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WEAK CHEBYSHEV SUBSPACES AND ALTERNATION

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Let T be a locally compact subset of R and $C_0(T)$ the space of continuous function which vanish at infinity. An n dimensional subspace G of $C_0(T)$ may possess one of the three alternation properties:

(A-1) For each $f \in C_0(T)$ which has a unique best approximation $g_0 \in G, f - g_0$ has $n+1$ alternating peak points;

(A-2) For each $f \in C_0(T)$, there exists a best approximation $g_0 \in G$ to f such that $f - g_0$ has $n+1$ alternating peak points;

(A-3) For each $f \in C_0(T)$ and each best approximation $g_0 \in G$ to $f, f - g_0$ has $n+1$ alternating peak points.

In this paper, for each $i \in \{1, 2, 3\}$ we give an intrinsic characterization of those subspaces G of $C_0(t)$ which have property (A-i).

1. Introduction. The classical alternation theorem states that if G is an n dimensional Chebyshev subspace of $C[a, b]$, then for each $f \in C[a, b]$ and its unique best approximation $g_0 \in G$, the error $f - g_0$ has $n+1$ alternating peak points. It is natural to ask whether such a result remains valid if we replace $C[a, b]$ by $C(T)$, where T is an arbitrary compact subset of the real line R or, more generally, by $C_0(T)$, where T is any locally compact subset of R. [Here $C_0(T)$ denotes the Banach space of all real-valued contionuous functions f on T "vanishing at infinity" (i.e., $\{t \in T \mid |f(t)| \geqq \varepsilon\}$ is compact for each $\varepsilon > 0$), and endowed with the supremum norm: $\|f\| = \sup_{t \in T} |f(t)|$. When T is actually compact, we often write $C(T)$ for $C_0(T)$.] And if such a result is not valid, characterize those n dimensional subspaces G of $C_0(T)$ for which the result does hold.

Properties (A-1) and (A-2) above, in the special case $T = [a, b]$, have been considered by Jones and Karlovitz [6] who proved that an n dimensional subspace G of $C[a, b]$ has property (A-1) if and only if G has property (A-2) if and only if G is "weak Chebyshev" (i.e. G has property (W-4) defined below). Furthermore, Handscomb, Mayers, and Powell [5; Theorem 8] showed that an n dimensional subspace G of $C[a, b]$ has property (A-3) (if and) only if G is a Chebyshev subspace. (The "if" part is just the classical alternation theorem.)

In this paper, for each $i \in \{1, 2, 3\}$, we give *intrinsic characterizations* of theose subspaces G of $C_0(T)$ which have property (A-i).

It turns out that, contrary to the case when $T = [a, b]$, properties (A-1) and (A-2) are *not* the same in general; and property (A-3) does *not* characterize Chebyshev subspaces. In giving our characterizations of the alternation properties, the following kinds of "weak Chebyshev" subspaces play the major role. (In the definition below, the letter "W" is an abbreviation for "weak Chebyshev".)

DEFINITION 1.1. An n dimensional subspace G of $C_0(T)$ is said have property

(W-1). If for each $1 \leqq m \leqq n$ and each set of points $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ with $t_i \in T (i = 1, 2, \cdots, m-1)$, there exists $0 \neq g \in G$ such that

$$(-1)^i g(t) \geqq 0 \text{ for all } t \in [t_i, t_{i+1}) \cap T \quad (i = 0, 1, \cdots, m-1) ;$$

(W-1'). If it satisfies the condition of property (W-1) only for $m = n$;

(W-2). If for each $1 \leqq m \leqq n$ and each set of points $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ with $t_i \in T$ $(i = 1, 2, \cdots, m-1)$, there exists $0 \neq g \in G$ such that

$$(-1)^i g(t) \geqq 0 \text{ for } t \in [t_i, t_{i+1}] \cap T \quad (i = 0, 1, \cdots, m-1) ;$$

(W-2'). If it satisfies the condition of property (W-2) only for $m = n$;

(W-3). If for each basis $\{g_1, g_2, \cdots, g_n\}$ of G and each set of points $t_1 < t_2 < \cdots t_n$ and $s_1 < s_2 < \cdots < s_n$ in T,

$$D\begin{pmatrix} g_1 g_2 \cdots g_n \\ t_1 t_2 \cdots t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix} \geqq 0 ,$$

where

$$D\begin{pmatrix} g_1 g_2 \cdots g_n \\ r_1 r_2 \cdots r_n \end{pmatrix} = \begin{vmatrix} g_1(r_1) \cdots g_1(r_n) \\ g_2(r_1) \cdots g_2(r_n) \\ \cdots \\ g_n(r_1) \cdots g_n(r_n) \end{vmatrix} \equiv \det [g_i(r_j)] ;$$

(W-4). If each $g \in G$ has at most $n-1$ sign changes, i.e., there do *not* exist $n+1$ points $t_1 < t_2 < \cdots < t_{n+1}$ in T with $g(t_i)g(t_{i+1}) < 0$ $(i = 1, 2, \cdots, n)$.

In §2, we study the various relationships between these weak Chebyshev properties. The main result here is Lemma 2.2. In §3, we establish that property (A-1) is equivalent to property (W-1) (Theorem 3.1). In §4, we prove that property (A-2) is equivalent to each of the (equivalent) properties (W-2), (W-2'), (W-3), and (W-4) (Theorem 4.1). In §5, we show that property (A-3) is equivalent to G being Chebyshev and having one of the equivalent properties (W-2), (W-2′), (W-3), and (W-4) (Theorem 5.1). This allows us to give an example (Example 5.4) showing that the Handscomb, Mayers, and Powell characterization of Chebyshev subspaces is *not* valid in general if T is not an interval. In §6, we give some examples of weak Chebyshev subspaces which are not Chebyshev. In §7, we characterize the n dimensional Chebyshev subspaces of $C_0(T)$ for certain locally compact Hausdorff spaces T (including T metric, but not necessarily a subset of $\boldsymbol{R}$).

It is worth mentioning here the motivation for the original use of the term "weak Chebyshev". Recall the classical result that *an n dimensional subspace G of $C[a, b]$ is Chebyshev if and only if for any basis $\{g_1, g_2, \cdots, g_n\}$* of G *and each set of points $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$ in* $[a, b]$,

$$D\begin{pmatrix} g_1 g_2 \cdots g_n \\ t_1 t_2 \cdots t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix} > 0 .$$

Karlin and Studden [7] generalized this determinant criterion and defined a *weak Chebyshev* subspace in $C[a, b]$ as one having what we have called property (W-3). It is mainly for this historical reason that we have kept the term "weak Chebyshev subspace". However, in contrast to the case when $T = [a, b]$, not every Chebyshev subspace of $C_0(T)$ has property (W-3) (see Example 3.3).

We conclude the introduction by recalling some basic terminology and notation. A *best approximation* to $f \in C_0(T)$ from G is any element $g_0 \in G$ such that $\|f - g_0\| = \inf_{g \in G} \|f - g\|$. The set of all best approximations to f from G will be denoted by $P_G(f)$. G is called a *Chebyshev subspace* if $P_G(f)$ is a single element for each $f \in C_0(T)$. An n dimensional subspace G of $C_0(T)$ is called a *Haar subspace* if 0 is the only element of G having n (or more) zeros in T. It is well known (at least when T is compact) that G is a Haar subspace if and only if it is Chebyshev. A *peak point* for $f \in C_0(T)$ is any $t \in T$ with $|f(t)| = \|f\|$. (This differs from what many authors call "peak points".) A set of points $t_1 < t_2 < \cdots < t_k$ in T are called *alternating peak points* for f if each t_i is a peak point of f and the $f(t_i)$ alternate in sign, i.e., $f(t_i) = \sigma(-1)^i \|f\| (i = 1, 2, \cdots, k)$ for some $\sigma \in \{-1, 1\}$. By an *interval* in $\boldsymbol{R}$, we shall mean any set of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$, where $-\infty \leqq a < b \leqq \infty$ and $a = -\infty$ or $b = \infty$ is possible on the open end. Note that every interval in $\boldsymbol{R}$ is locally compact.

Throughout this paper, unless explicitly stated otherwise, we assume that *n is some arbitrary but fixed positive integer and T is a locally compact subset of $\boldsymbol{R}$ which contains at least $n + 1$ points.*

2. Weak Chebyshev subspaces. We shall use the following topological result [1] (I. 9.7. Propositions 12 and 13, and I. 3.3, Proposition 5).

LEMMA 2.1. *A subset Y of a locally compact Hausdorff space X is locally compact* $\Leftrightarrow \bar{Y}\backslash Y$ *is closed. In particular,* $\bar{T}\backslash T$ *is closed.*

It is sometimes a useful technical device to extend the functions in $C_0(T)$ to functions defined on the smallest closed interval IT with contains T (i.e., the intersection of all closed intervals containing T). Since each function $f \in C_0(T)$ is uniformly continuous, it has a unique extension on to a continuous function $\tilde{f}$ on $\bar{T}$. Obviously, we must have $\tilde{f} = 0$ on $\bar{T}\backslash T$. Since $\boldsymbol{R}\backslash\bar{T}$ is open, it has a unique representation as a countable union of disjoint open intervals. Hence also $IT\backslash\bar{T} = \bigcup_j I_j$, where (I_j) is a countable collection of disjoint open intervals. We now define $\tilde{f}$ on $IT\backslash\bar{T}$ by extending $\tilde{f}$ linearly across each interval I_j. It is easy to verify that the resulting function $\tilde{f}$ is in $C_0(IT)$.

