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NONEXISTENCE OF F-MINIMIZING EMBEDDED DISKS

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There has been considerable interest recently in the question of when, given a smooth simple closed extreme curve in $\boldsymbol{R}^3$ as boundary, there exists an embedding of a disk having that boundary which is minimal or minimizing in some appropriate subclass of Lipschitz mappings of the disk. Almgren and Simon [2] showed that an area minimizing embedding of a disk exists in the class of all Lipschitz *embeddings* of disks. (They also showed that there exists an area minimizing embedding of a disk with k handles in the class of all such embeddings in case there exists some mapping of the disk with k handles whose area is less than that of any mapping with $k-1$ handles.) Tomi and Tromba [6] showed that there exists a minimal (not necessarily minimizing) embedding of a disk in the class of *all* Lipschitz mappings of the disk. Meeks and Yau [3] have shown that there exists an area minimizing embedding of the disk in the class of all Lipschitz mappings of the disk.

This paper shows that if one minimizes the integral of an essentially oriented integrand, it is possible for an immersion of the disk to have less integral than any embedding; such integrands arbitrarily closely approximate area.

An *integrand* (also called a *parametric functional*) or $\boldsymbol{R}^3$ is a continuous function

$$F\colon \boldsymbol{R}^3 \times S^2 \longrightarrow \boldsymbol{R}^+ ;$$

here S^2 denotes the unit sphere in $\boldsymbol{R}^3$ and $\boldsymbol{R}^+$ the positive real numbers. The integral $\boldsymbol{F}(S)$ over a surface S in $\boldsymbol{R}^3$ which is the image of a Lipschitz mapping of an oriented disk and which is 1-1 for almost all points in the disk (sums of such surfaces are all we need to consider in this paper) is defined by

$$\boldsymbol{F}(S) = \int_{x\in S} F(x, \nu_S(x))d\mathscr{H}^2x$$

here $\nu_S(x)$ is the oriented unit normal to S at x. The area integrand $F=1$ is one such integrand; others arise naturally, for example, when one considers the surface tension function of anisotropic solids (such as crystals) in contact with their melts or other substances.

An integrand F is defined to be *unoriented* if and only if $F(x, -v) = (x, v)$ for every $v \in S^2$ and x in $\boldsymbol{R}^3$. An integrand is defined to be *essentially oriented* if and only if there exists no triple

(c, a, G), where $c > 0$, $a: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ has divergence 0 and G is an unoriented integrand such that $F(x, v) = cG(x, v) + a(x) \cdot v$ for every $v \in \boldsymbol{S}^2$ and x in $\boldsymbol{R}^3$. (Such integrands arise for example as surface tension functions for crystals whose lattices do not have a center of symmetry; the relevance of this condition is seen in the first paragraph of the proof below.)

An integrand is defined to be *constant coefficient* if and only if $F(x, v) = F(p, v)$ for every x and p in $\boldsymbol{R}^3$ and every v in $\boldsymbol{S}^2$; in this case the integrand is usually written as a function of its second variable only. The theorem below is proved for constant coefficient integrands for the sake of simplicity; obvious modifications yield the theorem (on a small enough sphere) for any variable coefficient essentially oriented integrand.

THEOREM. *If $F: \boldsymbol{S}^2 \to \boldsymbol{R}^+$ is an essentially oriented, constant coefficient, elliptic integrand of class C^3, then there exists an oriented simple closed analytic curve C on the sphere and a Lipschitz immersion (which is not an embedding) of the oriented disk having C as boundary which has less F-integral than any Lipschitz embedding of an oriented disk having C as boundary.*

Proof. Define $F^-: \boldsymbol{S}^2 \to \boldsymbol{R}^+$ by $F^-(v) = F(-v)$ for each $v \in \boldsymbol{S}^2$. By [5], F and F^- do not have the same minimal surfaces, so (since the ellipticity of F implies that F-minimal surfaces are regular [1], [4]) there exists a smooth F-minimal surface S and a point p in S such that S is not F^--minimal in any neighborhood of p, with respect to the boundary given by intersecting S with the boundary of the neighborhood.

Let e be the ellipticity constant of F, and let

$$\varepsilon = \min\{(\max\{F(v): v \in \boldsymbol{S}^2\})^{-1}e, 10^{-4}\} .$$

Without loss of generality, assume that $v_0 = (0, 0, 1)$ is the upward unit normal to S at p, that $p \cdot v_0 \in (-1, -1 + \varepsilon)$, and that $S_1 = S \cap \boldsymbol{B}(0, 1)$ is the graph of a function. By [4], we may further assume that S_1 is the unique F-minimal surface with boundary $B = \partial S_1 = S_1 \cap \boldsymbol{S}^2$ and that there exists $= S_1'$ with boundary B and S_1' is the graph of a function. Since S_1' does not S_1, it must lie above it or below it somewhere; we may assume it lies above it somewhere (making the obvious changes below if it only lies below S_1). Then there exists an oriented simple closed curve B' on $\boldsymbol{S}^2$ which is C^2-close and flat-close to B but which lies below it, and with respect to which there is a unique F^--minimal surface S_2 which is the graph of a function ([4] again) and which crosses S somewhere in its interior, Note that $-S_2$ is uniquely F-minimal with respect to $-B'$.

