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FIBER HOMOLOGY AND ORIENTABILITY OF MAPS

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In this paper we introduce a concept of *fiber homology* for an arbitrary map $f: X \to Y$ and coefficient module G. This is a graded module denoted by $H_*(f_*; G)$ which reduces to $H_*(F; G)$ when f represents an orientable fiber bundel with standard fiber F. The concept of fiber homology permits us also to define a generalized notion of orientability, and these ideas turn out to be useful in the study of submersions. Our main theorem (obtained by means of a spectral sequence) asserts that if the fibers of a submersion $f: X \to Y$ are acyclic in dimensions smaller than q, then the rank r_q of the fiber homology $H_q(f_*; G)$ is bounded above by the sum of the q and $(q+1)$-dimensional Betti numbers of X and Y, respectively. In the orientable case, the q-dimensional Betti number of an arbitrary fiber $f^{-1}(y)$ is bounded above by r_q, and therefore also by the aforementioned sum. This leads to a number of more specialized results. For example, it is shown that the fibers of an orientable submersion $f: R^{2m-1} \to S^m$ must be either acyclic or homology spheres, and moreover, the subspace of points in S^m corresponding to the spherical fibers must have the homology of a point.

1. Basic concepts. Let $f: X \to Y$ denote a continuous map between topological spaces. By a *tubular neighborhood* belonging to f we will understand a homeomorphism

$$\Phi: B \times F \approx V$$

where B is an open connected subspace of Y, F a compact space and V a subset of X, such that $f \circ \Phi$ is the projection $B \times F \to B$. Given a point $y \in B$, we will write

$$F_y = V \cap f_y$$

where f_y denotes the preimage of y under f, and given two points $y, y' \in B$, we let

$$\Phi_y^{y'}: F_y \approx F_{y'}$$

denote the homeomorphism induced by Φ. The diagram

$$\begin{array}{ccc} H_*(f_y; G) & & H_*(f_{y'}; G) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ H_*(F_y; G) & \xrightarrow{\Phi_{y*}^{y'}} & H_*(F_{y'}; G) \end{array}$$

(where i, i' are inclusion induced, G is a coefficient module and H_* designates singular homology) gives rise to a linear relation

$$i' \circ \Phi_{y*}^{y'} \circ i^{-1} \colon H_*(f_y; G) \longrightarrow H_*(f_{y'}; G)$$

henceforth referred to as a *horizontal relation* induced by Φ. We will employ the notation

$$w \pitchfork w' \bmod \Phi \tag{1.1}$$

to indicate that the homology classes w, w' are horizontally related via Φ. Now let W denote the linear subspace of the direct sum

$$\sum_{t \in Y} H_*(f_t; G) \tag{1.2}$$

generated by elements of the form $[w - w']$ as (w, w') ranges over all pairs satisfying a relation (1.1). This makes sense, inasmuch as every element $w \in H_*(f_y; G)$ can be identified with a corresponding element in (1.2), an identification which we shall always assume. The *fiber homology* of f (with coefficients in G) is defined to be the quotient module

$$H_*(f_*; G) = \sum_{t \in Y} H_*(f_t; G)/W .$$

It is to be noted that $H_*(f_*; G)$ reduces to $H_*(F; G)$ when f represents an orientable fiber bundle (see [4]) with standard fiber F. The following simple and illustrative examples will exhibit the fiber homology in a more general case.

EXAMPLE 1. Let $X \subset R^3$ denote the open subspace consisting of points (x_1, x_2, x_3) with $|x_3| < 1$, excluding the origin, and let $f \colon X \to R$ be given by

$$f(x) = \log \|x\| .$$

This defines a differentiable submersion from a punctured 3-space to the real line whose fibers f_y are cylinders for $y \geqq 0$ and spheres for $y < 0$. It is easy enough to verify that the fiber homology is given by

$$H_p(f_*; G) \approx \begin{cases} G & p = 0, 2 \\ 0 & \text{otherwise} . \end{cases}$$

In particular, to see that the fiber homology vanishes in dimension 1, consider two points $y, y' \in R$ with $y' < 0 < y$. The cylindrical fiber f_y intersects the plane $x_3 = 0$ in a circle C_y, which (when oriented) represents a generator $w \in H_1(f_y; G)$. If we now subject

C_y to a contraction (via concentric circles) in the given plane, we obtain a corresponding family of homology classes $w_t \in H_1(f_t; G)$ as t varies from y to y'. At least for values t, t' which are not too far apart, the corresponding homology classes will clearly satisfy a horizontal relation (1.1). But by virtue of the fact that $f_{y'}$ is simply connected, this implies that w is annihilated by projection into $H_1(f_*; G)$.