Summarizing, each $f \in C_0(T)$ can be extended to a unique function $\tilde{f} \in C_0(IT)$ defined by $\tilde{f} = 0$ on $\bar{T}\backslash T$ and $\tilde{f}$ is linear on each of the disjoint open subintervals whose union is $IT\backslash\bar{T}$. In the sequel, the notation $\tilde{f}$ will be reserved for this unique extension of f to all of IT, and we let $\tilde{G} = \{\tilde{g} \mid g \in G\}$ denote the extension of the corresponding subspace G.

LEMMA 2.2. *Let G be an n dimensional subspace of* $C_0(T)$. *Consider the following statements*:

(1) *G has property* (W-1);
(1′) *G has property* (W-1′);
(2) *G has property* (W-2);
(2′) *G has property* (W-2′);
(3) *G has property* (W-3);
(4) *G has property* (W-4).

Then $(1') \Leftarrow (1) \Leftarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (4)$. *Moreover,* $(1) \not\Rightarrow (2)$ *and* $(1') \not\Rightarrow (1)$ *in general. In the case $n = 1$, all the properties are equivalent to the existence of a nonzero function $g \in G$ with $g(t) \geqq 0$ for all $t \in T$.*

Proof. The last statement is obvious as are the implications $(2) \Rightarrow (2')$ and $(2) \Rightarrow (1) \Rightarrow (1')$.

$(2') \Rightarrow (3)$. Let $s_1 < s_2 < \cdots < s_n$ in T be such that $D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix} \neq 0$. For each integer $k \in \{1, 2, \cdots, n\}$ define $u_k \in G$ by

$$(1) \qquad u_k(x) = D\begin{pmatrix} g_1 g_2 \cdots g_k & g_{k+1} \cdots g_n \\ x s_1 \cdots s_{k-1} & s_{k+1} \cdots s_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix}^{-1} .$$

Note that

$$(2) \qquad u_k(s_i) = (-1)^{k-1}\delta_{ki} \quad (i, k = 1, 2, \cdots, n) .$$

Set $x_0 = -\infty$, $x_i = s_i$ $(i = 1, \cdots, k-1)$, $x_i = s_{i+1}$ $(i = k, \cdots, n-1)$, and $x_n = \infty$. By property (W-2′), there exists $0 \neq v_k \in G$ such that

$$(-1)^i v_k(x) \geqq 0 \quad \text{for all} \quad x \in [x_i, x_{i+1}] \cap T$$

$(i = 0, 1, \cdots, n-1)$. In particular, $v_k(s_i) = 0$ for all $i \neq k$ and $(-1)^{k-1} v_k(s_k) \geqq 0$. Since $\{u_1, u_2, \cdots, u_n\}$ is a basis for G, it follows using eq. (2) that $v_k = \lambda_k u_k$ for some $\lambda_k > 0$.

Now let $t_1 < t_2 < \cdots < t_n$ in T with $D\begin{pmatrix} g_1 g_2 \cdots g_n \\ t_1 t_2 \cdots t_n \end{pmatrix} \neq 0$. Suppose there is a k such that $s_k \notin \{t_1, t_2, \cdots, t_n\}$. Since $u_k \neq 0$, it follows that there is an m such that $u_k(t_m) \neq 0$. From eq. (2), $t_m \notin \{s_1, \cdots, s_{k-1}, s_{k+1}, \cdots, s_n\}$. Set $\{r_1, r_2, \cdots, r_n\} = \{s_1, \cdots, s_{k-1}, s_{k+1}, \cdots, s_n\} \cup \{t_m\}$ with $r_i < r_{i+1}$ for all i. Then $t_m \in (x_i, x_{i+1})$ for some $i \in \{0, 1, \cdots, n-1\}$ implies

$$\begin{aligned} 0 &\leqq (-1)^i v_k(t_m) = (-1)^i \lambda_k u_k(t_m) \\ &= \lambda_k D\begin{pmatrix} g_1 g_2 \cdots g_n \\ r_1 r_2 \cdots r_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix}^{-1} \end{aligned}$$

so

$$D\begin{pmatrix} g_1 g_2 \cdots g_n \\ r_1 r_2 \cdots r_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix} > 0 .$$

By a repeated application of this argument, we obtain that

$$D\begin{pmatrix} g_1 g_2 \cdots g_n \\ t_1 t_2 \cdots t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1 g_2 \cdots g_n \\ s_1 s_2 \cdots s_n \end{pmatrix} > 0 .$$

Thus G has property (W-3).

At this point it is convenient to isolate some useful facts which will simplify the proof of Lemma 2.2 and are of independent interest.

CLAIM 1. G has property (W-3) (resp. ((W-4)) in $C_0(T) \Leftrightarrow \tilde{G}$ has property (W-3)(resp. (W-4)) in $C_0(IT)$.

Proof of Claim 1. If $\tilde{G}$ has property (W-3)(resp. (W-4)) in $C_0(IT)$, then the restriction $G = \tilde{G}|_T$ obviously has property (W-3) (resp. (W-4)) in $C_0(T)$.

Next suppose G has property (W-3) in $C_0(T)$. If $\tilde{G}$ fails to have property (W-3) in $C_0(IT)$, there exist points $\tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_n$ and $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n$ in IT such that

$$D\begin{pmatrix}\tilde{g}_1 \cdots \tilde{g}_n \\ \tilde{s}_1 \cdots \tilde{s}_n\end{pmatrix} < 0 < D\begin{pmatrix}\tilde{g}_1 \cdots \tilde{g}_n \\ \tilde{t}_1 \cdots \tilde{t}_n\end{pmatrix} .$$

Let k be the smallest index such that $\tilde{t}_k \notin T$, and set $t_i = \tilde{t}_i \in T$ $(i = 1, 2, \cdots, k-1)$. Define

$$\tilde{g}(t) = D\begin{pmatrix}\tilde{g}_1 \cdots \tilde{g}_{k-1}\tilde{g}_k\tilde{g}_{k+1} \cdots \tilde{g}_n \\ t_1 \cdots t_{k-1}t\tilde{t}_{k+1} \cdots \tilde{t}_n\end{pmatrix} \quad (t \in IT) .$$

Then $\tilde{g} \in \tilde{G}$ and $\tilde{g}(\tilde{t}_k) > 0$. Now $\tilde{t}_k \in I_m = (a, b)$ for some open interval I_m (where $IT\backslash\bar{T} = \bigcup_1^\infty I_j$), and $\tilde{g}$ is linear on I_m implies $\tilde{g}(a) > 0$ or $\tilde{g}(b) > 0$. We may assume $\tilde{g}(a) > 0$. But the endpoints of I_m lie in the boundary of $IT\backslash\bar{T}$, and hence in $\bar{T}$. Since $\tilde{g} = 0$ on $\bar{T}\backslash T$, $a \in T$. Set $t_k = a$. Then

$$D\begin{pmatrix}\tilde{g}_1 \cdots \tilde{g}_k \cdots \tilde{g}_n \\ t_1 \cdots t_k\tilde{t}_{k+1} \cdots \tilde{t}_n\end{pmatrix} = \tilde{g}(t_k) > 0 .$$

Continuing in this way with $\tilde{t}_{k+1}, \cdots, \tilde{t}_n$, we obtain points $t_1 < t_2 < \cdots < t_n$ in T such that

$$D\begin{pmatrix}g_1 \cdots g_n \\ t_1 \cdots t_n\end{pmatrix} = D\begin{pmatrix}\tilde{g}_1 \cdots \tilde{g}_n \\ t_1 \cdots t_n\end{pmatrix} > 0 ,$$

where $g_i = \tilde{g}_i|_T \in G$. Similarly, we obtain points $s_1 < s_2 < \cdots < s_n$ in T such that

$$D\begin{pmatrix}g_1 \cdots g_n \\ s_1 \cdots s_n\end{pmatrix} < 0 .$$

But this contradicts G having property (W-3). Thus $\tilde{G}$ must have property (W-3) in $C_0(IT)$.

