the coordinates of B and E respectively, and $y = f(x)$ be the equation of $Γ_{EB}$ with respect to this coordinate system, then

(2.1)
$$l = \int_0^{x_0} \sqrt{1 + g'^2}(x)\, dx + \int_{x_0}^b \sqrt{1 + f'^2}(x)\, dx$$

(2.2)
$$Δ = \text{area bounded by } Γ_{AB} \text{ and the } x\text{-axis}$$
$$= \int_0^{x_0} g(x)\, dx + \int_{x_0}^b f(x)\, dx \ .$$

We are going to maximize (2.2) under the constraint (2.1).

Let $η$ be a continuous differentiable function defined on $[0, b]$ with $η(b) = 0$, and let

$$I(\varepsilon) = \int_0^{x_0} [g(x) + \varepsilon\eta(x)] \, dx + \int_{x_0}^b [f(x) + \varepsilon\eta(x)] \, dx$$

$$+ \lambda \int_0^{x_0} \sqrt{1 + [g'(x) + \varepsilon\eta'(x)]^2} \, dx + \lambda \int_{x_0}^b \sqrt{1 + [f'(x) + \varepsilon\eta'(x)]^2} \, dx$$

then

$$(2.3) \qquad \frac{dI}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^{x_0} \eta(x) \, dx + \int_{x_0}^b \eta(x) \, dx + \lambda \int_0^{x_0} \frac{g'(x)\eta'(x)}{\sqrt{1 + g'^2(x)}} \, dx$$

$$+ \lambda \int_{x_0}^b \frac{f'(x)\eta'(x)}{\sqrt{1 + f'(x)^2}} \, dx$$

$$= \int_0^b \eta(x) \, dx - \lambda \int_0^{x_0} \frac{d}{dx}\left(\frac{g'(x)}{\sqrt{1 + g'^2(x)}}\right) \eta(x) \, dx$$

$$- \lambda \int_{x_0}^b \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1 + f'^2(x)}}\right) \eta(x) \, dx$$

$$+ \lambda\eta(x) \frac{g'(x)}{\sqrt{1 + g'^2(x)}}\bigg|_{x=0}^{x=x_0} + \lambda\eta(x) \frac{f'(x)}{\sqrt{1 + f'^2(x)}}\bigg|_{x=x_0}^{x=b}$$

the extreme curve has the property that $(dI/d\varepsilon)|_{\varepsilon=0} = 0$ and since $\eta(x_0)$ is arbitrary. (2.3) implies

$$\frac{g'(x_0)}{\sqrt{1 + g'(x_0)^2}} - \frac{f'(x_0)}{\sqrt{1 + f'(x_0)^2}} = 0$$

which is equivalent to

$$f'(x_0) = g'(x_0) \, .$$

This proves that Γ is tangent to Σ at E.

The above arguments show that if the extreme curve does exist then it is a curve of (i) or (ii) as described in the lemma. (We do not prove there is a unique extreme curve, in general, this may not be true.) We need to prove existence also.

[Existence]: Let a be the prescribed perimeter and let

$$S = \{\varDelta(\Gamma) \,|\, \Gamma \text{ joint } PQ \text{ in } \Omega, \, \Gamma = a\}$$
$$T = \{\varDelta(\Gamma) \,|\, \Gamma \text{ joint } PQ \text{ in } \Omega, \, \Gamma = a \text{ and } \Gamma \in C^\infty\}$$

then T is dense in S, hence

$$\sup S = \sup T \, .$$

Let $\varDelta_\infty = \sup T$, since T is bounded above by Ω, $\varDelta_\infty$ is finite and there exists a sequence $\{\varDelta(\Gamma_n)\}$ in T such that $\varDelta(\Gamma_n) \uparrow \varDelta_\infty$. If we can show that $\varDelta_\infty \in S$, i.e., $\varDelta_\infty = \varDelta(\Gamma_\infty)$ for some curve Γ_∞ join P, Q in Ω and $\Gamma_\infty = a$ then we are done.

To this purpose, let each Γ_n be perimetrized by arc length:

$$\Gamma_n = \{\vec{X}_n(s) = (x_n(s), y(s)) \,|\, 0 \leqq s \leqq a\}$$

since $x_n'^2(s) + y_n'^2(s) = 1$. $\{x_n\}$ and $\{y_n\}$ are bounded families of equicontinuous functions, hence by Ascoli's theorem, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that

$$x_{n_k} \longrightarrow x_\infty$$
$$y_{n_k} = y_\infty$$

uniformly on $[0, a]$ for some x_∞ and $y_\infty \in C[0, a]$. Let $\vec{X}_\infty(s) = (x_\infty(s), y_\infty(s))$ and let $\Gamma_\infty = \vec{X}_\infty(s)$. Then for any regular partition $0 = s_0 < s_1 < \cdots < s_N = a$ of $[0, a]$, we have

$$(2.4) \quad \sum_{i=1}^{N} |\vec{X}_\infty(s_i) - \vec{X}_\infty(s_{i-1})|$$

$$\leqq \sum_{i=1}^{N} |\vec{X}_\infty(s_i) - \vec{X}_{n_k}(s_i)| + \sum_{i=1}^{N} |\vec{X}_{n_k}(s_{i-1}) - \vec{X}_\infty(s_{i-1})|$$

$$+ \sum_{i=1}^{N} |\vec{X}_{n_k}(s_i) - \vec{X}_{n_k}(s_{i-1})|$$

$$\leqq \sum_{i=1}^{N} |\vec{X}_\infty(s_i) - \vec{X}_{n_k}(s_i)| + \sum_{i=1}^{N} |\vec{X}_{n_k}(s_{i-1}) - \vec{X}_\infty(s_{i-1})| + a \, .$$

Given $\varepsilon > 0$, we may choose n_k large enough such that

$$|\vec{X}_\infty(s) - \vec{X}_{n_k}(s)| < \frac{\varepsilon}{2N}$$

then (2.4) implies

$$\sum_{i=1}^{N} |\vec{X}_\infty(s) - \vec{X}_\infty(s_{i-1})| < \varepsilon + a \, .$$

Hence Γ_∞ is rectifiable and $\Gamma_\infty \leqq a$. On the other hand, $\Delta(\Gamma_\infty) \geqq \Delta(\Gamma)$ for all $\Delta(\Gamma) \in S$. This implies $\Gamma_\infty \geqq a$ and hence $\Gamma_\infty = a$.

The following lemma is the main lemma which will be used to prove theorems in the following sections.

LEMMA 2.2. *Let P and Q be two points in the plane, Γ a simple plane curve joining P and Q which encloses with the line segment $\overline{PQ}$ an area $\Delta(\Gamma)$, and λ be a constant such that $1/\lambda \geqq \overline{PQ}/2$. Then the functional*

$$\varphi(\Gamma) = \lambda \Delta(\Gamma) - \Gamma$$

achieves its maximum (local) at the smaller circular arc radius $1/\lambda$.

Note. There are two circular arcs of radius $1/\lambda$ passing through P, Q (Figure 6). Call the smaller one Γ_1 and the bigger one Γ_2. As we shall see from the following remarks, Γ_2 is a stationary point, not an extreme point of φ.

Proof. Let $\Gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$ be a parametrization of Γ with $\Gamma(0) = P$, $\Gamma(1) = Q$ and $x'^2(t) + y'^2(t) \neq 0$. Then

$$\varphi(\Gamma) = \frac{\lambda}{2} \int_{\partial A} x\,dy - y\,dx - \int_0^1 \sqrt{x'^2(t) + y'^2(t)}\,dt$$

$$= \frac{\lambda}{2} \int_0^1 (xy' - yx')\,dt - \int_0^1 \sqrt{x'^2 + y'^2}\,dt$$

$$= \int_0^1 F(t, x, y, x', y')\,dt$$

where

(2.5) $$F(t, x, y, x', y') = \frac{\lambda}{2}(xy' - yx') - \sqrt{x'^2 + y'^2}\,.$$

By the calculus of variations, the extreme curve must satisfy the Euler equations:

(2.6) $$F_x - \frac{d}{dt} F_{x'} = 0$$

and

(2.7) $$F_y - \frac{d}{dt} F_{y'} = 0\,.$$

Computing directly from (2.5) we get

(2.8) $$F_x = \frac{\lambda}{2} y'\,, \qquad F_{x'} = -\frac{\lambda}{2} y - \frac{x'}{\sqrt{x'^2 + y'^2}}$$

and

(2.9) $$F_y = \frac{\lambda}{2} x'\,, \qquad F_{y'} = \frac{\lambda}{2} y - \frac{y'}{\sqrt{x'^2 + y'^2}}\,.$$

Substituing (2.8) into (2.6) we have

$$\lambda y' + \frac{d}{dt} \frac{x'}{\sqrt{x'^2 + y'^2}} = 0$$

which is equivalent to

(2.10) $$\left[\frac{x''y' - x'y''}{(x'^2 + y'^2)^{3/2}} - \lambda \right] y' = 0\,.$$

Similarly, by substituting (2.9) into (2.7) we have

(2.11)
$$\left[\frac{x''y' - x'y''}{(x'^2 + y'^2)^{3/2}} - \lambda\right]x' = 0 .$$

Since $x'^2 + y'^2 \neq 0$, (2.10) and (2.11) together imply

(2.12)
$$\frac{x''y' - x'y''}{(x'^2 + y'^2)^{3/2}} = \lambda .$$

The left hand side of (2.12) represents the curvature of Γ. Thus Γ is a curve of constant curvature λ and therefore a circular arc of radius $1/\lambda$. There are two such circular arcs (Figure 7), say Γ_1 (the smaller one) and Γ_2 (the bigger one). To show that φ achieves its maximum on Γ_1, one can write down the second variation and show it is negative at Γ_1, however it is a tedious task. The following method seems much easier.