EXAMPLE 2. Modifying the preceding example by deleting the positive x_3-axis, one obtains a new submersion $f: R^3 \to R$ (after a suitable change of coordinates) whose fibers are cylinders for $y \geqq 0$ and planes for $y < 0$. This map is "fiber acyclic".

Returning to the case of an arbitrary map $f: X \to Y$, it is to be noted that for every point $y \in Y$ one has a natural homomorphism

$$H_*(f_y; G) \longrightarrow H_*(f_*; G) \tag{1.3}$$

(canonical injection into (1.2), followed by natural projection), and we will say that f is *orientable at* y (over G) if this homomorphism is injective. A map is said to be *orientable* if it is orientable at all points $y \in Y$, and it will also make sense to say that f is orientable in a particular dimension. Moreover, one observes that these definitions reduce to the classical conception of orientablity in the case of fiber bundles.

A novel feature of this generalized notion of orientability lies in the fact that a map $f: X \to Y$ may fail to be orientable over arbitrarily small neighborhoods of a given point $y \in Y$, a circumstance which is exemplified by both of the preceding examples. Not only do these submersions fail to be orientable, but this failure is concentrated (so to speak) in the immediate vicinity of a particular fiber (the fiber f_0, which constitutes the smallest or innermost cylinder in the given foliation of X). Fibers of this kind will be referred to as "separatrices", and one may define this concept as follows. Let us say that a homology class $w \in H_*(f_y; G)$ is *horizontally annihilated* over some neighborhood U of y, provided there exists a tubular neighborhood Φ with $B \subset U$ such that $w \not\sim 0'$ mod Φ, where $0'$ denotes the zero element in $H_*(f_{y'}; G)$ for some $y' \in B$. A fiber f_y will be called a *separatrix* (over G) if it admits a nonzero homology class $w \in H_*(f_y; G)$ which is horizontally annihilated over every neighborhood U of y.

In the present paper we shall restrict our attention to the class of submersions. Little will be lost if we presuppose a C^∞ setting and take "submersion" to mean a differentiable surjection $f: X \to Y$ between paracompact differentiable manifolds with $\dim X \geqq \dim Y$, such that the differential df has everywhere maximal rank. On the other hand, it should be pointed out that the proof of our Main Theorem does not require as much. The crucial property which is required for many steps can be formulated directly in terms of tubular neighborhoods. Given a map $f: X \to Y$, a subspace $C \subset X$ and a tubular neighborhood $\Phi: B \times F \to V$ belonging to f, we will say that Φ *cuts* C provided that

$$C \cap f_y \subset F_y$$

for all $y \in B$. The property in question is this:

(1.4) *Given $y \in Y$ and a compact subset $C \subset X$, there shall exist a tubular neighborhood Φ such that $y \in B$ and Φ cuts C.*

The proof of our Main Theorem also utilizes a spectral sequence theorem obtained in [1], the proof of which actually requires, in addition to (1.4), that X is second countable and locally compact, and that Y is triangulable. Although these conditions on the map $f: X \to Y$ suffice for the Main Theorem, some of the more specialized results which we shall present as corollaries presuppose also that the fibers f_y shall be manifolds. All these conditions are satisfied in the case of a differentiable submersion.

2. Main theorem and corollaries. We will now state our main result and develop a few of its implications. The notation R_p shall indicate Betti numbers, and in particular, $R_p(f_*; G)$ shall denote the rank of the p-dimensional fiber homology.

MAIN THEOREM. *If $f: X \to Y$ is a submersion and G a coefficient module, then*

$$R_0(f_*; G) = R_0(X; G) .$$

Moreover, if every fiber f_y is connected and t-acyclic over G for $q - m < t < q$, where $m = \dim Y$, then

$$R_q(f_*; G) \leqq R_q(X; G) + R_{q+1}(Y; G) .$$

If, in addition, $H_q(X; G)$ and $H_{q+1}(Y; G)$ vanish, then $H_q(f_*; G)$ vanishes.