Now let G have property (W-4) in $C_0(T)$. If $\tilde{G}$ fails to have property (W-4) in $C_0(IT)$, there exist $\tilde{g} \in \tilde{G}$ and points $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n$ in IT such that $\tilde{g}(\tilde{t}_i)\tilde{g}(\tilde{t}_{i+1}) < 0$ $(i = 1, 2, \cdots, n)$. If all $\tilde{t}_i$ are in T, then the function $g = \tilde{g}|_T \in G$ satisfies

$$g(\tilde{t}_i)g(\tilde{t}_{i+1}) < 0 \quad (i = 1, 2, \cdots, n)$$

which contradicts G having property (W-4) in $C_0(T)$. Thus let k be the smallest index such that $\tilde{t}_k \notin T$ and set $t_i = \tilde{t}_i \in T$ for $i = 1, 2, \cdots, k-1$. Since $\tilde{g} = 0$ on $\bar{T}\backslash T$, $\tilde{t}_k \in IT\backslash\bar{T}$ so $\tilde{t}_k \in I_m$ for some open interval I_m. We may assume $\tilde{g}(\tilde{t}_k) < 0$. By the same argument as in the above proof of the implication "G has (W-3) $\Rightarrow \tilde{G}$ has (W-3)", we obtain a point $t_k \in T$ such that

$$t_1 < t_2 < \cdots < t_k < \tilde{t}_{k+1} < \cdots < \tilde{t}_{n+1}$$

and $\tilde{g}(t_k) < 0$. Continuing in this way with $\tilde{t}_{k+1}, \cdots, \tilde{t}_{n+1}$, we obtain points $t_1 < t_2 < \cdots < t_{n+1}$ in T such that

$$g(t_i)g(t_{i+1}) = \tilde{g}(t_i)\tilde{g}(t_{i+1}) < 0 \quad (i = 1, 2, \cdots, n) ,$$

where $g = \tilde{g}|_T \in G$. But this contradicts G having property (W-4) in $C_0(T)$.

CLAIM 2. Let I be an interval in $\boldsymbol{R}$, H an n dimensional subspace of $C_0(I)$, and (I_j) an increasing sequence of compact intervals such that $I = \bigcup_1^\infty I_j$ and $H|_{I_1}$ is n dimensional. Then, for any given $i \in \{1, 1', 2, 2', 3, 4\}$, H has property (W-i) in $C_0(I) \Leftrightarrow H|_{I_j}$ has property (W-i) in $C_0(I_j)$ for each j.

Proof of Claim 2. Clearly, if H has property (W-i) in $C_0(I)$, then the restriction $H|_{I_j}$ has property (W-i) in $C_0(I_j)$ for each j.

Conversely, suppose first that $H|_{I_j}$ has property (W-2′) in $C_0(I_j)$ for each j. Let $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$, where $t_i \in I$ $(i = 1, 2, \cdots, n-1)$. Choose N sufficiently large that $t_i \in I_N$ $(i = 1, 2, \cdots, n-1)$. For each $k \geqq N$ there exists $h_k \in H|_{I_k} \backslash \{0\}$ such that

$$(-1)^i h_k(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}] \cap I_k$$

$(i = 0, 1, \cdots, n-1)$. Choose $g_k \in H$ such that $h_k = g_k|_{I_k}$ and let g be a cluster point of the sequence $(g_k/\|g_k\|)$. Then $g \in H\backslash\{0\}$ and

$$(-1)^i g(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}] \cap I$$

$(i = 0, 1, \cdots, n-1)$. Thus H has property (W-2′) in $C_0(I)$.

The proof of the implication "$\Leftarrow$" in the case when $i = 1, 1'$, or 2 is similar to the case $i = 2'$ proved above.

Next, assume that $H|_{I_j}$ has property (W-4) in $C_0(I_j)$ for each j. If H fails to have property (W-4) in $C_0(I)$, there exist points $t_1 < t_2 < \cdots < t_n$ in I and $h \in H$ such that $h(t_i)h(t_{i+1}) < 0$ $(i = 1, 2, \cdots, n)$. Choose N sufficiently large that $t_i \in I_N$ for all i. Then $h_N = h|_{I_N} \in H|_{I_N}$ satisfies

$$h_N(t_i)h_N(t_{i+1}) < 0 \quad (i = 1, 2, \cdots, n)$$

which contradicts $H|_{I_N}$ having property (W-4) in $C_0(I_N)$. Thus H has property (W-4) in $C_0(I)$.

The proof of the implication "$\Leftarrow$" in the case when $i = 3$ is similar to the above proof when $i = 4$.

CLAIM 3. Let I be an interval in $\boldsymbol{R}$ and H an n dimensional subspace of $C_0(I)$. Then H has one of the properties (W-2′), (W-3), or (W-4) $\Leftrightarrow H$ has them all.

Proof of Claim 3. There exists an increasing sequence of compact intervals (I_j) such that $I = \bigcup_1^\infty I_j$ and $H|_{I_1}$ is n dimensional. By Jones-Karlovitz [6], $H|_{I_j}$ has one of the properties (W-2′), (W-3), or (W-4) in $C_0(I_j) \Leftrightarrow H|_{I_j}$ has them all. The result now follows from Claim 2.

CLAIM 4. Let I be an interval in $\boldsymbol{R}$ and H an n dimensional subspace of $C_0(I)$. Then H has property (W-2′) $\Leftrightarrow H$ has property (W-2).

Proof of Claim 4. The implication "$\Leftarrow$" is obvious. Thus assume H has property (W-2′). Let $1 \leqq m \leqq n$ and $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ with $t_i \in I$ $(i = 1, 2, \cdots, m-1)$. We may assme $m < n$. If $t_{m-1} < \sup I$, choose points $t_m^{(k)} < t_{m+1}^{(k)} < \cdots < t_{n-1}^{(k)} < t_n^{(k)} = \infty$, with $t_{m-1} < t_m^{(k)}$, $t_i^{(k)} \in I$ $(i = m, m+1, \cdots, n-1)$, and $t_m^{(k)} \to \sup I$ as $k \to \infty$. Define $t_i^{(k)} = t_i$ if $0 \leqq i \leqq m-1$. If $t_{m-1} = \sup I$, choose points $t_{m-1}^{(k)} < t_m^{(k)} < \cdots < t_{n-1}^{(k)} < t_n^{(k)} = \infty$, with $t_{m-2} < t_{m-1}^{(k)}$, $t_i^{(k)} \in I$ $(i = m-1, m, \cdots, n-1)$, and $t_{m-1}^{(k)} \to \sup I$ as $k \to \infty$. Define $t_i^{(k)} = t_i$ if $0 \leqq i \leqq m-2$. In either case, there exists $h_k \in H \backslash \{0\}$ such that

$$(-1)^i h_k(t) \geqq 0 \quad \text{for all} \quad t \in [t_i^{(k)}, t_{i+1}^{(k)}] \cap I$$

$(i = 0, 1, \cdots, n-1)$, and all k. Let h be any cluster point of the sequence $(h_k/\|h_k\|)$. Then $h \in H \backslash \{0\}$ and

$$(-1)^i h(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}] \cap I$$

$(i = 0, 1, \cdots, m-1)$. Hence H has property $(W-2)$.

We can now easily complete the proof of Lemma 2.2.

(3) $\Rightarrow$ (4). If G has property (W-3) in $C_0(T)$, then Claim 1 implies $\tilde{G}$ has property (W-3) in $C_0(IT)$. By Claim 3, $\tilde{G}$ has property (W-4) in $C_0(IT)$. By Claim 1, G has property (W-4) in $C_0(T)$.

(4) $\Rightarrow$ (2). If G has property (W-4) in $C_0(T)$, then $\tilde{G}$ has property (W-4) in $C_0(IT)$ by Claim 1. By Claim 3, $\tilde{G}$ has property (W-2′) in $C_0(IT)$. By Claim 4, $\tilde{G}$ has property (W-2) in $C_0(IT)$. Clearly, $G = \tilde{G}|_T$ has property (W-2) in $C_0(T)$.

We show that (1) $\not\Rightarrow$ (2) and (1′) $\not\Rightarrow$ (1) in examples below. This completes the proof.

The proof of the implication (2′) $\Rightarrow$ (3) is an obvious modification of the proof given in [6] for the special case $T = [a, b]$. The implications (2′) $\Rightarrow$ (3) $\Leftrightarrow$ (4) have been verified independently by Zielke [10] using a different argument, and in the more general setting with G any n dimensional subspace of $\boldsymbol{R}^T$: the set of all real-valued functions on T, where T is any subset of $\boldsymbol{R}$.

The following two examples show that the the implications $(1) \Rightarrow (2)$ and $(1') \Rightarrow (1)$ in Lemma 2.2 are false in general.

2.3. Example of a subspace having property (W-1′) but not (W-1). Consider the set $T = \{1, 2, 3, 4\}$ and $G = \operatorname{span}\{g_1, g_2, g_3\} \subset C(T)$, where $g_1 = \delta_1 - \delta_4$, $g_2 = \delta_2 - \delta_4$, $g_3 = \delta_3 - \delta_4$, and $\delta_i(j) = 1$ if $i = j$, 0 if $i \neq j$. It is easy to see that there is no $g \neq 0$ in G such that $g \geqq 0$. Thus G fails (W-1). To see that G has property (W-1′), we show that for each pair of points $t_1 < t_2$ in T, there is a nonzero $g \in G$ such that $(-1)^i g(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, 2)$ (where $t_0 = -\infty$ and $t_3 = \infty$). We list all the possible choices of $t_1 < t_2$ and the corresponding g below. If $\{t_1, t_2\} = \{1, 2\}, \{1, 3\}$, or $\{1, 4\}$, take $g = -g_1$. If $\{t_1, t_2\} = \{2, 3\}$ or $\{2, 4\}$, take $g = g_1 - g_2$. If $\{t_1, t_2\} = \{3, 4\}$, take $g = -g_3$.