Fixing the arc length Γ, the classical isoperimetric inequality implies that $\Delta(\Gamma)$, and hence $\varphi(\Gamma) = \lambda \Delta(\Gamma) - \Gamma$, will achieve its maximum when Γ is a circular arc. Thus to maximize $\varphi(\Gamma)$, it suffices to consider the value of φ on the family of circular arcs passing through P and Q. Now let Γ be a circular arc in a neighborhood of Γ_1 with radius r and center 0. Let $\theta = (1/2)\angle POQ$ and let $a = (1/2)\overline{PQ}$ (Figure 7). Then we have

$$\Gamma = 2r\theta$$
$$\Delta = r^2\theta - ar\cos\theta$$
$$\theta = \sin^{-1}\frac{a}{r}$$

and

(2.13)
$$\varphi = \varphi(r) = \lambda\Delta(r) - \Gamma(r)$$
$$= (\lambda r^2 - 2r)\sin\frac{a}{r} - \lambda a\sqrt{r^2 - a^2}$$

(2.14)
$$\varphi'(r) = 2(\lambda r - 1)\left(\sin^{-1}\frac{a}{r} - \frac{a}{\sqrt{r^2 - a^2}}\right)$$

(2.15)
$$\varphi''(r) = 2\lambda\left(\sin^{-1}\frac{a}{r} - \frac{a}{\sqrt{r^2 - a^2}}\right)$$
$$+ 2(\lambda r - 1)\left[\frac{-1}{\sqrt{r^2 - a^2}}\frac{a}{r} + \frac{ar}{(r^2 - a^2)^{3/2}}\right]$$

(2.16)
$$\varphi''(\Gamma_1) = \varphi''\left(\frac{1}{\lambda}\right)$$
$$= 2\lambda\left(\sin^{-1}a\lambda - \frac{a\lambda}{\sqrt{1 - (a\lambda)^2}}\right) .$$

By assumption $1/\lambda > (1/2)\overline{PQ} = a$, so $0 < a\lambda < 1$. If we introduce the angle $\alpha = \sin^{-1} a\lambda$, then $0 < \alpha < \pi/2$ and $a\lambda/\sqrt{1 - (a\lambda)^2} = \tan\alpha$. (2.16) becomes

$$\varphi''(\Gamma_1) = 2\lambda(\alpha - \tan\alpha) < 0 .$$

This proves that $\varphi(\Gamma_1)$ is a local maximum.

REMARK.
(i) Equation (2.14) can be written as

$$(2.17) \qquad \varphi'(r) = 2(\lambda r - 1)(\theta - \tan\theta) , \qquad 0 \leqq \theta \leqq \frac{\pi}{2} .$$

Thus $\varphi'(r) = 0$ if and only if $r = 1/\lambda$ or $\theta = \tan\theta$; the second case can happen only if $\theta = 0$ and hence Γ would be the line segment $\overline{PQ}$, which is not a candidate for an extremal, as we have seen from the first part of the proof. Hence we obtain again that the circular arcs of radius $1/\lambda$ are extremal curves of φ.

(ii) $\varphi'(r) > 0$ if $a < r < 1/\lambda$. This means that φ is strictly increasing as r varies from a to $1/\lambda$ or geometrically, $\varphi(\Gamma_1) > \varphi(\Gamma)$ for all smaller circular arcs Γ lying "above" Γ_1 (Figure 8).

(iii) $\varphi'(r) < 0$ if $r > 1/\lambda$. This means that φ is strictly decreasing as r increases from $1/\lambda$ to ∞, or geometrically, $\varphi(\Gamma_1) > \varphi(\Gamma^*)$ for all circular arcs lying between Γ_1 and $\overline{PQ}$ ($r = \infty$) (Figure 8).

(iv) Γ_2 is a stationary curve but not an extreme curve.

For let C be any bigger circular arc which lies "inside of" Γ_2, that is r(radius of C) $< 1/\lambda$; and let P' and Q' be any two points on C such that the subarc $C_{P'Q'}$ of C between P' and Q' is a smaller circular arc (Figure 9). Now let $C^*_{P'Q'}$ be the small circular arc of radius $1/\lambda$ passing through P' and Q'. Applying our lemma to $C_{P'Q'}$ and $C^*_{P'Q'}$ we have:

$$(2.18) \qquad \lambda \varDelta(C_{P'Q'}) - C_{P'Q'} < \lambda \varDelta(C^*_{P'Q'}) - C^*_{P'Q'} .$$

Let S be the shaded area as shown in Figure 9. Adding the quantiy $\lambda S - C_{PP'} - C_{Q'Q}$ to both sides of (2.18) we have

$$\varphi(C) = \lambda \varDelta(C) - C \leqq \lambda \varDelta(C^*) - C^* = \varphi(C^*)$$

where $C^* = C_{PP'} + C^*_{P'Q'} + C_{Q'Q}$.

Now if we let C_1 be the circular arc through P and Q which has the same length as C^*, then by the classical isoperimetric inequality we have

$$(2.19) \qquad \varphi(C) < \varphi(C^*) < \varphi(C_1) .$$

By continuing this process, one will increase φ until C_1 shrink to Γ_1.

On the other hand, since C can be chosen to be very close to Γ_2 and C_1 can be made very close to C (by taking P' close enough to Q'), we have

$$\varphi(\Gamma_2) < \varphi(C_1)$$

by the continuity of φ.

If C is a bigger circular arc lying "outside of" Γ_2, by using the same argument as above, we will obtain the same result as (2.19). However this time C_1 is not a shrunken circular arc of C, but an enlarged circular arc of C. As this process continues, C_1 will grow bigger and bigger and φ will go to ∞, since $\lambda \Delta - \Gamma \sim \lambda 0(r^2) - 0(r) \to \infty$, as $r \to \infty$.

The above arguments show that $\varphi(\Gamma_2)$ is a minimum among the family of circular arcs. Obviously, it is not a minimum among all curves, since $\varphi(\Gamma) < \varphi(\Gamma_2)$ for noncircular curves Γ which have the same arc length as Γ_2.

The following lemma is an immediate result of Lemma 2.2.

LEMMA 2.3. *Let Λ be a plane curve joining P and Q, which does not meet the smaller circular arc C of radius $1/\lambda$. $\lambda > (1/2)\overline{PQ}$, and let Γ be any simple plane curve joining P and Q which encloses an area $\Delta(\Gamma)$ with Λ. Then the functional*

$$\varphi(\Gamma) = \lambda \Delta(\Gamma) - \Gamma$$

achieves its maximum on C.

3. Some results of Concus, Finn and Giusti. The following theorem of maximal principle is proved by Concus and Finn [3].

THEOREM 3.1. (Maximal principle). *Let Ω be a plane domain with piecewise smooth boundary Σ, and let u, v be two functions in $C^2(\Omega)$. Suppose that*

$$\operatorname{div} Tu \leqq \operatorname{div} Tv \quad in \quad \Omega$$
$$Tu \cdot \nu \geqq Tv \cdot \nu \quad on \quad \Sigma - \Sigma_0$$

where Σ_0 denotes the set of points in Σ at which the normal is not defined. Then $u \equiv v + constant$ in Ω.

Let Ω be as above and consider the following capillary free surface equations

$$(3.1) \qquad \operatorname{div} Tu = H \quad in \quad \Omega, \qquad Tu \cdot \nu = \cos \gamma \quad on \quad \Sigma - \Sigma_0,$$

where H and γ are constants, and ν denotes the outer normal of Σ. Theorem 3.1 implies that the solution of (3.1) is unique up to an additive of constant if it exists.

The following theorems for existence is proved by Giusti [13] and [15].

THEOREM 3.2. *Suppose that there exists an* $\varepsilon > 0$ *such that*

$$(3.2) \qquad \left| \frac{\Sigma^*}{\Sigma} - \frac{\Omega^*}{\Omega} \right| \cos \gamma < (1 - \varepsilon) \frac{\Gamma}{\Sigma}$$

for all curves Γ *in* Ω (Figure 2). *Then* (3.1) *has a solution.*

THEOREM 3.3. *Let* Ω *be a convex domain in the plane and let* $k(p)$ *be the curvature of* Σ *at* p. *Suppose that*

$$k(p) \leqq \frac{\Omega}{\Sigma}$$

for all $p \in \Sigma$, *then* (3.1) *has a solution for all* γ.

THEOREM 3.4. *Suppose that there holds*

$$(3.3) \qquad \left| \frac{\Sigma^*}{\Sigma} - \frac{\Omega^*}{\Omega} \right| < \frac{\Gamma}{\Sigma}$$

for all $\Omega^* \neq \varnothing$ *and* Ω, *then* (3.1) *has a solution for* $\gamma = 0$.

4. Geometric condition for the existence of solutions. In this section we give a geometric condition on Ω, which guarantees the existence of solutions of (3.1) for all contact angles γ.

DEFINITION 4.1. A plane domain Ω with piecewise smooth boundary has the interior rolling disk property of radius R if:

(i) for any regular point P on Σ, one can contact a circle of radius R inside Ω so that no antipodes of the circle lie on Σ.

(ii) for irregular points P *on* Σ, one can contact a circle of radius R inside Ω in any direction between $\nu(P+)$ and $\nu(P-)$ so that no antipodes of the circle lie on Σ, where $\nu(P+)$ and $\nu(P-)$ denotes the right hand and left hand normals of Σ at P.

Roughly speaking, a plane domain Ω has the interior rolling disk property of radius R if one can rotate a disk of radius R along every point of Σ so that no antipodes of the disk lie on Σ. Thus a triangle does not have the interior rolling disk property for any R. A domain with a narrow aperture fails to have the property if R is bigger

than half of the aperture.

We now state our main results.

THEOREM 4.1. *If Ω has the interior rolling disk property of radius $R = \Omega/\Sigma$, then (3.1) has a solution for any contact angle γ.*

THEOREM 4.2. *Let $\Gamma_1, \Gamma_2, \cdots$, be all of the circular arcs of radius $R = \Omega/\Sigma$ in Ω with end points on Σ which are tangent to Σ at regular points of Σ and contact to Σ at singular points of Σ. If there holds that*

$$(4.2) \qquad \left| \frac{\Sigma_i^*}{\Sigma} - \frac{\Omega_i^*}{\Omega} \right| \leqq \frac{\Gamma_i}{\Sigma}$$

for all $i = 1, 2, \cdots$, then (3.1) has a solution for all contact angle γ.

The proof of the above theorems is based on the following idea. Suppose we can show that

$$(4.3) \qquad \left| \frac{\Sigma^*}{\Sigma} - \frac{\Omega^*}{\Omega} \right| \leqq \frac{\Gamma}{\Sigma}$$

for all possible curve Γ in Ω, then for any γ with $0 < \gamma \leqq \pi/2$, Concus-Finn-Giusti's condition (3.2) holds by taking $\varepsilon = 1 - \cos(\gamma/2)$ and hence a solution exists.