In the orientable case one can say a good deal more since one has also an inequality

$$R_*(f_y; G) \leqq R_*(f_*; G)$$

which holds for every point $y \in Y$. By a successive application of our theorem one can therefore obtain

COROLLARY 1. *Let $f: X \to Y$ be a submersion which is orientable over G in dimensions $t \leqq p$. Let X and Y be t-acyclic over G for $t \leqq p$ and $t \leqq p+1$, respectively. Then f_y is t-acyclic over G for all $t \leqq p$ and every point $y \in Y$.*

It will be shown in § 6 that a real-valued submersion $f: X \to R$ without separatrices is orientable. This fact, together with Corollary 1, implies

COROLLARY 2. *Let $f: X \to R$ be a submersion, and let X be t-acyclic over G for $t \leqq p$. If for some $t \leqq p$ and some point $y \in Y$ the fiber f_y is not t-acyclic over G, then f admits a separatrix over G in some dimension $s \leqq t$.*

This result shows, in particular, that the separatrices which have been exhibited in our Examples 1 and 2 were necessitated by the existence of cylindrical fibers and the topology of X.

It is easy to see that a compact fiber cannot be a separatrix, since such a fiber can be imbedded in a tubular neighborhood (in consequence of (1.4)). This observation leads immediately to

COROLLARY 3. *Let $f: X \to R$ be a submersion, and let X be t-acyclic over G for $t < p$. If every noncompact fiber is t-acyclic over G for $t \leqq p$, then*

$$R_p(f_y; G) \leqq R_p(X; G)$$

for all $y \in Y$.

Our assumptions imply that there are no separatrices in dimensions $t \leqq p$, and this implies orientability in the same range. One may therefore conclude by Corollary 1 that the fibers are acyclic in dimensions $t < p$. The Main Theorem now gives $R_p(f_*; G) \leqq R_p(X; G)$, and since f is also orientable in dimension p, one obtains the inequality in question.

Apart from real-valued submersions, another particularly simple class is given by submersions of codimension 1, which is to say, submersions having 1-dimensional fibers. Here we can mention the following result:

COROLLARY 4. *Let $f: X \to Y$ be a submersion of codimension 1 having connected fibers. If X and Y are acyclic over Z_2 in dimensions 1 and 2, respectively, then f admits no compact fiber.*

It should be noted that if the fibers of a submersion of codimension 1 are assumed to be connected, then these fibers can only be circles or lines. Under the given acyclicity conditions on X and Y, our corollary concludes that there can be no circular fibers, and this actually implies that f is equivalent to the natural projection $Y \times R \to Y$. For a detailed study of the codimension 1 case, we may refer to [2].

To establish Corollary 4, we first apply our Main Theorem with $q = 1$ to conclude that $R_1(f_*; Z_2) = 0$. It therefore suffices to show that f is orientable in dimension 1. The easiest way to see this, perhaps, is to consider the subset $U \subset Y$ corresponding to circular fibers and observe that the restriction g of f to the preimage of U represents a circle bundle, and this is certainly orientable over Z_2. Now f will be orientable if for any $y \in U$, the homomorphism (1.3) turns out to be injective. But this homomorphism clearly factors as follows:

$$H_*(f_y; Z_2) \xrightarrow{\phi} H_*(g_*;\ Z_2) \xrightarrow{\psi} H_*(f_*; Z_2)$$

where ϕ and ψ are the obvious homomorphisms. Since g is orientable over Z_2, ϕ will be injective, and it is also evident that ψ will be an isomorphism. The composition is therefore injective.

It may be of interest to point out in connection with Corollary 4 that a submersion $f\colon R^3 \to S^2$ with connected fibers can be obtained from the Hopf fibration $S^3 \to S^2$ by deleting a point in S^3. Moreover, we have shown in [2] that every submersion $f\colon X \to S^2$ of codimension 1 with connected fibers arises from this Hopf fibration by deleting a subspace $W \subset S^3$.

In the remaining corollaries we will be concerned with submersions $f\colon R^n \to S^m$ which are orientable over some coefficient module G. It should also be noted that these results depend on the fact that the fibers are manifolds.

COROLLARY 5. *There does not exist an orientable submersion* $f\colon R^n \to S^m$ *for* $n < 2m - 1$.