2.4. Example of a subspace having property (W-1) but not (W-2). Let T be the set of natural numbers and let $G = \operatorname{span}\{g_1, g_2\} \subset C_0(T)(= c_0)$, where $g_1 = \delta_1$, $g_2 = \delta_2 - \delta_3$, and $\delta_i(j) = 1$ if $i = j$, 0 otherwise. It is easy to check that G has property (W-1). However, the function $g = g_1 - g_2$ has two sign changes so G fails (W-4). By Lemma 2.2, G fails (W-2).

Under certain conditions on T (e.g., if T is an interval or if T is unbounded), the properties (W-1) and (W-1′) *are* equivalent. This is the content of the following result.

PROPOSITION 2.5. *Suppose that either T is unbounded or* $\inf T$ *or* $\sup T$ *is an accumulation point of T (in $\boldsymbol{R}$). Then an n dimensional subspace G of $C_0(T)$ has property* (W-1) $\Leftrightarrow$ *it has property* (W-1′).

Proof. By Lemma 2.2 it suffices to verify that (W-1′) $\Rightarrow$ (W-1). Let G have property (W-1′). If $\sup T$ is an accumulation point of T or if $\sup T = \infty$, then the same proof as given in Claim 4 of Lemma 2.2 shows that G has property (W-1). If $\inf T$ is an accumulation point of T or if $\inf T = -\infty$, a similar proof works.

We next give a condition which insures that property (W-1) is equivalent to (W-2).

DEFINITION 2.6. A function $\delta: T \to \boldsymbol{R}$ is called a *delta function* if δ is the characteristic function of a point in T. That is, for some $t_0 \in T$, $\delta = \chi_{t_0}$, where $\chi_{t_0}(t) = 0$ if $t \neq t_0$ and $\chi_{t_0}(t_0) = 1$.

Since a delta function χ_{t_0} is continuous iff t_0 is an isolated point of T, $C_0(T)$ contains delta functions iff T contains isolated points.

Observe also that a subspace G of $C_0(T)$ *cannot* contain a delta function if some $f \in C_0(T)\backslash G$ has a unique best approximation $g_0 \in G$. (For otherwise, some scalar multiple of the delta function, added to g_0, would another best approximation to f.) In particular, a Chebyshev subspace of $C_0(T)$ cannot contain delta functions.

PROPOSITION 2.7. *Let G be an n dimensional subspace of $C_0(T)$ which does not contain any delta function. Then G has property* (W-1) $\Leftrightarrow$ *G has property* (W-2).

Proof. It suffices by Lemma 2.2 to show that if G has property (W-1), it has property (W-2). Fix any integer m, $1 \leqq m \leqq n$. We must show that for each set of points $-\infty = t_0 < t_1 < \cdots < t_m = \infty$ with $t_i \in T$ $(i = 1, \cdots, m-1)$, there exists $0 \neq g \in G$ such that

(a) $(-1)^i g(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$

and

(b) $g(t_i) = 0$ $(i = 1, 2, \cdots, m-1)$.

If $m = 1$, condition (b) is vacuously satisfied and (a) follows by property (W-1). Thus we may assume $m > 1$. What we will show is that for each integer k, with $1 \leqq k \leqq m-1$, and each set of points $-\infty = t_0 < t_1 < \cdots < t_m = \infty$ with $t_i \in T$ $(i = 1, \cdots, m-1)$, there is $0 \neq g_k \in G$ such that $(-1)^i g_k(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$ and $g_k(t_i) = 0$ $(i = 1, 2, \cdots, k)$. Then the function $g = g_{m-1}$ will satisfy (a) and (b). We proceed by induction on k.

Assume first that $k = 1$. By property (W-1), there exists $0 \neq g \in G$ such that (a) holds. If $g(t_1) = 0$, set $g_1 = g$ and we are done. If $g(t_1) \neq 0$, then $g(t_1) < 0$.

Case 1. $t_1 = \sup T$.
Choose $g_0 \in G\backslash\{0\}$ such that $g_0(t) \geqq 0$ for all $t \in T$.

Case 2. $t_1 < \sup T$.
Let $\tau = \inf\{t \in T \mid t > t_1\}$. Then $t_1 \leqq \tau \leqq t_2$.
We consider three subcases.

Case 2.1. $\tau = t_1$.
Choose a sequence (τ_j) in T, $t_1 < \tau_j < t_2$, such that $\tau_j \to t_1 = \tau$. Set $t_i^{(j)} = t_i$ if $i \neq 1$ and $t_1^{(j)} = \tau_j$. Choose $g^{(j)} \in G$, $\|g^{(j)}\| = 1$, such that $(-1)^i g^{(j)}(t) \geqq 0$ for all $t \in [t_i^{(j)}, t_{i+1}^{(j)}) \cap T$ $(i = 0, 1, \cdots, m-1)$. Let g_0 be a cluster point of the sequence $(g^{(j)})$.

Case 2.2. $\tau = t_2$.
Set $\tilde{t}_0 = -\infty$ and $\tilde{t}_i = t_{i+2}$ $(i = 1, 2, \cdots, m-2)$. Choose $0 \neq g_0 \in G$ such that $(-1)^i g_0(t) \geqq 0$ for all $t \in [\tilde{t}_i, \tilde{t}_{i+1}] \cap T$ $(i = 0, 1, \cdots, m-3)$.

Case 2.3. $t_1 < \tau < t_2$.

If $\tau \in T$, let $\tilde{t}_i = t_i$ if $i \neq 1$ and $\tilde{t}_1 = \tau$. Choose $0 \neq g_0 \in G$ such that $(-1)^i g_0(t) \geqq 0$ for all $t \in [\tilde{t}_i, \tilde{t}_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$.

If $\tau \notin T$, choose a sequence (τ_j) in T with $\tau < \tau_j < t_2$ and $\tau_j \to \tau$. Set $t_i^{(j)} = t_i$ if $i \neq 1$ and $t_1^{(j)} = \tau_j$. Choose $g^{(j)} \in G$, $||g^{(j)}|| = 1$, such that $(-1)^i g^{(j)}(t) \geqq 0$ for all $t \in [t_i^{(j)}, t_{i+1}^{(j)}) \cap T$ $(i = 0, 1, \cdots, m-1)$. Let g_0 be a cluster point of the sequence $(g^{(j)})$.

In every case, we have obtained a function $g_0 \in G \backslash \{0\}$ satisfying $g_0(t)g(t) \geqq 0$ for all $t \in T \backslash \{t_1\}$ and $g_0(t_1) \geqq 0$. If $g_0(t_1) = 0$, set $g_1 = g_0$. If $g_0(t_1) \neq 0$, then $g_0(t_1) > 0$ and set $g_1 = g - (g(t_1)/g_0(t_1))g_0$. Since $\alpha = g(t_1)/g_0(t_1) < 0$, it follows that g_1 satisfies $(-1)^i g_1(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$, and $g_1(t_1) = 0$. If $g_1 = 0$, then $g = \alpha g_0$. Since G contains no delta functions, there exists a point $\tilde{t} \in T \backslash \{t_1\}$ such that $g_0(\tilde{t}) \neq 0$. Hence

$$0 \leqq g(\tilde{t})g_0(\tilde{t}) = \alpha[g_0(\tilde{t})]^2 < 0$$

which is absurd. Thus $g_1 \neq 0$ and the proof of the first step $k = 1$ is complete.

The proof of the induction step is analogous.

REMARK 2.8. Example 2.3 above is of a subspace G which does not contain any delta function, has property (W-1′), but not (W-1), and hence not (W-2), (W-2′), (W-3), or (W-4).

COROLLARY 2.9. *Let I be any interval in $\boldsymbol{R}$ and let G be n dimensional subspace of $C_0(I)$. Then G has one of the following properties $\Leftrightarrow$ it has then all*: (W-1), (W-1′), (W-2), (W-2′), (W-3), (W-4).

Proof. Since I contains no isolated point, $C_0(I)$ contains no delta function.

If I were bounded, then both $\sup I$ and $\inf I$ would be accumulation points of I. The result now follows by Lemma 2.2, Proposition 2.5, and Proposition 2.7.

We note that when I is an interval, the equivalent properties (W-1), (W-1′), (W-2), and (W-2′) simplify somewhat. For example, if I is a bounded interval and $a = \inf(I)$, $b = \sup(I)$, then an n dimensional subspace G of $C_0(I)$ has property (W-2′) iff for each set of points

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\ ,$$

there exists $g \in G \backslash \{0\}$ such that

$$(-1)^i g(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}]$$

$(i = 0, 1, \cdots, n - 1)$.

3. A characterization of property (A-1).

THEOREM 3.1. *Let G be an n dimensional subspace of $C_0(T)$. The following statements are equivalent*:

(1) *G has property* (W-1);

(2) *For each $f \in C_0(T)$ which has a unique best approximation $g_0 \in G$, $f - g_0$ has at least $n + 1$ alternating peak points*;

(3) *Each $f \in C_0(T)$ which has 0 as its unique best approximation in G must have at least $n + 1$ alternating peak points.*

REMARK. In particular, if G is "very non-Chebyshev" (i.e., each $f \in C_0(T)\backslash G$ has more than one best approximation in G), then G has property (W-1).