However (4.3) is equivalent to

$$(4.4) \qquad -\Gamma + \frac{1}{R}\Omega^* \leqq \Sigma^* \leqq \Gamma + \frac{1}{R}\Omega^*$$

where $R = \Omega/\Sigma$.

Let Σ^* be fixed and let P, Q be its end points on Σ, consider the functionals

$$f(\Gamma) = \Gamma + \frac{1}{R}\Omega^*(\Gamma)$$

and

$$g(\Gamma) = -\Gamma + \frac{1}{R}\Omega^*(\Gamma)$$

defined on all curves Γ in Ω which join P and Q. If we can show that

$$(4.5) \qquad \operatorname*{Max}_{\Gamma} g(\Gamma) \leqq \Sigma^* \leqq \operatorname*{Min}_{\Gamma} f(\Gamma)$$

for all $P, Q \in \Sigma$, then we are done.

On the other hand, let $\Omega' = \Omega - \Omega^*$, $\Sigma' = \Sigma - \Sigma^*$ then

$$f(\Gamma) = \Gamma + \frac{1}{R}(\Omega - \Omega')$$

$$= \frac{1}{R}\Omega - \left(-\Gamma + \frac{1}{R}\Omega'\right)$$

$$= \Sigma - \left(-\Gamma + \frac{1}{R}\Omega'\right)$$

thus minimizing $f(\Gamma)$ is equivalent to maximizing $-\Gamma + (1/R)\Omega'$, which is just maximizing $g(\Gamma)$ with Σ^* replaced by Σ' and Ω^* replaced by Ω'. Therefore the only thing we need to do is to maximize g.

We need only consider the case $\Gamma \leqq \Sigma'$, for if $\Gamma > \Sigma'$

$$g(\Gamma) = -\Gamma + \frac{1}{R}\Omega^* < -\Sigma' + \frac{1}{R}\Omega = g(\Sigma') \ .$$

Now let Γ be any curve with $d(P, Q) \leqq \Gamma \leqq \Sigma'$ where $d(P, Q)$ is the distance between P and Q. By Lemma 2.1, among all curves of arc length Γ, Ω^* and thus $g(\Gamma)$ would achieve its maximum in one of the following cases:

(i) a circular arc if one can draw a circular arc in Ω with arc length Γ (3.8).

(ii) a curve which coincides with some subarcs of Σ and circular arcs of equal radii with the property that if the end points of the circular arcs are smooth points of Σ then the circular arcs are tangent to Σ (3.9).

Let

$$\mathscr{F} = \{\Gamma \,|\, d(P, Q) \leqq \Gamma \leqq \Sigma', \ \Gamma \text{ is a curve satisfying (3.8) or (3.9)}\} \ .$$

Then we have

$$\operatorname*{Max}_{\text{any } \Gamma} g(\Gamma) = \operatorname*{Max}_{\Gamma \in \mathscr{F}} g(\Gamma) \ .$$

This reduces the maximization over a very complicated class of curves to one over a much simpler class.

To prove Theorem 4.1, we shall show that g achieves its maximum on the boundary Σ' and since

$$g(\Sigma') = -\Sigma' + \frac{1}{R}\Omega = -\Sigma' + \Sigma = \Sigma^*$$

(4.4) hold for Σ' and hence will hold for all possible curve.

Before proving the theorems, let us define some notation: Let C be a simple closed curve, the interior of which is divided by Σ

into subregions $\Delta_1, \Delta_2, \cdots, \Delta_k$, we will call Δ_i positive $(+)$ if Δ_i lies inside Ω, negative $(-)$ if Δ_i lies outside Ω.

We need the following lemmas.

LEMMA 4.3.   *Let Ω be as in Theorem 4.1; then no circle of radius less than R can have its interior divided by Σ into subregions containing more than one negative region.*

Proof.   Suppose there exists a circle $S_r(0)$ of radius $r \leqq R$ centered at 0, which is divided by Σ into more than one negative regions, let $\Delta_1, \Delta_2, \Delta_3$ be three adjacent regions of the type $-, +, -$, and let Σ_{AB}, Σ_{CD} be two of the subarcs of Σ bounding Δ_2 (Figure 10). Shrink $S_r(0)$ with respect to 0 until the first point of contact (with $\Sigma_{AB} \cup \Sigma_{CD}$) happens, say at $P \in \Sigma_{CD}$, i.e., there exists a circle $S_t(0)$, $t \leqq r \leqq R$, which contacts Σ_{CD} at P and intersects Σ_{AB} at some points, say A', B'. Now take the circle S_R of radius R containing $S_t(0)$ and tangent to $S_t(0)$ at P. S_R contacts Σ at P but does not lie inside Ω, contradicting the hypothesis that Ω has the interior rolling disk property of radius R.

DEFINITION 4.2.   A subregion $\Omega' \subset \Omega$ is convex in Ω if for every two points $P, Q \in \Omega'$, the line segment $\overline{PQ} \subset \Omega$ implies $\overline{PQ} \subset \Omega'$.

LEMMA 4.4.   *Let Ω be as in Theorem 4.1 and Γ be a circular arc of radius R in $\overline{\Omega}$ which is not contained in Σ, and let the subregion Ω^* be convex in Ω. Then*

$$g(\Gamma) < g(\Gamma') \quad (strictly) .$$

Proof.   Let Γ' be the completing circular arc of Γ. Then by Lemma 4.3, either Σ^* or Σ' lies inside the circle $\Gamma \cup \Gamma'$.

If Σ^* lies inside the circle $\Gamma \cup \Gamma'$, then $\Omega^* \leqq \pi R^2$, and there are two possible cases $\Sigma^* > \Gamma'$ or $\Sigma^* \leqq \Gamma'$.

(i)   If $\Sigma^* > \Gamma'$ (Figure 11a), then

$$\frac{1}{R}\Omega^* \leqq \frac{1}{R}\pi R^2 = \pi R < 2\pi R = \Gamma + \Gamma' < \Gamma + \Sigma^*$$

hence we have

$$g(\Gamma) = -\Gamma + \frac{1}{R}\Omega^* < \Sigma^* = g(\Sigma') .$$

(ii)   If $\Sigma^* \leqq \Gamma'$ (Figure 11b), then by the isoperimetric inequality

$$\frac{1}{R}\Omega^* \leqq \frac{1}{R}\frac{(\Sigma^* + \Gamma)^2}{4\pi} \leqq \frac{\Gamma' + \Gamma}{4\pi R}\cdot(\Sigma^* + \Gamma) = \frac{2\pi R}{4\pi R}(\Sigma^* + \Gamma) < \Sigma^* + \Gamma .$$

We still have

$$g(\Gamma) = -\Gamma + \frac{1}{R}\Omega^* < \Sigma^* = g(\Sigma') \ .$$

If Σ' lies inside the circle $\Gamma \cup \Gamma'$ (Figure 11c), then by Lemma 2.3 $\Sigma' \leqq -\Gamma + (1/R)\Omega'$, hence we have

$$g(\Gamma) = -\Gamma + \frac{1}{R}\Omega^* = -\Gamma + \frac{1}{R}(\Omega - \Omega')$$

$$= \Sigma - \Gamma - \frac{1}{R}\Omega' < \Sigma + \Gamma - \frac{1}{R}\Omega'$$

$$= \Sigma - \left(-\Gamma + \frac{1}{R}\Omega'\right) \leqq \Sigma - \Sigma'$$

$$= \Sigma^* = g(\Sigma') \ .$$

This proves the lemma.

LEMMA 4.5. *Let Γ be a curve in $\bar{\Omega}$ whose end points P and Q lie on Σ, let Ω' and Ω^* be the two disjoint regions cut by Γ. Suppose that one can draw a piecewise smooth curve Γ_1 in $\bar{\Omega}'$ which join P and Q and with the property that a circle of radius R can be rotated on it in such a way that the circular arc between Γ and Γ_1 is always a smaller circular arc of the circle (Figure 12). Then*

$$g(\Gamma) < g(\Gamma_1) \ .$$

Proof. Let Ω_1' and Ω_1^* be the corresponding subregions cut by Γ_1 such that $\Omega_1' \subset \Omega'$ and $\Omega_1^* \supset \Omega^*$. Let A and B be two points on Γ_1 which are close enough such that the two circles of radius R contact to Γ_1 at A and B from the side of Ω_1^* have nonempty intersection in $\Omega_1^* - \Omega^*$. Let C be the intersection of these two circles and let D, E, F, G be the intersections of them with Γ as shown in Figure 12. Let $S, \varDelta_A, \varDelta$ be the regions enclosed by AB, BC, CA; DAF, FD; FG, GBC, CF respectively, and consider the new curve $\Gamma(A), \Gamma(AB)$ defined as follows

$$\Gamma(A) = PD + DA + AF + FQ$$
$$\Gamma(AB) = PD + DA + AB + BG + GQ \ .$$

Here we simplify the notation by writing $PD = \Gamma_{PD}$ (the restriction of Γ between P and D), $DA =$ circular arc between D and A, $AB = \Gamma_{1AB}$ (the restriction of Γ_1 between A and B) $\cdots$ etc. Then

(4.8) $g(\Gamma(A)) - g(\Gamma) = -DA - AF + DF + \dfrac{1}{R}\varDelta_A$

$$= \left(\frac{1}{R}\varDelta_A - DAF\right) - \left(\frac{1}{R}\cdot 0 - DF\right)$$

$$= \varphi(DAF) - \varphi(DF) \geqq 0$$

$$\text{(Lemma 2.3 with } \varDelta = DF\text{)}$$

$g(\Gamma(AB)) - g(\Gamma(A))$

$$= -AB - BG + AF + FG + \frac{1}{R}(\varDelta + S)$$

$$= -AB - BG + (AC + CF) + FG + \frac{1}{R}(\varDelta + S)$$

$$= (AC + BC - AB) + \frac{1}{R}S + \left(\frac{1}{R}\varDelta - BC - BG + CF + FG\right)$$

$$= (AC + BC - AB) + \frac{1}{R}S + \left[\frac{1}{R}\varDelta - CBG - \left(\frac{1}{R}\cdot 0 - CFG\right)\right]$$

$$= AC + BC - AB + \frac{1}{R}S + [\varphi(CBG) - \varphi(CFG)]$$

$$\geqq AC + BC - AB ,$$

since $(1/R)S \geqq 0$ and $\varphi(CBG) \geqq \varphi(CFG)$ (Lemma 2.3 with $\varDelta = CFG$).