Using this uniqueness, one notes further that there exists $c' > 0$ such that if W_1 is any embedding of an oriented disk with boundary B, if W_2 is any embedding of an oriented disk with boundary B', and if $W_1 \cap W_2$ is empty, then

$$\boldsymbol{F}(W_1) + \boldsymbol{F}(-W_2) - \boldsymbol{F}(S_1) - \boldsymbol{F}(-S_2) > c' \tag{1}$$

(without loss of generality one may assume that W_1 and W_2 are the graphs of Lipschitz functions with bounds on their slopes and then use the compactness of that space of functions). The idea now is to construct a curve which forces a mapping of an oriented disk which has close to the least F integral to be approximately $S_1 - S_2$ near $x_3 = -1$.

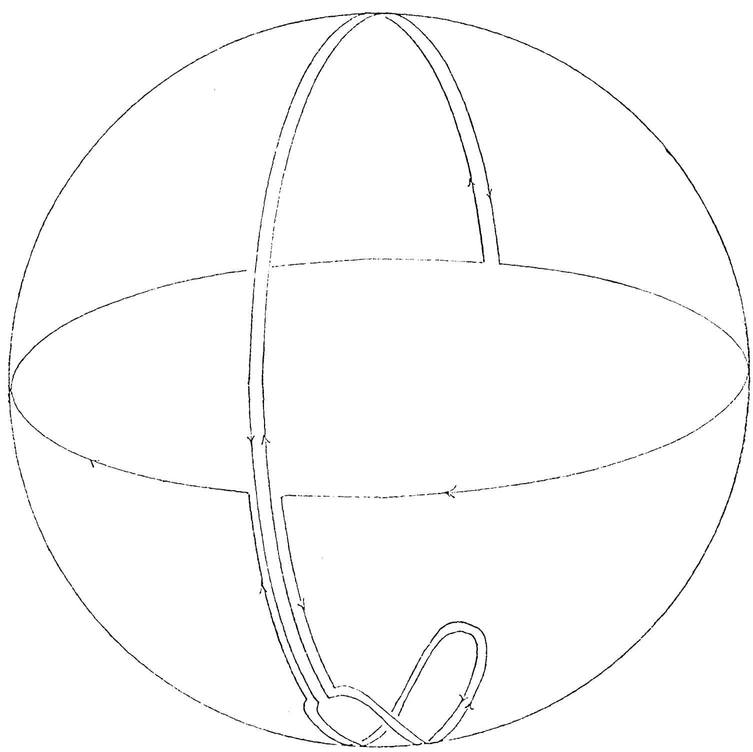
For each $0 < \delta < \varepsilon^2$, construct an oriented simple closed curve C_δ as follows. Define

$B_1 = B' \sim \{x: |x_2| \langle \delta, x_1 \rangle 0\}$, oriented as a subset of B'

$B_2 = B \sim \{x: |x_2| \langle 2\delta, x_1 \rangle 0\}$, oriented as a subset of B

$B_3 = \boldsymbol{S}^2 \cap \{x: x_3 = 0$, and either $x_1 \geqq 0$ and $|x_2| \geqq 2\delta$ or $x_1 < 0$ and $|x_2| > \delta\}$, oriented as part of the boundary of a downward-oriented disk.

For $i \in \{-2, -1, 1, 2\}$, define z_i by $(x_i, i\delta, z_i) \in B_{|i|}$ for some $x_1 > 0$. For $i = 1$ or -1 let



$b_i = S^2 \cap \{(x_1, i\delta, x_3)$: either $x_1 > 0$ and $x_3 \geqq z_i$ or $x_1 \leqq 0$ and $x_3 > 0\}$, oriented up at $((1-\delta^2)^{1/2}, \delta, 0)$ if $i = 1$ and down at $((1-\delta^2)^{1/2}, -\delta, 0)$ if $i = -1$,

and for $i = 2$ or -2, let

$b_i = S^2 \cap \{(x_1, i\delta, x_3)$: $x_1 > 0$ and $0 \geqq x_3 \geqq z_i\}$, oriented up if $i = 2$ and down if $i = -2$.

Let $C_\delta = b_1 + b_{-1} + b_2 + b_{-2} + B_3 + B_2 - B_1$ (see the figure).

We define T_δ as follows. For $i = 1$ and 2, let T_i be the narrow strip on S^2 between b_i and b_{-i}, oriented outward if $i = 1$ and inward if $i = 2$.

Let T_3 be the disk $\{x: |x| = 1, x_3 = 0\}$, oriented down. Let $T_\delta = T_1 + T_2 + T_3 + S_1 - S_2$. Then $\partial T_\delta = C_\delta$.