Proof. (1) $\Rightarrow$ (2). Suppose G has property (W-1). Fix any $f \in C_0(T)$ with $P_G(f) = \{g_0\}$ a singleton. If $f = g_0$, any $n + 1$ points in T work. Thus we assume $f \neq g_0$. Choose a compact set K in T so that

$$|(f - g_0)(t)| < \frac{1}{2}\|f - g_0\| \quad \text{for all} \quad t \in T\backslash K .$$

We define a set of points in T inductively as follows. Let

$$t_1 = \min\{t \in K \mid |(f - g_0)(t)| = \|f - g_0\|\} .$$

We may assume $(f - g_0)(t_1) = \|f - g_0\|$. Having chosen t_i, we set

$$t_{i+1} = \min\{t \in K \cap [t_i, \infty) \mid (f - g_0)(t) = -(f - g_0)(t_i)\} .$$

Now either this procedure yields $n + 1$ points t_i (which clearly satisfy $(-1)^{i+1}(f - g_0)(t_i) = \|f - g_0\|$) and we are done, or this process ends with m points t_i, $1 \leqq m \leqq n$ (i.e., the set $K \cap [t_m, \infty) \cap \{t \in T \mid (f - g_0)(t) = -(f - g_0)(t_m)\}$ is empty). Thus we assume the latter case.

Set $z_0 = -\infty$ and $z_m = \infty$. If $m > 1$ we define additional z_i as follows: for each $i = 1, 2, \cdots, m - 1$, set

$$z_i = \max\{t \in K \cap [t_i, t_{i+1}] \mid (-1)^{i+1}(f - g_0)(t) \geqq 0\} .$$

It follows that $t_i \leqq z_i < t_{i+1}$ and so

$$-\infty = z_0 < z_1 < \cdots < z_{m-1} < z_m = \infty .$$

For $i = 0, 1, \cdots, m - 1$, let

$$M_i = \max\{(-1)^{i+1}(f - g_0)(t) \mid t \in [z_i, z_{i+1}] \cap K\}$$

and $M = \max\{M_i \mid i = 0, 1, \cdots, m-1\}(\geqq 0)$. By compactness of K and the choice of the t_i and z_i, it follows that $M < \|f - g_0\|$.

Since G has property (W-1), there is $0 \neq g \in G$ such that $(-1)^{i+1}g(t) \geqq 0$ for all $t \in [z_i, z_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$. By scaling g we may assume

$$0 < \|g\| \leqq \min\left\{\frac{1}{2}\|f - g_0\|, \|f - g_0\| - M\right\} .$$

If $t \in T \backslash K$, then

$$\begin{aligned} |(f - g_0 - g)(t)| &\leqq |(f - g_0)(t)| + |g(t)| < \frac{1}{2}\|f - g_0\| + \|g\| \\ &\leqq \|f - g_0\| . \end{aligned}$$

If $t \in [z_i, z_{i+1}) \cap K$ $(i = 0, 1, \cdots, m-1)$, then

$$\begin{aligned} (-1)^{i+1}(f - g_0 - g)(t) &= (-1)^{i+1}(f - g_0)(t) - (-1)^{i+1}g(t) \\ &\leqq M + \|g\| \leqq \|f - g_0\| \end{aligned}$$

and

$$(-1)^{i+1}(f - g_0 - g)(t) \geqq (-1)^{i+1}(f - g_0)(t) \geqq -\|f - g_0\| .$$

Thus $|(f - g_0 - g)(t)| \leqq \|f - g_0\|$ implies that $\|f - g_0 - g\| \leqq \|f - g_0\|$ and hence $g_0 + g \in P_G(f) = \{g_0\}$, a contradiction.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Assume (3) holds and let $1 \leqq m \leqq n$ and $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$, where $t_i \in T$ $(i = 1, 2, \cdots, m-1)$. We show there exists $g \in G \backslash \{0\}$ such that $(-1)^i g(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$. Let $T_0 = \bar{T} \backslash T$ and for each positive integer N, let

$$T_N = \left\{t \in T \mid \operatorname{dist}(t, T_0) \geqq \frac{1}{N}\right\}$$

if $T_0 \neq \emptyset$ and $T_N = T$ if $T_0 = \emptyset$. It is easy to verify that T_N is a closed subset of $\boldsymbol{R}$, hence of T, $T_N \subset T_{N+1}$, and $T = \bigcup_1^\infty T_N$.

Thus the sets

$$K_{N,0} = \left[t_1 - N, t_1 - \frac{1}{N}\right] \cap T_N ,$$

$$K_{N,i} = \left[t_i, t_{i+1} - \frac{1}{N}\right] \cap T_N \quad (i = 1, 2, \cdots, m-2) ,$$

and

$$K_{N,m} = [t_{m-1}, t_{m-1} + N] \cap T_N$$

are disjoint compact subsets of T. ($K_{N,0}$ may be empty for all N in which case we simply ignore it in the subsequent argument.) By Urysohn's lemma (or our linearization procedure) there exists functions $f_{N,i} \in C_0(T)$ $(i = 0, 1, \cdots, m-1)$ such that $0 \leqq f_{N,i} \leqq 1$, $f_{N,i} = 1$ on $K_{N,i}$, $f_{N,i} = 0$ off $(t_i - 1/2N, t_{i+1} - 1/2N)$ $(i = 1, 2, \cdots, m-2)$, $f_{N,0} = 0$ off $(t_1 - N - 1/N, t_1 - 1/2N)$, and $f_{N,m-1} = 0$ off $(t_{m-1} - 1/N, t_{m-1} + N + 1/N)$. Set $f_N = \sum_{i=0}^{m-1} (-1)^i f_{N,i}$. Then $f_N \in C_0(T)$ and f_N has only $m - 1 \leqq n - 1$ changes of sign so f_N has at most n alternating peak points. Thus 0 cannot be the unique best approximation to f_N. Choose any $g_N \in P_G(f_N) \backslash \{0\}$. Then g_N must have the same sign as f_N on the sets $K_{N,i}$. That is, $(-1)^i g_N(t) \geqq 0$ for all $t \in K_{N,i} (i = 0, 1, \cdots, m-1)$. Let $h_N = g_N / \|g_N\|$. Let g be a cluster point of (h_N). Then $g \in G$, $\|g\| = 1$. Since $K_{N,0} \nearrow (-\infty, t_1) \cap T$ as $N \to \infty$, $K_{N,i} \nearrow [t_i, t_{i+1}) \cap T$ $(i = 1, 2, \cdots, m-2)$, and $K_{N,m} \nearrow [t_{m-1}, \infty) \cap T$, it follows that $(-1)^i g(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, m-1)$.

COROLLARY 3.2. *Let G be an n dimensional subspace of $C_0(T)$ which does not contain any delta function* (e.g., *if T is an interval or if G is Chebyshev*). *Then*:

(a) *The following statements are equivalent*:

(1) *G has any one of the equivalent properties* (W-1), (W-2), (W-2′), (W-3), *or* (W-4);

(2) *For each $f \in C_0(T)$ which has a unique best approximation $g_0 \in G$, $f - g_0$ has at least $n + 1$ alternating peak points*;

(3) *East $f \in C_0(T)$ having 0 as its unique best approximation in G has at least $n + 1$ alternating peak points.*

(b) *If T satisfies the hypothesis of Proposition* 2.5 (e.g., *if T is an interval*), *and if G is a Chebyshev subspace, then the above statements are equivalent to*

(4) *G has property* (W-1′).

(c) *If T is an interval and G is a Chebyshev subspace, then G has all of the weak Chebyshev properties* (W-1), (W-1′), (W-2), (W-2′), (W-3), *and* (W-4).

Proof. The proof of (a) follows from Theorem 3.1, Lemma 2.2, and Proposition 2.7.

The proof of (b) follows from part (a) and Proposition 2.5.

To prove (c), let T be an interval and G a Chebyshev subspace. Then for each $f \in C_0(T)$, $f - P_G(f)$ has $n + 1$ alternating peak points. (This follows, essentially, by a result of Bram [2]. It can also be deduced just as Remez did in the classical case $T = [a, b]$; see e.g., [8].) The result now follows from (b).

There seems to be a commonly held belief that statement (2) of Corollary 3.2 is *always* valid for Chebyshev subspaces G of $C(T)$, if T is a compact subset of $\boldsymbol{R}$. The following example shows this to be false.

3.3. Example of a one dimensional Chebyshev subspace of $C(T)$ which fails to have any weak Chebyshev property. Let $T = \{1, 2\}$. Consider the one dimensional subspace G spanned by the function g_1 defined by $g_1(t) = (-1)^t$ $(t \in T)$. Then G is Chebyshev since g_1 has no zeros. Since g_1 changes sign, it follows by Lemma 2.2 (when $n = 1$) that G fails to have any of the weak Chebyshev properties. Hence by Corollary 3.2(a), some $f \in C_0(T)$ has the property that $f - P_G(f)$ has less than two alternating peak points. (More explicity, define $f \in C_0(T)$ by $f(1) = 0$ and $f(2) = 2$. Then $P_G(f) = g_1$ and $f - g_1$ is of one sign.)