Suppose that we can show that

(4.9) $AC + BC > AB$

provided A and B are sufficiently close, then

$$g(\Gamma(AB)) \geqq g(\Gamma(A))$$

and using this together with (4.8) and the standard compactness argument we obtain $g(\Gamma_1) > g(\Gamma)$.

Let $k(A)$ be the curvature of Γ_1 at A, since Γ_1 can be rotated on it form the side of Ω_1^* a circle of radius R, we have $k(A) \leqq 1/R$ provided that A is a regular point of Γ_1. If A is not a regular point of Γ_1, then by hypothesis, we may consider $k(A^+)$ and $k(A^-)$, the curvatures of the two smooth curves meeting at A, we still have $k(A^+) \leqq 1/R$ and $k(A^-) \leqq 1/R$. We shall prove (4.9) under these curvature conditions. We consider here only the case where A is a regular point of Γ_1; for nonregular points, essentially the same arguments work.

Case (i). $|k(A)| < 1/R$.

There exists a neighborhood $N(A)$ of A on Γ_1 on which $|k(B)| < 1/R$ for all $B \in N(A)$. Pick a point $B \in N(A)$ such that the two circles

of radius R contact to Γ_1 at A and B from the side of Ω_1^* have nonempty intersection, and let C be as above. Let (O, X, Y) be a local coordinate system centered at $O \in \Gamma_1$ with O between A and B in which OX and OY coincide with the tangent and normal directions of Γ_1 at O respectively. As O moves from A to B, C will first lie on one side of OY and then on the other side. Since Γ_1 is smooth between A and B there exists a point O between A and B at which OY passes through C (Figure 13). Let AO and AC be expressed in this coordinate system as $y(x)$ and $z(x)$ respectively, and let a, b be the x-axis components of A, B respectively. Then we have

$$(4.10) \qquad y(a) = z(a) \,, \qquad y'(a) = z'(a) \,, \qquad y'(0) = 0$$

the curvature condition gives

$$(4.11) \qquad -\frac{d}{dx}\frac{z'}{\sqrt{1+z'^2}} \leqq \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} \leqq \frac{d}{dx}\frac{z'}{\sqrt{1+z'^2}} \,.$$

Integrate (4.11) from a to x and use the initial condition $y'(a) = z'(a)$ we get

$$-\frac{z'(x)}{\sqrt{1+z'^2(x)}} \leqq \frac{y'(x)}{\sqrt{1+y'^2(x)}} \leqq \frac{z'(x)}{\sqrt{1+z'^2(x)}}$$

for all $a \leqq x \leqq 0$; this is equivalent to

$$y'^2(x) \leqq z'^2(x)$$

and hence

$$AC = \int_a^0 \sqrt{1+z'^2(x)}\,dx \geqq \int_a^0 \sqrt{1+y'^2(x)}\,dx = A0 \,.$$

Similarly, $BC \geqq BO$ and we get

$$AC + BC \geqq AB \,.$$

Case (ii). $k(A) \leqq -1/k$ (this case may happen on Σ).

There exists a neighborhood $N(A)$ of A on Γ_1 on which $k(B) < 0$ for all $B \in N(A)$. Pick a point $B \in N(A)$ and set up the coordinate system (O, X, Y) as in Case (i) (Figure 14), and let $y(x)$, $z(x)$, a, b be defined in this coordinate system as in Case (i). Then we have

$$y'(a) = z'(a) \,, \qquad y'(0) = 0 \,.$$

Since y has negative curvature for all $a \leqq x \leqq 0$, $y''(x) < 0$ for all $a \leqq x \leqq 0$. That is, y' is strictly decreasing on $[a, 0]$. Similarly, z has positive curvature for all $a \leqq x \leqq 0$, z' is strictly increasing on $[a, 0]$.

Thus for $x \in [a, 0]$ we have

$$z'(x) > z'(a) = y'(a) > y'(x) > y'(0) = 0$$

which implies that $z'^2(x) > y'^2(x)$ and we have

$$AC = \int_a^0 \sqrt{1 + z'^2(x)}\, dx > \int_a^0 \sqrt{1 + y'^2(x)}\, dx = AO \ .$$

Similarly $BC > BO$ and we obtain

$$AC + BC > AB \ .$$

This completes the proof.

REMARK. The lemma still is true if we replace "always a smaller circular arc" by "not a bigger circular arc" i.e., we allow semi-circles to happen between Γ and Γ_1. As we can take a sequence of curves $\{\Gamma_{1,n}\}$ satisfies the hypothesis of the lemma, such that

$$\Gamma_{1,n} \longrightarrow \Gamma_1 \quad \text{(in arc length)}$$
$$\Omega_{1,n}^* \longrightarrow \Omega_1^*$$

then

$$g(\Gamma) < \lim_{n \to \infty} g(\Gamma_{1,n}) = g(\Gamma_1) \ .$$

LEMMA 4.6. *Let Ω be as in Theorem 4.1 and Γ be a circular arc of radius $> R$. Then*

$$g(\Gamma) < g(\Sigma')$$

still holds.

Proof. Let $P, Q \in \Sigma$ be the end points of Γ. There are two cases to consider:

(i) $d(P, Q) \leqq 2R$ and Γ is not a bigger circular arc. In this case, one can draw a nonbigger circular arc Γ_1 of radius R passing through P, Q and lying "outside" Γ (i.e., $\Gamma_1 \subset \Omega' = \Omega - \Omega^*$). By the remark of Lemma 4.5 and Lemma 4.4 we have

$$g(\Gamma) < g(\Gamma_1) < g(\Sigma') \ .$$

(ii) $d(P, Q) > 2R$ or Γ is a bigger circular arc. Let $\Omega' = \Omega - \Omega^*$, let Γ_t be any curve joining P, Q in $\bar{\Omega}'$ with the property that a circle of radius R can be rotated on it such that the circular arc of the circle which lies between Γ_t and Γ is not a bigger circular arc. Let Ω_t^* be the region bounded by Γ_t and Σ^*, and let $\Omega_t' = \Omega - \Omega_t^*$. Consider the set $S_1 = \{\Omega_t^*\}$, partially order S_1 by set inclusion. It is

easy to see that S_1 is closed in every linear ordered subset, i.e., every linear ordered subset is bounded above by an element in S_1. Hence, by Zorn's lemma, there exists a maximal element Ω_1^* in S_1. Since Γ is a curve of curvature $< R$, S_1 is nonempty, and we have $\Omega_1^* \supset \Omega^*$ strictly. (Actually, $\Omega_1^* = \Omega^*$ if and only if Γ is a circular arc of radius R or $\Gamma = \Sigma'$.) Let $\Gamma_1 = \partial\Omega_1^* \cap \Omega$, by the remark of Lemma 4.5, we have

$$g(\Gamma) < g(\Gamma_1) .$$

In other words, g can be increased by "moving Γ towerd Σ'" to a new curve Γ_1 in a certain distance which depends on the situation of Γ and the geometry of Σ.

Now repeat the argument on Γ_1. (The argument can be repeated if a circle of radius R can be rotated on it from the side of Ω_1^*. It is clear that Γ_1 has this property from the construction of Γ_1.) We obtain a curve Γ_2 and a subregion $\Omega_2^* \supset \Omega_1^*$ such that

$$g(\Gamma_1) \leqq g(\Gamma_2) .$$

Continue this process we get a monotone sequence of curves Γ_n and an increasing sequence of regions Ω_n^* such that

$$g(\Gamma_n) \leqq g(\Gamma_{n+1}) .$$

Since $\Omega_n^* \subset \Omega_{n+1}^* \subset \Omega$ and Ω is bounded we have $\lim_{n \to \infty} \Omega_n^* = \Omega_\infty^* \cap \Omega$. Let $\Gamma_\infty = \partial\Omega_\infty^* \cap \Omega'$; by our construction of Γ_n, we have $\lim_{n \to \infty} \Gamma_n = \Gamma_\infty$ uniformly in pointwise convergence and also converges in the corresponding arc length. Hence

$$g(\Gamma_1) < \lim_{n \to \infty} g(\Gamma_n) = g(\Gamma_\infty) .$$

We claim that $\Gamma_\infty = \Sigma'$.

Let $E = \Gamma_\infty - (\Gamma_\infty \cap \Sigma') = \bigcup_{i=1}^\infty E_i$, where each E_i is a connected component of E. Since Γ_∞ is the monotone limit of Γ_n and each Γ_n has the property that a circle of radius R can be contacted at each point of it from the side of Ω_n^*, Γ_n inherits the property too, and so does E_i for each i.

If $E_i \neq \emptyset$ and is not a circular arc of radius R, then there exists a point $x \in E_i$ with the property that a circle $B(R)$ of radius R can be contacted on it from the side of Ω_∞^* such that $E_i \cap B(R) = \{x\}$. Thus, it is possible to move $B(R)$ across E_i slightly so that $B(R) \cap \Omega_\infty'$ is a smaller circular arc of $B(R)$, (Figure 15). Since $\Gamma_n \to \Gamma_\infty$ uniformly, there exists Γ_{n-1} such that $B(R) \cap \Omega_{n-1}'$ is also a smaller circular arc of $B(R)$. Let $\{A, B\} = B(R) \cap \Gamma_n$, $\widehat{AB} = B(R) \cap \Omega_n'$, $\Gamma_{n,t} = \Gamma_n - \Gamma_{*|AB} + \widehat{AB}$, and $\Omega_{n,t}^*$ be the region bounded by $\Gamma_{n,t}$ and

Σ^*. We have $\Omega^*_{n,t} \supset \Omega^*_n$ and $\Gamma_{n,t}$ has the property that a circle of radius R can be contacted at each point of it such that the circular arc between Γ_n and Γ_{n-1} is always a smaller circular arc of the circle. This contradicts the maximality of Ω^*_n.

