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FIXED POINT SETS OF 1-DIMENSIONAL PEANO CONTINUA

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It is shown that every nonempty closed subset of a 1-dimensional Peano continuum X is the fixed point set of some continuous self-mapping of X.

1. Introduction. A topological space X is said to have the *complete invariance property* (CIP) if every nonempty closed subset of X is the fixed point set of some continuous self-mapping of X. The term CIP was suggested by L. E. Ward, Jr. in [5, p. 553] where it was asked if every Peano continuum had CIP. Examples have been given in [3], [4, 3.1] which show that n-dimensional Peano continua need not have CIP if $n > 1$. In [4, 3.4] it is asked if every 1-dimensional Peano continuum has CIP. The purpose of this note is to answer that question in the affirmative by showing that every 1-dimensional Peano continuum has CIP.

2. Preliminaries. Let M be a metric space. A sequence of subsets of M is called a *null sequence* provided that for any $\varepsilon > 0$ at most a finite number of its elements has diameter greater than ε. The space M is said to have *property* S provided that for each $\varepsilon > 0$, M is the union of a finite number of connected sets each of diameter less than ε. A *partitioning* of M is a finite collection $\mathscr{U}$ of pairwise disjoint connected open subsets of M whose union is dense in M. If the mesh of $\mathscr{U}$ is less than ε (each element of $\mathscr{U}$ is of diameter less than ε), $\mathscr{U}$ is called on *ε-partitioning*. A sequence $\mathscr{U}_1$, $\mathscr{U}_2, \cdots$ of partitionings is called a *decreasing sequence* if, for each positive integer i, $\mathscr{U}_{i+1}$ is a refinement of $\mathscr{U}_i$ and the mesh of $\mathscr{U}_i$ approaches 0 as i increases without limit. It is well-known [1, p. 545] that every Peano continuum has a decreasing sequence of partitionings.

A *dendron* is a connected, simply connected, finite graph. The closure of a subset A of a topological space shall be denoted by $\mathrm{Cl}(A)$.

3. The result.

THEOREM *Every* 1-*dimensional Peano continuum has the complete invariance property.*

Proof. Let X be a 1-dimensional Peano continuum and let A be a closed subset of X. Let $\mathscr{U}_1, \mathscr{U}_2, \cdots$ be a decreasing sequence of $(1/i)$-partitionings of X. Then each $\mathscr{U}_i$ is a finite collection of open connected pairwise disjoint sets of diameter less than $1/i$ such that for each i $\bigcup\{U \mid U \in \mathscr{U}_i\}$ is dense in X and $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$. We suppose

$$\mathscr{U}_1 = \{U_{1,1}, \cdots, U_{1,m_{0,1}}\}$$
$$\mathscr{U}_2 = \{U_{2,i,j} \mid U_{2,i,j} \subset U_{1,i} \in \mathscr{U}_1 \text{ and } j \in \{1, \cdots, m_{1,i}\}\}$$

and for $i > 1$

$$\mathscr{U}_i = \{U_{i,j_1,\cdots,j_i} \mid U_{i,j_1,\cdots,j_i} \subset U_{i-1,j_1,\cdots,j_{i-1}} \in \mathscr{U}_{i-1} \text{ and } j_i \in \{1, \cdots, m_{i-1,j_1,\cdots,j_{i-1}}\}\} .$$

For each $i = 1, 2, \cdots$ let

$$\mathscr{U}_i' = \{U \in \mathscr{U}_i \mid \mathrm{Cl}(U) \cap A \neq \varnothing\} .$$

Without loss of generality,

$$\mathscr{U}_1' = \{U_{1,1},\ldots, U_{1,n_{0,1}}\} \quad \text{and for} \quad i > 1$$
$$\mathscr{U}_i' = \{U_{i,j_1,\cdots,j_i} \mid U_{i,j_1,\cdots,j_i} \subset U_{i-1,j_1,\cdots,j_{i-1}} \in \mathscr{U}_{i-1}' \text{ and } j_i \in \{1, \cdots, n_{i-1,j_1,\cdots,j_{i-1}}\}\} .$$

Notice that $A \subset \mathrm{Cl}(\bigcup \mathscr{U}_i')$ for each i.

Let $A_{1,1}$ be an arc in X which meets $U_{1,1}$ and $U_{1,2}$. If $A_{1,1} \cap U_{1,3} \neq \varnothing$ let $A_{1,2} = \varnothing$. If $A_{1,1} \cap U_{1,3} = \varnothing$ let $A_{1,2}$ be an arc such that $A_{1,2}$ meets $U_{1,3}$ and $A_{1,1} \cap A_{1,2}$ is an endpoint of $A_{1,2}$. Suppose $A_{1,1} \cup \cdots \cup A_{1,i}$ is a finite dendron such that $A_{1,1} \cup \cdots \cup A_{1,i}$ meets $U_{1,j}$ for each $j \in \{1, \cdots, i+1\}$. If $i + 2 \leqq n_{0,1}$ let $A_{1,i+1} = \varnothing$ if $(A_{1,1} \cup \cdots \cup A_{1,i}) \cap U_{1,i+2} \neq \varnothing$, otherwise, let $A_{1,i+1}$ be an arc which meets $U_{1,i+2}$ and such that $(A_{1,1} \cup \cdots \cup A_{1,i}) \cap A_{i,i+1}$ is an endpoint of $A_{1,i+1}$. By induction $A_{1,i}$ is defined for each $i \in \{1, \cdots, n_{0,1} - 1\}$. Let