4. A characterization of property (A-2).

THEOREM 4.1. *Let G be an n dimensional subspace of $C_0(T)$. The following statements are equivalent.*

(1) *G has any one of the equivalent properties* (W-2), (W-2′), (W-3), or (W-4);

(2) *For each $f \in C_0(T)$, there exists a $g_0 \in P_G(f)$ such that $f - g_0$ has at least $n + 1$ alternating peak points.*

Proof. (1) ⇒ (2). Suppose $G = \text{span}\{g_1, g_2, \cdots, g_n\}$ has property (W-4) and $f \in C_0(T)\backslash G$. Assume first that T is a compact interval $[a, b]$. Then the result follows from Jones-Karlovitz [6]. Next let T be an arbitrary interval. Then there is an increasing sequence of compact intervals T_k such that $\bigcup_1^\infty T_k = T$. For k sufficiently large, $G|_{T_k}$ will be n dimensional and we assume this to be the case. Set $G_k = G|_{T_k}$. Then each G_k has property (W-4) (in $C_0(T)|_{T_k} \subset C(T_k)$) since G does. By the first part, there exist $h_k \in P_{G_k}(f|_{T_k})$ and $n + 1$ points $t_{k_1} < t_{k_2} < \cdots < t_{k,n+1}$ in T_k such that

$$(1) \qquad \sigma_k(-1)^i(f - h_k)(t_{ki}) = \|f - h_k\|_k \quad (i = 1, 2, \cdots, n + 1)$$

for some $\sigma_k \in \{-1, 1\}$, where $\|h\|_k = \|h\|_{T_k} = \sup_{t \in T_k} |h(t)|$ Since $h_k \in P_{G_k}(f|_{T_k})$, we have

$$\|h_k\|_k \leqq 2\|f\|_k \leqq 2\|f\| \quad \text{for all} \quad k .$$

By passing to a subsequence, we may assume all the σ_k to be the same, say $\sigma_k = 1$. Write $h_k = \sum_{i=1}^n \alpha_{ki} g_i |_{T_k}$.

Claim. $\sup_k |\alpha_{ki}| < \infty$ $(i = 1, 2, \cdots, n)$.

Indeed, we may assume that $\dim G_1 = n$. Since $T_1 \subset T_k$ and $h_k \in P_{G_k}(f|_{T_k})$ $(k = 1, 2, \cdots)$, we have

$$||f - h_k||_1 \leqq ||f - h_k||_k \leqq ||f||_k \leqq ||f|| \quad (k = 1, 2, \cdots),$$

so

$$||h_k||_1 \leqq ||f|| + ||f||_1 \leqq 2||f|| \qquad (k = 1, 2, \cdots).$$

Now G_1 is isomorphic to l_∞^n (the space of all n-tuples of real numbers endowed with the maximum norm) by the mapping

$$\sum_{i=1}^n \beta_i g_i |_{T_1} \longrightarrow (\beta_1, \beta_2, \cdots, \beta_n).$$

Hence there exists a constant M_n (depending only on n) such that

$$\max_{1 \leqq i \leqq n} |\beta_i| \leqq M_n \left\| \sum_{i=1}^n \beta_i g_i |_{T_1} \right\|$$

for all real numbers $\beta_1, \beta_2, \cdots, \beta_n$. Since $h_k|_{T_1} = \sum_{i=1}^n \alpha_{ki} g_i |_{T_1}$ and $||h_k||_1 \leqq 2||f||$, we obtain

$$\max_{1 \leqq i \leqq n} |\beta_{ki}| \leqq M_n ||h_k||_1 \leqq 2M_n ||f|| \qquad (k = 1, 2, \cdots)$$

from which the claim follows.

Using the claim, we can pass to a subsequence and obtain $\alpha_{ki} \to \alpha_i$ $(i = 1, 2, \cdots, n)$ for some $\alpha_i \in \boldsymbol{R}$. Let $g_0 = \sum_1^n \alpha_i g_i \in G$ and $\bar{h}_k = \sum_{i=1}^n \alpha_{ki} g_i \in G$. Then $\bar{h}_k|_{T_k} = h_k$ and $\bar{h}_k \to g_0$. Since $\bar{h}_k|_{T_k} = h_k \in P_{G_k}(f|_{T_k})$, we have,

$$||f - \bar{h}_k||_k \leqq ||f - g_0||_k \leqq ||f - g_0|| \quad (k = 1, 2, \cdots).$$

On the other hand, we can choose $t_0 \in T$ such that $|(f - g_0)(t_0)| = ||f - g_0||$. Since $\bigcup_1^\infty T_k = T$ and the T_k are increasing, we have $t_0 \in T_k$ for all k sufficiently large. Also $\bar{h}_k \to g_0$ implies that for each $\varepsilon > 0$, there exists k_ε such that $|(f - \bar{h}_k)(t_0)| \geqq |(f - g_0)(t_0)| - \varepsilon$ for all $k \geqq k_\varepsilon$. Thus for all k sufficiently large,

$$||f - \bar{h}_k||_k \geqq |(f - \bar{h}_k)(t_0)| \geqq |(f - g_0)(t_0)| - \varepsilon = ||f - g_0|| - \varepsilon.$$

Hence $||f - \bar{h}_k||_k \to ||f - g_0||$. Thus for $i = 1, 2, \cdots, n + 1$,

$$\begin{aligned} |(f - g_0)(t_{ki})| &= |(f - \bar{h}_k)(t_{ki}) - (g_0 - \bar{h}_k)(t_{ki})| \\ &\geqq ||f - \bar{h}_k||_k - ||g_0 - \bar{h}_k||_k \longrightarrow ||f - g_0||, \end{aligned}$$

and hence

$$t_{ki} \in K \equiv \left\{t \in T \mid |(f - g_0)(t)| \geqq \frac{1}{2}\|f - g_0\|\right\}$$

for k sufficiently large. Since K is compact, by passing to a subsequence, we may assume that $t_{ki} \to t_i \in T$ $(i = 1, 2, \cdots, n + 1)$.

By passing to the limit in eq. (1),

$$(-1)^i(f - g_0)(t_i) = \|f - g_0\| \quad (i = 1, 2, \cdots, n + 1) ,$$

and $t_1 < t_2 < \cdots < t_{n+1}$. If $g_0 \notin P_G(f)$, then for any $g \in P_G(f)$ we obtain $|(f - g)(t_i)| < \|f - g_0\|$ and

$$\begin{aligned}(-1)^i(g - g_0)(t_i) &= (-1)^i(f - g_0)(t_i) - (-1)^i(f - g)(t_i) \\ &= \|f - g_0\| - (-1)^i(f - g)(t_i) > 0\end{aligned}$$

for $i = 1, 2, \cdots, n + 1$. Thus $g - g_0$ has n sign changes, contradicting property (W-4). This shows that $g_0 \in P_G(f)$ and proves the result when T is any interval.

Finally, let T be any locally compact subset of $\boldsymbol{R}$. Let IT, $\tilde{f} \in C_0(IT)$, and $\tilde{G}$ be as described prior to Lemma 2.2. Since G has property (W-4) and since each $\tilde{g} \in \tilde{G}$ is linear, hence monotonic, on each interval in $IT\backslash\bar{T}$, $\tilde{G}$ also has property (W-4). Fix any $f \in C_0(T)\backslash G$. Then $\tilde{f} \in C_0(IT)\backslash\tilde{G}$ and by the result proved for intervals there exist $\tilde{g}_0 \in P_{\tilde{G}}(\tilde{f})$ and points $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_{n+1}$ in IT such that

$$\sigma(-1)^i(\tilde{f} - \tilde{g}_0)(\tilde{t}_i) = \|\tilde{f} - \tilde{g}_0\|_{IT} \neq 0 \quad (i = 1, 2, \cdots, n + 1)$$

for some $\sigma \in \{-1, 1\}$.

Claim. We may assume $\tilde{t}_i \in T$ $(i = 1, 2, \cdots, n + 1)$.

If $\tilde{t}_i \in T$, set $t_i = \tilde{t}_i$. If some $\tilde{t}_i \notin T$, then (since $\tilde{f} - \tilde{g}_0 = 0$ on $\bar{T}\backslash T$) $\tilde{t}_i$ is in one of the disjoint open intervals I_m whose union is $IT\backslash\bar{T}$. But $\tilde{f} - \tilde{g}_0$ is linear on each such subinterval and $|(\tilde{f} - \tilde{g}_0)(\tilde{t}_i)| = \|\tilde{f} - \tilde{g}_0\|_{IT}$ imply that $\tilde{f} - \tilde{g}_0$ is constant on I_m. We then replace $\tilde{t}_i$ with either one of the endpoints t_i of I_m. Clearly, the resulting $t_i \in T$, $t_1 < t_2 < \cdots < t_{n+1}$, and

$$\sigma(-1)^i(\tilde{f} - \tilde{g}_0)(t_i) = \|\tilde{f} - \tilde{g}_0\|_{IT} \quad (i = 1, 2, \cdots, n + 1)$$

which proves the claim.