Thus, each E_i should be a circular arc of radius R or an empty set. The first case is impossible since Ω has the interior rolling disk property of radius R. We conclude that $E_i = \oslash$ for all i, and hence $\Gamma_\infty = \Sigma'$.

LEMMA 4.7. *Let Γ be a curve consisting of subcurves Γ_1, Γ_2, $\cdots$, and coinciding with Σ^* on $\Gamma - \bigcap_{i=1}^\infty \Gamma_i$. Let Γ_i, Σ_i^* and Ω_i^* be the corresponding connected components of $\Gamma - (\Gamma \cap \Sigma^*)$, $\Sigma^* - (\Gamma \cap \Sigma^*)$ and Ω^*. Then $g(\Gamma) \leqq \Sigma^*$ if $g(\Gamma_i) \leqq \Sigma_i^*$ for all $i = 1, 2, \cdots$.*

Proof. (Figure 16)

$$(4.12) \qquad \sum_{i=1}^\infty \Sigma_i^* \geqq \sum_{i=1}^\infty g(\Gamma_i) = -\sum_{i=1}^\infty \Gamma_i + \frac{1}{R}\sum_{i=1}^\infty \Omega_i$$

$$= -\Gamma + \frac{1}{R}\Omega + (\Gamma \cap \Sigma^*) .$$

On the other hand, $\Sigma^* = \sum_{i=1}^\infty \Sigma_i^* + (\Gamma \cap \Sigma^*)$. This and (4.12) imply the lemma.

LEMMA 4.8. *Let Ω be as in Theorem 4.1, Γ a circular arc of radius $r < R$, then $g(\Gamma) < g(\Sigma')$ still holds.*

Proof. Let P and Q be the end points of Γ on Σ.

(i) Suppose we can draw in Ω a smaller circular arc $\Gamma(R)$ of radius R with P and Q as its end points (Figure 17a). Then by Lemma 2.2 and Lemma 2.3 we have

$$g(\Gamma) < g(\Gamma(R)) < g(\Sigma') .$$

(ii) Suppose $\Gamma(R)$ does not lie inside Ω. Then we can "shrink Γ toward Σ^*" until the first point of contact happens, i.e., consider all circular arcs joining P and Q, which lie between Γ and Σ^*. Let $\Gamma(t_1)$ be the circular arc which has the first point of contact with Σ^*, at P_1 (Figure 17b). Then by the remarks of Lemma 2.2 we have

$$g(\Gamma) < g(\Gamma(t_1)) .$$

If Case (i) holds on each piece of $\Gamma_{PP_1}(t)$ and $\Gamma_{P_1Q}(t)$, then by Lemma 4.7, we will have

$$g(\Gamma) < g(\Sigma') .$$

If Case (i) does hold on both of $\Gamma_{PP_1}(t_1)$ and $\Gamma_{P_1Q}(t_1)$, say it fails on $\Gamma_{PP_1}(t)$; then repeat the argument on $\Gamma_{PP_1}(t)$, we get a second point of contact and increase g on the new circular arc. Since Σ^* is of finite arc length, after at most countable process, we will get a sequence of circular arcs $\Gamma_1, \Gamma_2, \cdots$, with each Γ_i satisfying the situation in (i). Let $\Gamma_i(R)$ and Σ_i^* be the circular arc of radius R and boundary curves respectively corresponding to Γ_i, then Case (i) and Lemma 4.7 imply the lemma.

Lemmas 4.4, 4.5, 4.6 and 4.7 together imply the following lemma.

LEMMA 4.9. *Let Ω be as in Theorem 4.1. Then $g(\Gamma)$ always achieves its maximum on Σ'. More precisely, if $\Gamma \neq \varnothing$ or Σ' (i.e., $\Omega^* \neq \varnothing$ or Ω), then $g(\Gamma) < \Sigma^*$ strictly.*

Proof of Theorem 4.1. By Lemma 4.9

$$(4.13) \qquad\qquad g(\Gamma) \leqq g(\Sigma') = \Sigma^* .$$

On the other hand,

$$f(\Gamma) = \Gamma + \frac{1}{R}\Omega^*$$

$$= \Sigma - \left(-\Gamma + \frac{1}{R}\Omega'\right), \quad \Omega' = \Omega - \Omega^* .$$

Again by Lemma 4.9, $-\Gamma + (1/R)\Omega'$ achieves its maximum on Σ^*, hence

$$(4.14) \qquad\qquad f(\Gamma) \geqq \Sigma - \left(-\Sigma^* + \frac{1}{R}\Omega'(\Sigma^*)\right)$$

$$= \Sigma - \left(-\Sigma^* + \frac{1}{R}\Omega\right)$$

$$= \Sigma - (-\Sigma^* + \Sigma)$$

$$= \Sigma^*$$

(4.13) and (4.14) imply

$$(4.15) \qquad\qquad g(\Gamma) \leqq \Sigma^* \leqq f(\Gamma)$$

for all possible Γ. If $\Omega^* \neq \varnothing$ or Ω (4.15) holds strictly and hence by Theorem (3.4), Equation 3.1 has a solution for all contact angle γ.

LEMMA 4.10. *Let Γ be a circular arc of radius R in Ω which contacts to Σ at P and Q (Figure 18). Suppose that no points of Σ'*

can be contacted by a circle of radius $R = \Omega/\Sigma$ in the interior of Ω. Then $g(\Gamma) > \Sigma^*$.

Proof. We will prove that g increases strictly as Σ' shrinks to Γ, and hence $g(\Gamma) > g(\Sigma') = \Sigma^*$.

Let $C_1, C_2, \cdots, C_n$ be a sequence of circular arc of radius R in Ω which meet Σ' at $P_1, Q_1; P_2, Q_2; \cdots; P_n, Q_n$ respectively, and let $\Gamma_i = \Sigma_{PP_i} + C_i + \Sigma_{Q_iQ}$, Δ_i be the region between C_i and C_{i+1} (write $C_0 = \Sigma'_{P_1Q_1}$). (Figure 18). Then

$$g(\Sigma') - g(\Gamma_1) = -\Sigma'_{P_1Q_1} + \frac{1}{R}\Delta_0 - C_1$$

$$= \varphi(C_0) - \varphi(C_1) < 0$$

(Lemma 2.3 with $\Delta = C_1$)

if C_1 is properly chosen (all we require here is that $\Sigma_{P_1Q_1} <$ bigger circular arc passing through P_1 and Q_1).
Similarly

$$g(\Gamma_1) - g(\Gamma_2) = -\Sigma_{P_1P_2} - C_1 - \Sigma_{Q_1Q_2} + \frac{1}{R}\Delta_1 + C_2$$

$$= \varphi(\Sigma_{P_1P_2} + C_1 + \Sigma_{Q_1Q_2}) - \varphi(C_2) < 0$$

(Lemma 3.2 with $\Delta = C_2$)

if Γ_2 is properly chosen.
Repeat the argument we have

$$g(\Gamma_i) < g(\Gamma_{i+1})$$

provided Γ_{i+1} is properly chosen. For these sequence of Γ_i, we then have

$$g(\Sigma') < g(\Gamma_1) < \cdots < g(\Gamma_n) < g(\Gamma) .$$

Proof of Theorem 4.2. We are not going to prove this theorem in detail since much of the proof is essentially that of Theorem 4.1. We give here a sketch of the proof.

Let $\Gamma \in \mathscr{F}$, i.e., Γ is a circular arc or a curve composed of circular arcs of equal radii which are tangent to Σ at regular points of Σ, and Γ coincides with Σ on the rest parts of Γ. By Lemma 4.7 it suffices to consider g on each circular arc.

Let C be one of the circular arc and let r be its radius.
(i) If $r > R$, then repeat the argument as in the proof of

Lemma 4.6. We can increase g by moving C towards Σ'. We may not, however, reach Σ' since we do not know whether Σ' satisfies the interior rolling disk property of radius. If it does, then we will reach Σ' and hence $g(C) < g(\Sigma') = \Sigma^*$. If it does not, then as we have seen from the proof of Lemm 4.6, we shall stop at a curve which is the trajectory from rotating a circle of radius R on Σ', i.e., a curve consisting of circular arcs of radius R, all of which contact to Σ' (tangent to Σ' on regular points) and coinciding with Σ' on the complement of those circular arcs. By the hypothesis and Lemma 4.7 again, we still have $g(C) < \Sigma^*$.

(ii) If $r \leqq R$, then by Lemma 4.9, we can always increase g by moving C toward Σ^* until it consists of circular arcs of radius R which contact Σ^*. Again by the hypothesis and Lemma 4.7, we have $g(C) < \Sigma^*$. This completes the proof.

5. **Existence and nonexistence of solutions on neck domains and tail domains.** In § 4 we proved the existence of solutions of (3.1) for all γ on domains having the interior rolling disk property of radius $R = \Omega/\Sigma$. We will show in this section that this condition is also necessary for domains with no tail of radius R but not necessary for some neck domains.

Let Ω be a plane domain with piecewise smooth boundary Σ and Ω_0 be any subregion of Ω with the interior rolling disk property of radius R. Consider the set

$$S(\Omega, \Omega_0) = \{\Omega_t \,|\, \Omega_0 \subset \Omega_t \subset \Omega, \ \Omega_t \text{ has the interior rolling}$$
$$\text{disk property of radius } R\} \,.$$

Partially order S by set inclusion and by Zorn's lemma, there exists a maximal element Ω_1 in $S(\Omega, \Omega_0)$. If Ω has interior rolling disk property of radius R, then $\Omega_1 = \Omega$ for every choice of Ω_0. If Ω does not have the property, then $\Omega_1 \subset \Omega$ strictly, and different choices of Ω_0 may result in different Ω_1. More precisely, Σ_1 (the boundary of Ω_1) consists of circular arcs of radius R and coincides with some subarc of Σ on which a circle of radius R can be contacted (rotated) in the interior of Ω.

DEFINITION 5.1. A plane domain Ω is called a neck domain of radius R if it contains more than one maximal subregion with the interior rolling disk property of radius R (Figure 19).

DEFINITION 5.2. A plane domain Ω is said to have a tail of radius R if there exists a subarc Σ_{PQ} of Σ, such that

(i) P and Q can be contacted simultaneously by a circle of

radius R in the interior of Ω.