$$B_1 = A_{1,1} \cup \cdots \cup A_{1,n_{0,1}-1} .$$

Suppose $B_1, \cdots, B_k$ are finite dendrons such that $B_1 \subset B_2 \subset \cdots \subset B_k$, B_k meets U for each $U \in \mathscr{U}_k'$ and

$$B_k - B_{k-1} \subset \bigcup\{U \mid U \in \mathscr{U}_k'\} .$$

For each $U_{k,j_1,\cdots,j_k} \in \mathscr{U}_k'$ let $A_{k+1,j_1,\cdots,j_k,1} = \varnothing$ if B_k meets $U_{k+1,j_1,\cdots,j_k,1}$, otherwise, let $A_{k+1,j_1,\cdots,j,1}$ be an arc in $U_{k,j_1,\cdots,j_k}$ which meets $U_{k+1,j_1,\cdots,j_k,1}$ and such that $B_k \cap A_{k+1,j_1,\cdots,j_k,1}$ is an endpoint of $A_{k+1,j_1,\cdots,j_k,1}$. Let $U_{k,j_1,\cdots,j_k} \in \mathscr{U}_k'$ and suppose $A_{k+1,j_1,\cdots,j_k,i}$ is defined for $i \in \{1, \cdots, m\}$ where $m < n_{k,j_1,\cdots,j_k}$. If $B_k \cup \bigcup_{i=1}^m A_{k+1\ j_1,\cdots,j_k,i}$ meets $U_{k+1,j_1,\cdots,j_k,m+1}$ let $A_{k+1,j_1,\cdots,j_k,m+1} = \varnothing$, otherwise, let $A_{k+1,j_1,\cdots,j_k,m+1}$ be an arc in $U_{k,j_1,\cdots,j_k}$ which meets $U_{k+1j,\,1,\cdots,j_k,m+1}$ and such that $(B_k \cup \bigcup_{i=1}^m A_{k+1,j_1,\cdots,j_k,i}) \cap A_{k+1,j_1,\cdots,j_k,m+1}$ is an endpoint of $A_{k+1,j_1,\cdots,j_k,m+1}$. Let

$$B_{k+1} = B_k \cup \bigcup \{A_{k+1\, j_1,\cdots,j_k,j_{k+1}} \mid U_{k+1\, j_1,\cdots,j_k,j_{k+1}} \in \mathscr{U}'_{k+1}\} .$$

By induction B_k is defined for each $k = 1, 2, \cdots$.

Let $B = A \cup B_1 \cup B_2 \cup \cdots$. Then B is connected since $B_1 \subset B_2 \subset \cdots$, each B_i is connected and $\bigcup B_i$ is dense in B. The set B is compact since $B - U$ is contained in a finite dendron for each open neighborhood U of A and A is compact. It is easy to show that B has property S. To see this, let $\varepsilon > 0$ and let n be a positive integer such that $3/n < \varepsilon$. Since B_n has property S, there is a positive integer m and continua $K_1, \cdots, K_m$ such that $B_n = K_1 \cup \cdots \cup K_m$ and each K_i has diameter $< 1/n$. Let $U \in \mathscr{U}_n'$. Let $K_{i_1}, \cdots, K_{i_r}$ be the members of $\{K_1, \cdots, K_m\}$ which meet U. Then $(K_{i_1} \cup \cdots \cup K_{i_r} \cup U) \cap B$ has at most i_r components, and each of these has diameter $< 3/n < \varepsilon$. It follows that B has property S and hence is locally connected (see [6, p. 20]). By [2, p. 174] B is a retract of X.

It suffices to prove that there is a continuous mapping $f: B \to B$ such that $f(x) = x$ if and only if $x \in A$. Since B is locally connected, each component of $B - A$ is open in B. Hence, $B - A$ has at most countably many components $C_1, C_2, \cdots$. Notice that every component of $B - A$ is a simply connected local graph. It follows from the last sentence and from the construction of the sets B_k that every sequence of pairwise disjoint arcs in $B - A$ is a null sequence. Hence, the sequence $C_1, C_2, \cdots$ is null. It suffices to prove, therefore, that for each $i \geqq 1$ there exists a continuous mapping $g_i: Cl(C_i) \to Cl(C_i)$ such that $g_i(x) = x$ if and only if $x \in Cl(C_i) - C_i$. The existence of g_i follows easily from the fact that C_i is a simply connected local graph in which every sequence of pairwise disjoint arcs is null.

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