Since $\|\tilde{f} - \tilde{g}_0\|_{IT} \leqq \|\tilde{f} - \tilde{g}\|_{IT}$ for all $\tilde{g} \in \tilde{G}$ and since $\|\tilde{h}\|_{IT} = \|h\|$ for every $h \in C_0(T)$ (because $\tilde{h}$ is linear on each of the subintervals whose union is $IT\backslash\bar{T}$), it follows that $\|f - g_0\| \leqq \|f - g\|$ for all $g \in G$. That is, $g_0 \in P_G(f)$, and using the claim,

$$\sigma(-1)^i(f - g_0)(t_i) = \|f - g_0\| \quad (i = 1, 2, \cdots, n + 1) .$$

(2) $\Rightarrow$ (1). Suppose (2) holds. We will show that G has property (W-2′). We first prove the

Claim. For each set of points $-\infty = t_0 < t_1 < \cdots < t_n = \infty$, with $t_i \in T$ $(i = 1, 2, \cdots, n-1)$, there exists $g \in G\backslash\{0\}$ such that

$$(-1)^i g(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}) \cap T$$

$(i = 1, 2, \cdots, n-1)$, and $g(t) \neq 0$ for some $t \in T\backslash\{t_1, t_2, \cdots, t_{n-1}\}$.

We construct the sets $K_{N,i} (i = 0, 1, \cdots, n)$ and functions $f_N \in C_0(T)$ exactly as in the proof of the implication (3) $\Rightarrow$ (1) of Theorem 3.1. Then f_N has at most n alternating peak points. Choose $g_N \in P_G(f_N)$ such that $f_N - g_N$ has at least $n+1$ alternating peak points. Then $g_N \neq 0$ and $g_N f_N \geqq 0$ on each set $K_{N,i}$ $(i = 0, 1, \cdots, n)$. Let $h_N = (g_N/\|g_N\|)$ and let g be a cluster point of the sequence (h_N). Then $g \in G$, $\|g\| = 1$, and $(-1)^i g(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, n-1)$. If $g(t) = 0$ for all $t \in T\backslash\{t_1, t_2, \cdots, t_{n-1}\}$, then for such t, $|h_N(t)| < 1/8$ eventually (i.e., for N large enough) and hence $|g_N(t)| < 1/4$ eventually (since $\|g_N\| \leqq 2\|f_N\| \leqq 2$). But if $t \in T\backslash\{t_1, t_2, \cdots, t_{n-1}\}$, then $t \in K_{N,i}$ eventually for some i implies that, eventually,

$$|f_N(t) - g_N(t)| \geqq |f_N(t)| - |g_N(t)| \geqq 1 - \frac{1}{4} = \frac{3}{4} .$$

Thus for some integer N_0, we have $\|f_{N_0} - g_{N_0}\| \geqq 3/4$. It follows using the definition of f_{N_0} and the above properties of g_{N_0} that $f_{N_0} - g_{N_0}$ cannot have more than n alternating peak points, a contradiction to the choice of g_N. This proves the claim.

We next show that for each set of points $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$, with $t_i \in T$ $(i = 1, 2, \cdots, n-1)$, and each integer k, with $1 \leqq k \leqq n-1$, there exists $g_k \in G\backslash\{0\}$ such that

(a) $(-1)^i g_k(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, n-1)$, and
(b) $g_k(t_i) = 0$ $(i = 1, 2, \cdots, k)$.

Once we have this, it is clear that the function $g = g_{n-1}$ satisfies

$$(-1)^i g(t) \geqq 0 \quad \text{for all} \quad t \in [t_i, t_{i+1}] \cap T$$

$(i = 0, 1, \cdots, n-1)$ and this shows that G has property (W-2′). We proceed by induction on k. The induction step is similar to the case $k = 1$ so we only prove the latter. By the claim, there exist $g \in G\backslash\{0\}$ which satisfies (a) and $g(s) \neq 0$ for some $s \in T\backslash\{t_1, t_2, \cdots, t_{n-1}\}$. If $g(t_1) = 0$, set $g_1 = g$ and we are done. If $g(t_1) \neq 0$, then $g(t_1) < 0$.

Proceeding exactly as in the proof of Proposition 2.7 we obtain a function $g_0 \in G\backslash\{0\}$ satisfying $g_0(t)g(t) \geqq 0$ for all $t \in T\backslash\{t_1\}$ and $g_0(t_1) \geqq 0$. If $g_0(t_1) = 0$, set $g_1 = g_0$. If $g_0(t_1) \neq 0$, then $g_0(t_1) > 0$ and set $g_1 = g - \alpha g_0$, where $\alpha = (g(t_1)/g_0(t_1)) < 0$. Then $(-1)^i g_1(t) \geqq 0$ for all $t \in [t_i, t_{i+1}) \cap T$ $(i = 0, 1, \cdots, n-1)$, and $g_1(t_i) = 0$. If $g_1 = 0$, then $g = \alpha g_0$. Hence

$$0 \leqq g_0(s)g(s) = \frac{1}{\alpha}[g(s)]^2 < 0$$

which is absurd. Thus $g_1 \neq 0$ and we are done.

Example 2.4 above, along with Theorems 3.1 and 4.1, show that properties (A-1) and (A-2) are not the same in general. However, there is one important case when they are the same.

COROLLARY 4.2. *Let I be any interval in $\boldsymbol{R}$ and G an n dimensional subspace in $C_0(I)$. Then the following statements are equivalent:*

(1) *G has any one of the equivalent properties* (W-1), (W-1′), (W-2), (W-2′), (W-3), *or* (W-4);

(2) *For each $f \in C_0(I)$, there exists $g_0 \in P_G(f)$ such that $f - g_0$ has at least $n + 1$ alternating peak points;*

(3) *For each $f \in C_0(I)$ which has a unique best approximation $g_0 \in G$, $f - g_0$ has at least $n + 1$ alternation peak points;*

(4) *Each $f \in C_0(I)$ with $P_G(f) = \{0\}$ has at least $n + 1$ alternating peak points.*

Proof. Corollary 2.9, Theorem 3.1, and Theorem 4.1.

In the particular case when I is the compact interval $[a, b]$, the equivalence of properties (W-2′), (W-3), (W-4), (A-1), and (A-2) was first proved by Jones-Karlovitz [6].

5. A characterization of property (A-3).

THEOREM 5.1. *Let G be an n dimensional subspace of $C_0(T)$. The following statements are equivalent.*

(1) *G is Chebyshev and has one of the equivalent properties* (W-2), (W-2′), (W-3), *or* (W-4);

(2) *For each $f \in C_0(T)$ and each $g_0 \in P_G(f)$, $f - g_0$ has at least $n + 1$ alternating peak points.*

Proof. (1) $\Rightarrow$ (2). This follows from Theorem 4.1.

(2) $\Rightarrow$ (1). If statement (2) holds, Theorem 4.1 implies that G has each of the equivalent properties (W-2), (W-2′), (W-3), and (W-4). If G were not Chebyshev, G would not be Haar so there would exist some nonzero $g_0 \in G = \text{span}\{g_1, g_2, \cdots, g_n\}$ which vanishes on a set of n distinct points $T_0 = \{t_1, t_2, \cdots, t_n\}$ of T. We may assume $\|g_0\| = 1$. Then $\det[g_i(t_j)]_1^n = 0$ so there are scalars α_i not all zero such that $\sum_{i=1}^n \alpha_i g_j(t_i) = 0$ $(j = 1, 2, \cdots, n)$, i.e., $\sum_1^n \alpha_i g(t_i) = 0$ for all $g \in G$. Let $\sigma_i = \text{sgn}\, \alpha_i$ if $\alpha_i \neq 0$ and $\sigma_i = 1$ if $\alpha_i = 0$. Since each t_i is a G_δ, Urysohn's lemma (or our linearization procedure) implies that for each $i = 1, 2, \cdots, n$, there exist $f_i \in C_0(T)$ and disjoint neighborhoods U_i of t_i such that $0 \leqq f_i \leqq 1$, $f_i(t_i) = 1$, $f_i = 0$ off U_i, and $f_i(t) < 1$

for all $t \neq t_i$ (see e.g., [3; Cor. 4.2, p. 148]). Let $z = \sum_1^n \sigma_i f_i$. Then $z \in C_0(T)$, $z(t_i) = \sigma_i (i = 1, 2, \cdots, n)$, and $|z(t)| < 1$ if $t \in T \backslash T_0$. Set $f = z(1 - |g_0|) + g_0$. Then $f \in C_0(T)$, $f(t_i) - g_0(t_i) = z(t_i) = \sigma_i$ $(i = 1, 2, \cdots, n)$, and

$$|f(t) - g_0(t)| = |z(t)|(1 - |g_0(t)|)$$
$$= \begin{cases} <1 & \text{if} \quad t \in T \backslash T_0 \\ 1 & \text{if} \quad t \in T_0 . \end{cases}$$

Thus $||f - g_0|| = 1$ and the set of peak points for $f - g_0$ is T_0. If $g \in G$ and $||f - g|| < 1$, then since $|f(t_i)| = 1$ for all i,

$$\operatorname{sgn} g(t_i) = \operatorname{sgn} f(t_i) = z(t_i) = \sigma_i \quad (i = 1, 2, \cdots, n)$$

so

$$0 = \sum_1^n \alpha_i g(t_i) = \sum_1^n |\alpha_i| \, |g(t_i)| > 0$$

which is absurd. Thus $||f - g|| \geqq 1 = ||f - g_0||$ for all $g \in G$ and hence $g_0 \in P_G(f)$. But $f - g_0$ has only n peak points (viz. the set T_0). This contradiction to (2) shows that G must be Chebyshev.