Suppose for each $0 < \delta < \varepsilon^2$ there were an embedding E_δ of the disk with boundary C_δ satisfying $\boldsymbol{F}(E_\delta) < \boldsymbol{F}(T_\delta)$. For almost all $d \in (-1, 1)$, $E_\delta \cap \{x: x_3 = d\}$ must consist of cycles and curves connecting boundary components. We claim first that b_1 must be connected to b_{-1} and b_2 to b_{-2} for any such slice of E_δ by a horizontal plane. Assume to the contrary that for some $0 > d > -1 + \varepsilon$, $E_\delta \cap \{x: x_3 = d\}$ consists of curves connecting b_1 to b_2 and b_{-1} to b_{-2} (and possible cycles). Then, since E_δ is an embedding of a disk, all other horizontal slices for $0 > d > -1 + \varepsilon$ also connect b_1 to b_2 and b_{-1} to b_{-2}, and the components of E_δ which are bounded by those connecting pieces in $E_\delta \cap \{x: x_3 = -1 + \varepsilon\}$ together with $C_\delta \cap \{x: x_3 > -1 + \varepsilon\}$ consist of two topological oriented disks, E_1 and E_2. If E_1 is the disk containing most of b_1, then the projection of E_1 onto the plane with normal $(0, 2^{-1/2}, 2^{-1/2})$ must cover all but at most area $2\pi\varepsilon$ of an ellipse with major axis 1 and minor axis $2^{-1/2}$; thus E_1 must have area at least $\sqrt{2}\pi/2 - 2\pi\varepsilon$. Similarly E_2 must have area at least $\sqrt{2}\pi/2 - 2\pi\varepsilon$. By the ellipticity of F,

$$\boldsymbol{F}(E_1 + E_2 - T_1) + \boldsymbol{F}(-T_2) - \boldsymbol{F}(T_3) \geqq e(\mathscr{H}^2(E_1 + E_2 - T_1) + \mathscr{H}^2(T_2) - \mathscr{H}^2(T_3))$$

so

$$\begin{aligned} \boldsymbol{F}(E_\delta) &> \boldsymbol{F}(E_1 + E_2) \\ &\geqq \boldsymbol{F}(T_3) - \boldsymbol{F}(-T_3) - \boldsymbol{F}(T_2) + e\pi(\sqrt{2} - 1 - 4\varepsilon) \\ &\geqq \boldsymbol{F}(T_\delta) + e\pi[\sqrt{2} - (1 + 4\varepsilon + e^{-1}\max\{F(v): v \in S^2\}(4\delta + 5\varepsilon))] \\ &> \boldsymbol{F}(T_\delta)\,. \end{aligned}$$

This contradicts the $\boldsymbol{F}$-minimizing property E_δ is assumed to have; therefore for each $d \in (-1 + \varepsilon, 0)$, the slice of E_δ by horizontal plane at height d must connect b_1 to b_{-1} and b_2 to b_{-2}.

We have then that for almost every $d \in (-1 + \varepsilon, 0)$, the part of E bounded by $C_\delta L\{x: x_3 < d\}$ together with the curves connecting the b's, consists of two oriented topological disks. Let E_{1d} be the one having part of b_1 in its boundary. Using as comparison surfaces the appropriate parts of the strips between b_1 and b_{-1}, together with patches in horizontal slicing planes, we see that the length of the curve in $\{x: x_3 = d\}$ for most d between 0 and $-1 + \varepsilon$, is less than $\delta^{1/2}$. Since $-S_2$ is uniquely $\boldsymbol{F}$-minimal as an integral current, we see that the $\boldsymbol{F}$-minimal currents with boundary equal to E_{1d} must be close in the flat norm and in F-integral to $-S_2$ and $\boldsymbol{F}(-S_2)$ for small δ. Thus *any* embedding of a disk with boundary ∂E_{1d} must have $\boldsymbol{F}$ integral at least $\boldsymbol{F}(-S_2) - O(\delta)$. The same is true for the other topological disk E_{2d} and S_1. Now inequality (1) says that if δ is small enough, E_{1d} and E_{2d} being disjoint implies $\boldsymbol{F}(T) < \boldsymbol{F}(E)$, contradicting our assumption on E_δ.

Thus if δ is small enough, there is no embedding of the oriented disk with boundary C_δ whose $\boldsymbol{F}$ integral is less than that of T_δ.

The analytic results are obtained by approximation.

References

1. F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure,* Ann. Math., **87** (1968), 321-391.
2. F. J. Almgren, Jr. and L. Simon, *Existence of embedded solutions of Plateau's problem,* Ann. Scuola Nor. Sup. Pisa **6** Ser. 4 (1979), 447-495.
3. W. H. Meeks, III and S. T. Yau, *The classical Plateau problem and the topoljgy of 3-manifolds,* Arch. Rational Mech. Anal., (to appear).
4. R. Schoen, L. Simon, and F. J. Almgren, Jr., *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals*, Acta Math., **139** (1977), 217-265.
5. J. E. Taylor, *F-minimal hypersurfaces characterize F up to a point,* preprint.
6. F. Tomi and A. J. Tromba, *Extreme curves bound embedded minimal surfaces of the type of the disk,* Math. Zeit., **158** (1978), 137-145.

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