(ii) No points of Σ_{PQ} other than P and Q can be contacted by a circle of radius R in the interior of Ω (Figure 19). With these definitions, we can now state our theorems.

LEMMA 5.1. *For tail domains of radius* $R = \Omega/\Sigma$, *there exists an* $\gamma_0 > 0$ *such that* (3.1) *has no solution for any* γ *with* $0 \leqq \gamma < \gamma_0$.

Proof.   Lemma 4.10.

Theorem 4.1 and Lemma 5.1 together imply the following theorem.

THEOREM 5.2. *Let* Ω *be a piecewise smooth domain with no necks of radius* $R = \Omega/\Sigma$, *then* (3.1) *has solutions for all* $\gamma \geqq 0$ *if and only if* Ω *satisfies the interior rolling disk condition of radius* R.

For neck domains the interior rolling disk property is not a necessary condition for existence. A counterexample can be constructed as follows:

EXAMPLE 5.3. Let Ω_1 and Ω_2 be two unit disks with nonempty intersection, let $\Omega = \Omega_1 \cup \Omega_2$, $\{P, Q\} = \Sigma_1 \cap \Sigma_2$, and 2θ be the angle made by P, Q with respect to the center of the disk (Figure 20). Then we have

$$\Omega = 2\pi - 2\theta + \sin 2\theta$$

(5.1) $$\Sigma = 4(\pi - \theta)$$

$$R = \frac{\Omega}{\Sigma} = \frac{1}{2}\left[1 + \frac{\sin 2\theta}{2(\pi - \theta)}\right] \geqq \frac{1}{2}.$$

Thus if the aperture $\overline{PQ}$ is less than 1, Ω will be a neck domain of radius R. However, as we will see from the following proof, the solution surface always exists.

Let Γ be the circular arc of radius R passing through the points P and Q and call the angle subtended by the arc 2φ (Figure 20). Let $S(R)$ be the region enclosed by Γ and $\overline{PQ}$.

We have:

$$S(R) = \frac{R^2}{2}(2\varphi - \sin 2\varphi)$$

$$\Gamma = 2R\varphi$$

$$\Omega^* = \frac{1}{2}\Omega + S(R)$$

$$\Sigma^* = \frac{1}{2}\Sigma.$$

Thus

(5.2) $$\left| \frac{\Omega^*}{\Omega} - \frac{\Sigma^*}{\Sigma} \right| = \frac{S(R)}{\Omega} = \frac{\dfrac{R^2}{2}(2\varphi - \sin 2\varphi)}{2\pi - 2\theta + \sin 2\theta}$$

(5.3) $$\frac{\Gamma}{\Sigma} = \frac{R\varphi}{2(\pi - \theta)}$$

(5.1), (5.2) and (5.3) imply:

$$\frac{\Gamma}{\Sigma} - \left| \frac{\Omega^*}{\Omega} - \frac{\Sigma^*}{\Sigma} \right| = \frac{R}{2}\left[\frac{\varphi}{\pi - \theta} - \frac{2\varphi - \sin 2\varphi}{4(\pi - \theta)} \right]$$

$$= \frac{R}{2(\pi - \theta)}[2\varphi + \sin 2\varphi] > 0$$

hence by Theorem 4.2, the solution exists.

In the case that Ω is the union of two disks of different radii, it was shown by Finn and Giusti [11] that the solution of (3.1) will fail to exist if the aperture is small enough. The following theorem is a generalization of this result.

THEOREM 5.4. *Let Ω_1, Ω_2 be two plane domains with one region of overlap such that (3.1) has solutions on each domain for all $\gamma \geqq 0$. Suppose that $R_1 \neq R_2 (R_i = \Omega_i/\Sigma_i)$. There exists a positive number γ_0 such that (3.1) fails to have solutions on $\Omega = \Omega_1 \cup \Omega_2$ for all $0 \leqq \gamma < \gamma_0$ when the aperture is sufficiently small.*

Proof. Let P, Q be the points of intersection of Σ_1 and Σ_2, let Γ be the line segment joining P and Q, and let $\Sigma^* = \Sigma \cap \Sigma_1$ (Figure 21), then we have

$$R = \frac{\Omega}{\Sigma} \longrightarrow \frac{\Omega_1 + \Omega_2}{\Sigma_1 + \Sigma_2},$$

$$\Sigma^* \longrightarrow \Sigma_1, \quad \text{and} \quad \Omega^* \longrightarrow \Omega_1 \quad \text{as} \quad \Gamma \longrightarrow 0$$

and therefore

$$\left| \Sigma^* - \frac{1}{R}\Omega^* \right| \longrightarrow \left| \Sigma_1 - \frac{\Sigma_1 + \Sigma_2}{\Omega_1 + \Omega_2} \cdot \Omega_1 \right|$$

$$= \left| \frac{\Sigma_1 \Omega_2 - \Sigma_2 \Omega_1}{\Omega_1 + \Omega_2} \right| = \frac{\Omega_1 \Omega_2}{\Omega_1 + \Omega_2} \left| \frac{1}{R_1} - \frac{1}{R_2} \right|$$

$$= \text{positive constant.}$$

Hence

$$\left| \Sigma^* - \frac{1}{R}\Omega^* \right| > \Gamma \quad \text{if} \quad \Gamma \text{ is small enough.}$$

This shows that solutions cannot exist for all $\gamma \geqq 0$ and the proof is completed.

With the results of Theorem 5.4 and Example 5.3, it seems natural to ask the following question: Can we expect the existence of solutions if $R_1 = R_2$? The answer is still negative. Before giving a counterexample, we need the following lemma.

LEMMA 5.5. *Let Ω_1 and Ω_2 be as is Theorem 5.4 except that $R_1 = R_2$ and let $\Omega = \Omega_1 \cup \Omega_2$. If $R = \Omega/\Sigma$, then $R \geqq R_1$, equality holds only if $\Omega_1 \cap \Omega_2 = \varnothing$ or $\Omega_1 = \Omega_2$.*

Proof. Let Δ_i, $i = 1, \cdots, n$ be the components of $\Omega_1 \cap \Omega_2$ such that each Δ_i is bounded by $a_i \subset \Sigma_i$ and $b_i \subset \Sigma_2$ (Figure 22). Then we have

$$\Omega = \Omega_1 + \Omega_2 - \sum_{i=1}^{n} \Delta_i$$

$$\Sigma = \Sigma_1 + \Sigma_2 - \sum_{i=1}^{n} (a_i + b_i)$$

hence

$$R - R_1 = \frac{\Omega \Sigma_1 - \Omega_1 \Sigma}{\Sigma \Sigma_1}$$

$$= \frac{1}{\Sigma \Sigma_1} \left[\Omega_2 \Sigma_1 - \Omega_1 \Sigma_2 + \sum_{i=1}^{n} ((a_i + b_i) \Omega_1 - \Delta_i \Sigma_1) \right].$$

Since $R_1 = R_2$, we have $\Omega_2 \Sigma_1 - \Omega_1 \Sigma_2 = 0$. Thus

$$R - R_1 = \frac{\Omega_1}{\Omega} \sum_{i=1}^{n} \left(\frac{a_i + b_i}{\Sigma_1} - \frac{\Delta_i}{\Omega_1} \right)$$

$$\geqq \frac{\Omega_1}{\Sigma} \sum_{i=1}^{n} \left(\frac{b_i}{\Sigma_1} - \left| \frac{a_i}{\Sigma_1} - \frac{\Delta_i}{\Omega_1} \right| \right).$$

By hypothesis, a solution exists in Ω_1 for each γ, hence (1.5) holds with $\Sigma^* = a_i$, $\Omega^* = \Delta_i$ and $\Gamma = b_i$. Therefore we have

(5.4) $$\left| \frac{a_i}{\Sigma_i} - \frac{\Delta_i}{\Omega_1} \right| \leqq \frac{b_i}{\Sigma_1}$$

and hence $R - R_1 \geqq 0$.

If $\Omega_1 \cap \Omega_2 \neq \varnothing$ and $\Omega_1 \neq \Omega_2$ then (5.4) holds strictly for some i and hence $R - R_1 > 0$ strictly.

From Lemma 4.5, it is not difficult to construct a counterexample to our equation. The idea is as follows: Suppose we can construct a noncircular domain Ω_1 which satisfies the interior rolling disk condition of radius 1 but does not satisfy the condition for any radius

> 1. Let Ω_2 be the disk of radius 2. We then have $R_1 = R_2 = 1$ and $R > 1$, thus $\Omega = \Omega_1 \cup \Omega_2$ would be a tail domain of radius R if the locations of Ω_1 and Ω_2 are properly chosen. Hence, by Theorem 5.2 no solution exists for $\gamma = 0$.

EXAMPLE 5.6. Let Ω_1 be the figure shown in (Figure 23), whose boundary consists of two circular arcs of radius 1 and $a > 1$ and the two line segments tangent to the circular arcs. Let 2θ be the angle formed by the extension of these two line segments. We then have:

$$\Sigma_1 = \pi(a + 1) + 2(a - 1)(\theta + \cot \theta)$$

$$\Omega_2 = \frac{1}{2}\pi(a^2 + 1) + (a^2 - 1)(\theta + \cot \theta) \, .$$

We want to choose a and θ so that $R_1 = \Omega_1/\Sigma_1 = 1$, or equivalently, we want a and θ to satisfy the following equation:

$$(5.5) \qquad f(\theta) = \theta + \cot \theta = \frac{\pi}{2} \frac{1 + 2a - a^2}{(1 - a^2)} \, .$$

Picking $a = 3/2$ in (5.5), equation becomes:

$$\theta + \cot \theta = \frac{7\pi}{2} \, .$$

Since $f(0) > 7\pi/2$, $f(\pi/4) < 7\pi/2$, there exists $\theta \in (0, \pi/4)$ satisfying the above relation, more precisely, θ lies between $5°$ and $6°$.

Now let Ω_2 be the disk of radius 2, and let $\Omega = \Omega_1 \cup \Omega_2$ with Ω_2 overlapping Ω_1 over the circular arc of radius a (Figure 24). Then $R > 1$, and Ω is a tail domain of radius R. Hence no solution exists for $\gamma = 0$.