In a result related to Theorem 5.1, Gopinath and Kurshan [4] essentially proved that an n dimensional subspace G of $C_0(T)$ is Chebyshev and has property (W-4) $\Leftrightarrow$ it has the property $(G - K)$: For each set of points $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$ with $t_i \in T$ $(i = 1, 2, \cdots, n - 1)$, there exists $g \in G$ such that

$$(-1)^i g(t) > 0 \quad \text{if} \quad t \in (t_i, t_{i+1}) \cap T$$

$(i = 0, 1, \cdots, n - 1)$, and

$$g(t_i) = 0 \quad (i = 1, 2, \cdots, n - 1) .$$

Theorem 5.1 can be strengthened in case T is an interval.

COROLLARY 5.2. *Let I be an interval in $\boldsymbol{R}$ and G an n dimensional subspace of $C_0(I)$. Then the following statements are equivalent:*

(1) *G is Chebyshev;*

(2) *For each $f \in C_0(I)$ and each $g_0 \in P_G(f)$, $f - g_0$ has at least $n + 1$ alternating peak points;*

(3) *For each $f \in C_0(I)$ and each $g_0 \in P_G(f)$, $f - g_0$ has at least $n + 1$ peak points.*

Proof. (1) $\Rightarrow$ (2). By Corollary 3.2(c), G has all the weak Chebyshev properties so the result follows from Theorem 5.1.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). This follows exactly as in the proof of (2) $\Rightarrow$ (1) in Theorem 5.1.

REMARK 5.3. The implication (1) $\Rightarrow$ (2) in Corollary 5.2, in the particular case when $I = [a, b]$, is just the classical alternation theorem. Also, when $I = [a, b]$, the implication (2) $\Rightarrow$ (1) was proved by Handscomb, Mayers, and Powell [5]. We now show that implication (1) $\Rightarrow$ (2) *fails* in general if I is not an interval. Indeed, it fails if one adjoins a single point outside the interval.

EXAMPLE 5.4. Let $T = [a, b] \cup \{c\}$, where $c \notin [a, b]$. Let $G = \operatorname{span}\{g_1\}$, where $g_1(t) = 1$ if $t \in [a, b]$ and $g_1(c) = -1$. Then G is Chebyshev in $C(T)$ since g_1 has no zeros. But if $f(t) \equiv 1$, then $P_G(f) = 0$ and $f - P_G(f)$ does not have two alternating peak points.

6. Examples of weak Chebyshev subspaces. There are a number of important subspaces which are not Chebyshev but are weak Chebyshev. We give a few examples below.

6.1. (Polynomial Splines). Let $T = [a, b]$ and fix any $k(\geqq 1)$ points $s_1 < s_2 < \cdots < s_k$ in T. For any integer $m \geqq 0$ let

$$S_{m,k} = \operatorname{span}\{1, t, \cdots, t^m, (t - s_1)_+^m, \cdots, (t - s_k)_+^m\} ,$$

where

$$(t - s)_+^m = \begin{cases} (t - s)^m & \text{if } t \geqq s \\ 0 & \text{if } t < s . \end{cases}$$

$S_{m,k}$ is the $n \equiv m + k + 1$ dimensional subspace of $C[a, b]$ known as "the polynomial splines of degree m with k fixed knots." It is known (see e.g., [7], p. 18) that $S_{m,k}$ has property (W-3), and thus by Corollary 4.2 has all of the weak Chebyshev properties (W-i) ($i = 1, 1', 2, 2', 3$, and 4) as well as the alternation properties (A-1) and (A-2).

6.2. (Weighted Polynomial Splines). The example in (1) can be modified as follows. Let $T = \boldsymbol{R}$ and k points $s_1 < s_2 < \cdots < s_k$ in T be given. Let $w \in C_0(T)$ be any positive function such that $w \cdot p \in C_0(T)$ for any polynomial p (e.g., $w(t) = e^{-t^2}$). Then the $n \equiv m + k + 1$ dimensional subspace

$$S_{m,k}^0 = \{wg \mid g \in S_{m,k}\} ,$$

where $S_{m,k}$ is defined as in 6.1, obviously has property (W-3) since $S_{m,k}$ does. Thus by Corollary 4.2, $S_{m,k}^0$ has all the Chebyshev properties (W-i) (i = 1, 1', 2, 2', 3, and 4) and the alternation properties (A-1) and (A-2).

6.3. (Weighted Chebyshev subspace). Let $T = I$ be any interval in R and P any n dimensional Chebyshev subspace of $C_0(T)$. Let $w \in C_0(T)$ be any nonnegative function which does not vanish identically, and set

$$G = \{wp \mid p \in P\} .$$

Then since P is Chebyshev, it follows that G is an n dimensional subspace of $C_0(T)$ having property (W-3), and hence, by Corollary 4.2, all the weak Chebyshev properties (W-i) ($i = 1, 1', 2, 2', 3$, and 4) as well as the alternation properties (A-1) and (A-2).

6.4. Let $T = N$ denote the set of natural numbers (so $C_0(T) = c_0$) and let any n points $k_1 < k_2 < \cdots < k_n$ in T be given. Define $g_i \in c_0$ by

$$g_i(t) = \begin{cases} 1 & \text{if} \quad t = k_i \\ 0 & \text{otherwise} \end{cases}$$

($i = 1, 2, \cdots, n$). Then $G = \text{span}\{g_1, g_2, \cdots, g_n\}$ is an n dimensional subspace of c_0 which is easily seen to have properties (W-1) and (W-2). Thus by Proposition 2.5, Lemma 2.2, and Theorems 3.1 and 4.1, G has all the weak Chebyshev properties (W-i) ($i = 1, 1', 2, 2', 3$, and 4) and the alternation properties (A-1) and (A-2).

We note that *none* of the above four examples is a Chebyshev subspace in general.

7. **A generalization.** We can give the following generalization of the equivalence (1) $\Leftrightarrow$ (3) of Corollary 5.2. In particular, it provides another characterization of Chebyshev subspaces in $C_0(T)$ for certain T (including T metric). However, unlike Haar's characterization concerning the number of zeros of elements of the subspace, our characterization is not intrinsic.

THEOREM 7.1. *Let T be any locally compact Hausdorff space containing at least $n + 1$ points and let G be an n dimensional subspace of $C_0(T)$. If each point of T is a G_δ* (e.g., *if T is metric*), *or if each nonzero element of G has only finitely many zeros, then the following statements are equivalent*:

(1) *G is Chebyshev;*

(2) *For each $f \in C_0(T)$ and each $g_0 \in P_G(f)$, $f - g_0$ has at least $n + 1$ peak points.*

The proof of the implication (2) $\Rightarrow$ (1) is similar to the proof of the implication (2) $\Rightarrow$ (1) of Theorem 5.1. The implication (1) $\Rightarrow$ (2)

is well-known and due, for compact T, to Remez (see e.g., [8]).

It is worth noticing that this characterization of Chebyshev spaces is no longer valid in general if both the conditions on T and G (viz. (i) each point of T be a G_δ, and (ii) each nonzero element of G have only finitely many zeros) are dropped. To see this, let $\beta\boldsymbol{R}$ denote the Stone-Čech compactification of $\boldsymbol{R}$ and consider the space $T = \beta\boldsymbol{R}\backslash\boldsymbol{R}$. It is well-known that T is a compact Hausdorff space in which no point is a G_δ (see e.g., [9; p. 150, prob 112]). A simple induction shows that no finite subset of T is a G_δ. Thus each $f \in C(T)$ has an infinity of peak points (since the set of all peak points is a G_δ). By Urysohn's lemma there exists a nonzero $g \in C(T)$ which has a zero. Thus $G = \text{span}\{g\}$ is not Chebyshev, but for each $f \in C(T)$ and $g_0 \in P_G(f)$, $f - g_0$ has infinitely many peak points.

References

1. N. Bourbaki, *Éléments de Mathématique, Livre III, Topologie Générale*, Chs. 1–2, Hermann, Paris, 1965.
2. J. Bram, Chebyshev approximation in locally compact spaces. Proc. Amer. Math. Soc., **9** (1958), 133–136.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
4. B. Gopinath and R. P. Kurshan, The existence in T-spaces of functions with prescribed alternations, J. Approximation Theory, **21** (1977), 143–150.
5. D. C. Handscomb, D. F. Mayers, and M. J. D. Powell, The general theory of linear approximation, in *Methods of Numerical Approximation* (ed. by D. C. Handscomb), Pergamon Press, New York, 1966.
6. R. C. Jones and L. A. Karlovitz, Equioscillation under nonuniqueness in the approximation of continuous functions, J. Approximation Theory, **3** (1970), 138–145.
7. S. Karlin and W. J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Interscience, New York, 1966.
8. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, New York, 1970.
9. A. Wilansky, *Topology for Analysis*, Ginn, Waltham, Mass., 1970.
10. R. Zielke, Einige Eigenschaften schwacher Tschebyscheff-Systeme, preprint (1977).

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