6. Two applications. In this section we shall derive two simple applications of our results. The first one is a purely geometrical property of plane figures. There may be a direct proof, however it can be derived easily from Theorem 4.1. The proof here is originally from Finn's proof for Bernstein's problem for constant mean curvature $H \neq 0$. The second one is an extension of the result of Concus and Finn [1], who proved that the solution of the capillary surface equation exists near a wedge of angle 2α, $(0 < \alpha < \pi/2)$ if and only if $\alpha + \gamma \geq \pi/2$. We extend this result to the case $\alpha > \pi$.

THEOREM 6.1. *If Ω is a piecewise smooth domain which satisfies the interior rolling disk condition of radius $R = \Omega/\Sigma$, then Ω cannot contain a disk of radius $2R$ unless Ω is itself a disk of radius $2R$.*

Proof. By Theorem 4.1, the equation

$$(6.1) \qquad \operatorname{div} Tu = H \quad \text{in} \quad \Omega$$

$$(6.2) \qquad Tu \cdot \nu = 1 \quad \text{on} \quad \Sigma$$

has a solution, where H is implicitly determined by (6.1) and (6.2) and is equal to $\Sigma/\Omega = 1/R$.

Suppose $\Omega \supset B_{2R}$, the disk of radius $2R$, and $\Omega \neq B_{2R}$. Then $\partial B_{2R} \cap \Omega$ has positive one dimensional measure. Integrating (6.1) over B_{2R} and applying the divergence theorem, we get

$$(6.3) \qquad \begin{aligned} H \cdot B_{2R} &= \int_{\partial B_{2R} \cap \Sigma} Tu \cdot \nu + \int_{\partial B_{2R} \cap \Omega} Tu \cdot \nu \\ &< \partial B_{2R} \cap \Sigma + \partial B_{2R} \cap \Omega \\ &= \partial B_{2R} \ . \end{aligned}$$

Or

$$4\pi R < 4\pi R$$

which is impossible.

Equality can hold in (6.3) only if $\partial B_{2R} \cap \Omega$ has one dimensional measure zero. The hypothesis that Σ is piecewise smooth then implies $\Omega = B_{2R}$.

REMARK. It is natural to generalize this result to n dimensions. Suppose an n-dimensional set Ω has the property that a ball of radius $R = \Omega/\Sigma$ can be rotated along every point of its boundary Σ in the interior of Ω, then Ω cannot contain a ball of radius nR, unless Ω is itself a ball of radius nR. In this case our proof is no longer valid, since we only proved Theorem 4.1 in the 2 dimensional case.

THEOREM 6.2. *In a wedge of angle $2\alpha > \pi$, the equation*

$$\operatorname{div} Tu = H \quad \text{inside the wedge}$$

$$Tu \cdot \nu = \cos \gamma \quad \text{on the boundary}$$

has solutions for all contact angles γ.

Proof. Let O be the vertex of the wedge and let A and B be two points on the boundary such that $\overline{OA} = \overline{OB} = 1$. Consider the region

$$\Omega = \{(x, y) \mid (x, y) \text{ lie inside the wedge and the distance from} \\ (x, y) \text{ to } OA \text{ or } OB \text{ is less than } a\}$$

(Figure 25), then

$$\Sigma = 4 + 4\alpha a$$

$$\Omega = 4a + 4\alpha a^2 - \pi a^2$$

$$R = \frac{\Omega}{\Sigma} = a - \frac{\pi a^2}{4 + 4\alpha a} < a$$

thus Ω satisfies the interior rolling disk condition of radius R, and hence by Theorem 4.1 solutions exists for all $\gamma \geqq 0$.

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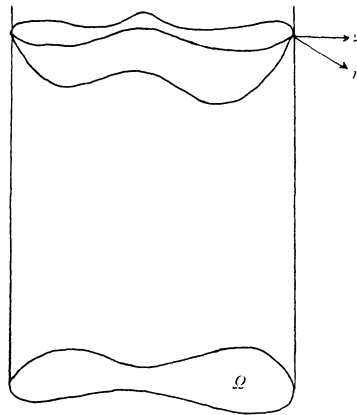
$$\vec{n} = \left\langle \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}, \; \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}, \; \frac{-1}{\sqrt{1 + u_x^2 + u_y^2}} \right\rangle$$

$$= \left\langle Tu, \; \frac{-1}{\sqrt{1 + u_x^2 + u_y^2}} \right\rangle$$

$$\vec{n} \cdot \nu = Tu \cdot \nu = \cos \gamma$$

FIGURE 1
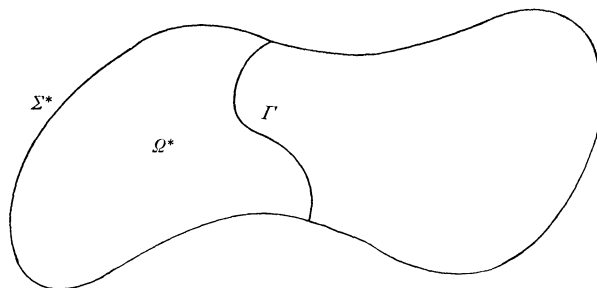
FIGURE 2
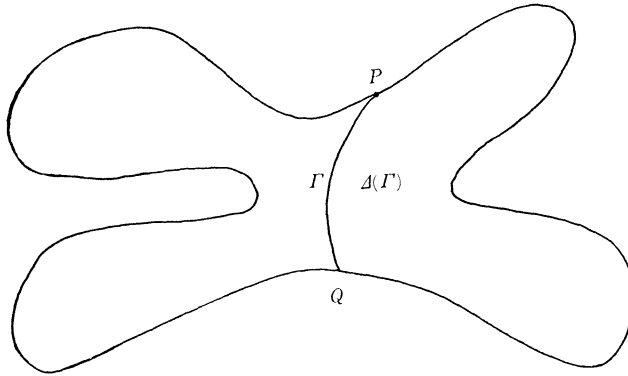
Γ takes circular arc to bound the maximum area.

FIGURE 3a
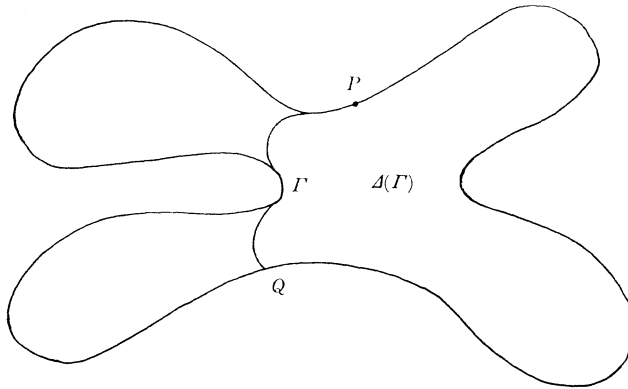
Γ coincides with Σ and circular arcs of equal radius, the circular arcs are tangent to Σ at smooth points of Σ except possibly for the two end points P and Q.

FIGURE 3b
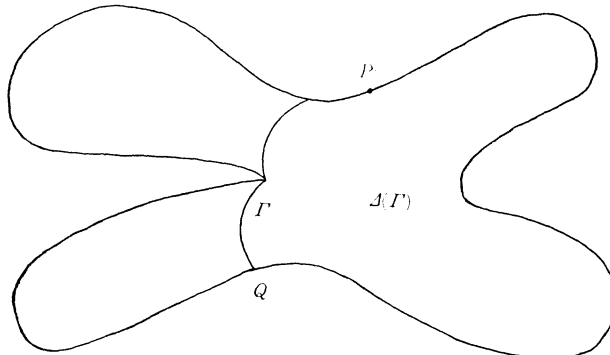
At singular points of Σ, Γ may not be tangent to Σ.
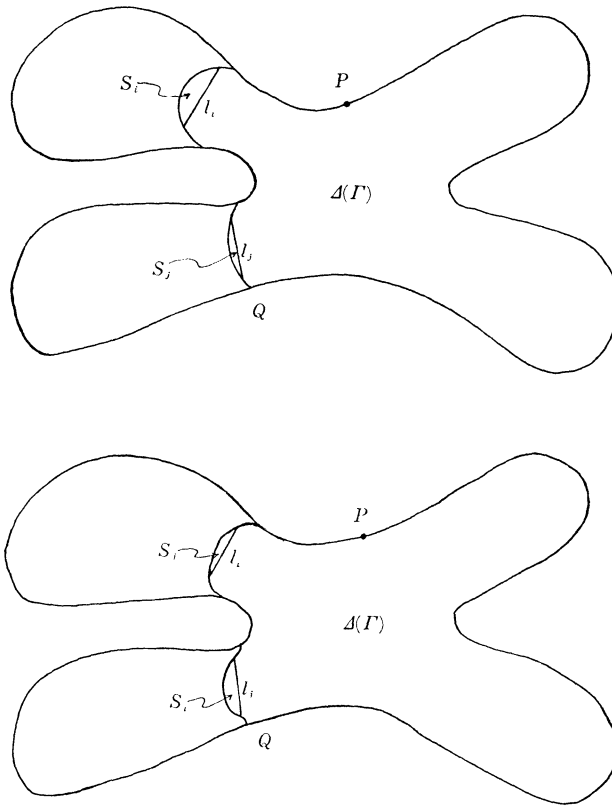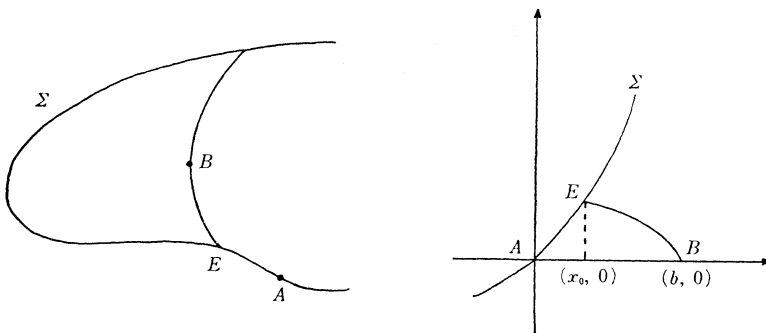
FIGURE 3c

FIGURE 4

FIGURE 5
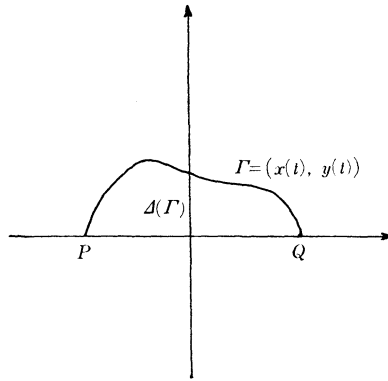
FIGURE 6
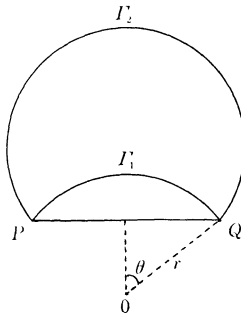
φ achieves its local maximum on Γ_1.
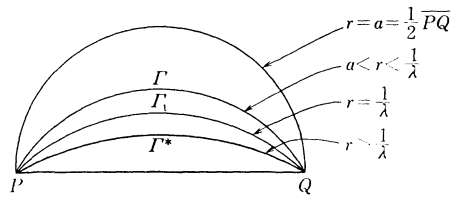
FIGURE 7

$\varphi(\Gamma) < \varphi(\Gamma_1)$, $\varphi(\Gamma^*) < \varphi(\Gamma_1)$.

FIGURE 8
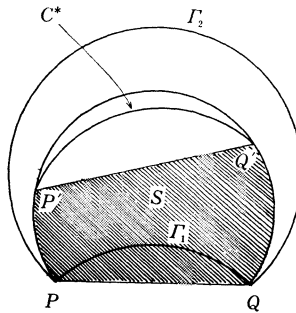
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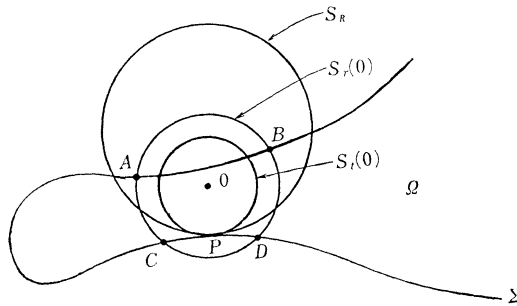
FIGURE 10
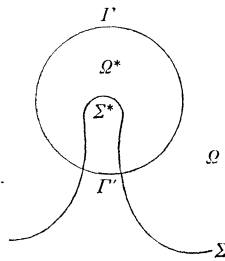
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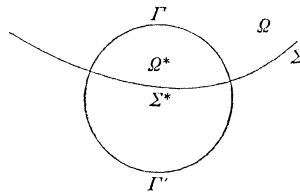
(b)
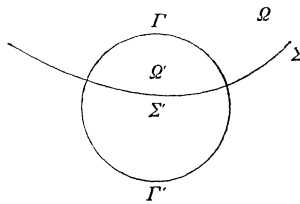
(c)

FIGURE 11
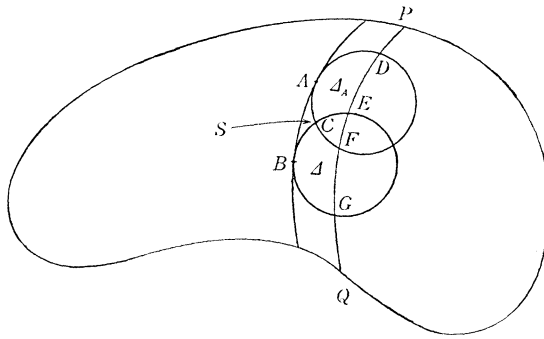
FIGURE 12
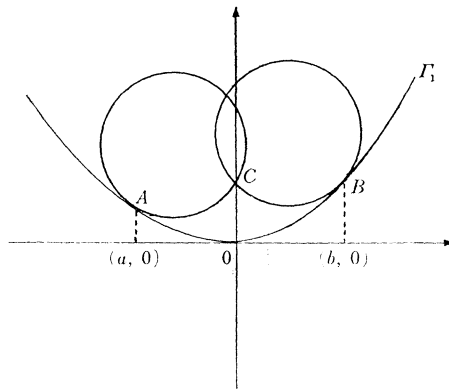
FIGURE 13

FIGURE 14

part of Γ_n

part of Γ_{n-1}
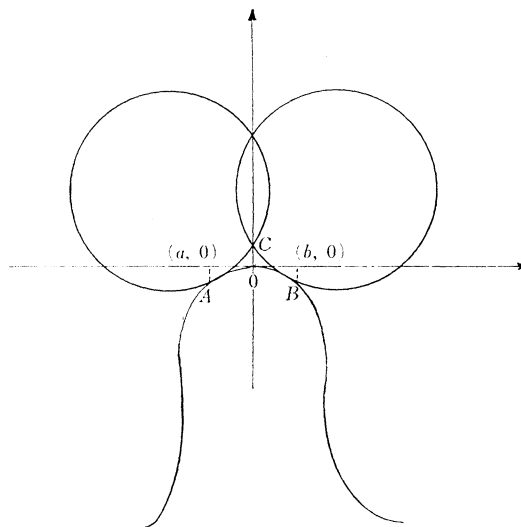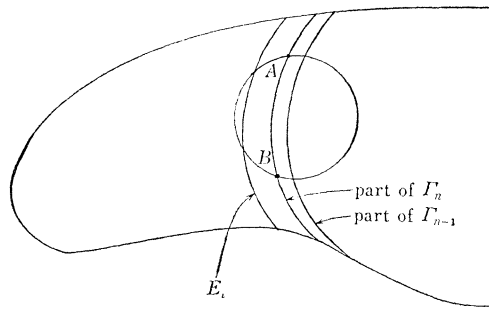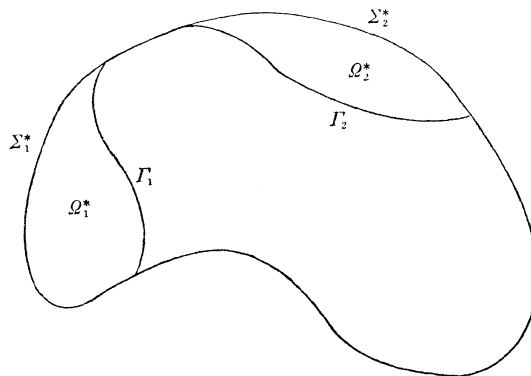
FIGURE 15

FIGURE 16
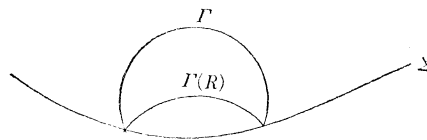
(a)
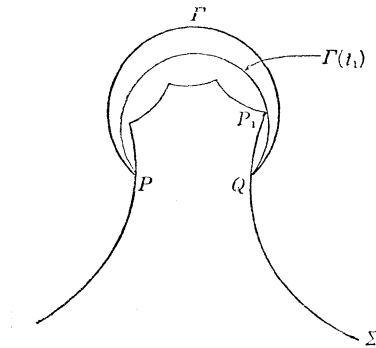
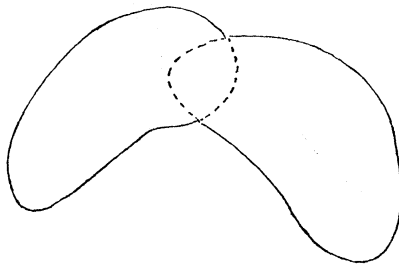(b)

FIGURE 17
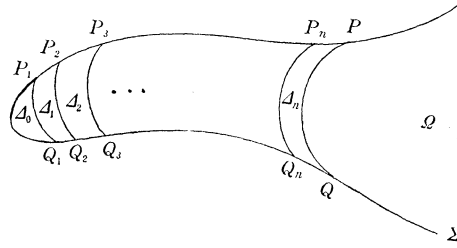
FIGURE 18
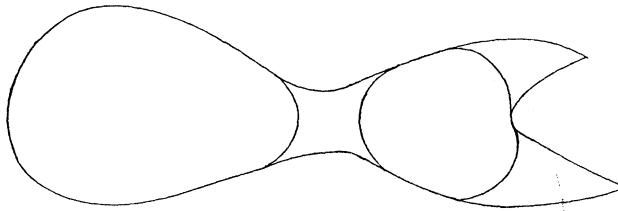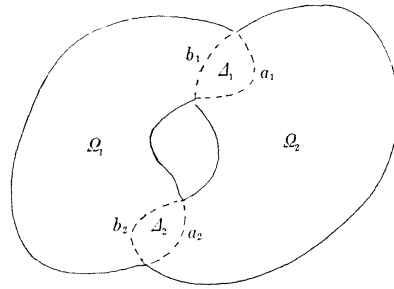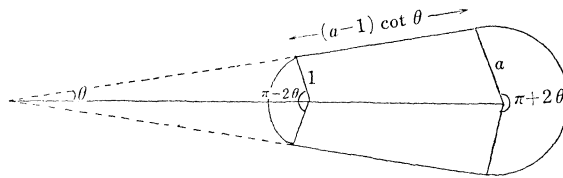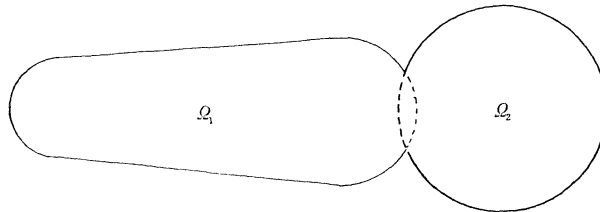

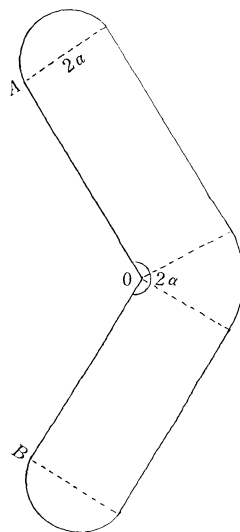A neck domain with 2 tails.

FIGURE 19
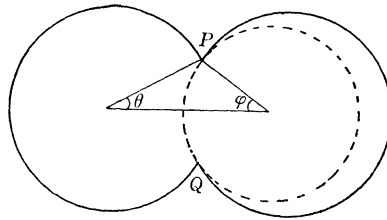
FIGURE 20

FIGURE 21

FIGURE 22

FIGURE 23

FIGURE 24

FIGURE 25