For such a submersion one could conclude by Corollary 1 that all fibers are t-acyclic over G for $t < m - 1$. But since the fibers are $(n - m)$-manifolds and $n - m < m - 1$, it follows that the fibers are t-acyclic over G for all t. By the Vietoris-Begle theorem for submersions [3], this would imply that S^m is acyclic.

COROLLARY 6. *For an orientable submersion* $f\colon R^{2m-1} \to S^m$ *every compact fiber must be a homology sphere and every noncompact fiber must be acyclic. Moreover, the subspace* $U \subset S^m$ *corresponding to compact fibers must have the homology of a point.*

One sees as before that every fiber is t-acyclic for $t < m - 1$. For the noncompact fibers this implies total acyclicity. The compact fibers will be homology $(m-1)$-spheres, provided they are orientable (in the sense of manifolds). This is easily established in the differentiable case if we consider the tangent bundle τ and normal bundle ν to f_y. Clearly ν is a trivial bundle (since it maps to the tangent vector space S_y^m under df) and is therefore orientable. On the other hand,

$$\tau \oplus \nu \approx \bar{\tau} \mid f_y$$

where $\bar{\tau}$ denotes the tangent bundle to R^{2m-1}. Since ν and $\bar{\tau}$ are both orientable, τ must be orientable, and this implies that f_y is also orientable. The conclusion about U follows now from a generalized Thom-Gysin sequence (Theorem 4.9 of [1]).

We have previously observed that submersions satisfying the hypothesis of Corollary 6 can be obtained in the case $m = 2$ from the Hopf fibration $S^3 \to S^2$, and it is clear that one can do likewise for $m = 4$ and 8. It is an open question whether orientable submersions $f: R^{2m-1} \to S^m$ exist for $m \neq 1, 2, 4$, and 8.

COROLLARY 7. *Let $f: R^n \to S^m$ be an orientable submersion with $2m - 1 \leqq n \leqq 3(m-1)$. Then every compact fiber must be a homology sphere.*

As before, every fiber must be t-acyclic for $t < m - 1$, and every compact fiber f_y must be orientable. By Poincaré duality applied to f_y, t-acyclicity for $0 < t < m - 1$ implies t-acyclicity for $n - 2m + 1 < t < n - m$. But the stipulated relation between n and m implies that the union of the two intervals is precisely the interval $0 < t < n - m$. Using once more the fact that f_y is orientable (together with 0-acyclicity), one also has $H_{n-m}(f_y; G) \approx G$, as was to be shown.

3. Homology of submersions.

With every submersion $f: X \to Y$ and coefficient module G one can associate certain homology groups $H_{s,t}(f; G)$ which arise quite naturally and can be studied on an axiomatic basis. The importance of these groups in the context of the present paper stems from the following theorem, which is the main result established in [1].

THEOREM. *Let $f: X \to Y$ be a submersion and G a coefficient module. There exists a convergent E^2 spectral sequence with*

$$E^2_{s\,t} \approx H_{s,t}(f; G)$$

and E^∞ isomorphic to the bigraded group associated with a filtration of $H_*(X, G)$.

It is to be noted that the given spectral sequence reduces to the Serre sequence in the case of a fibration.

All that we shall need to know about the homology groups $H_{s,t}(f; G)$ is that they can be computed by means of a certain algorithm, which is fully described in [1]. For the convenience of the reader we will now give a brief summary of this algorithm.

Firstly, one requires the notion of a *simplicial bundle* over a simplicial complex K, which may be defined as a map $p: E \to |K|$, together with a function which to every simplex $\sigma \in K$ assigns a homeomorphism ${}_\sigma\Phi: |\sigma| \times F_\sigma \approx E_\sigma$, where F_σ is compact and ${}_\sigma\Phi$ is the restriction of a tubular neighborhood belonging to p, such that

$$E = \bigcup_{\sigma \in K} E_\sigma \tag{B1}$$

$${}_\tau\Phi \text{ cuts } E_\sigma \text{ for every face } \tau < \sigma\,. \tag{B2}$$

With every simplicial bundle $p: E \to |K|$ and coefficient module G we will associate a chain complex $C_*(K; H_t(F; G))$ in which the homology groups $\{H_t(F_\sigma; G): \sigma \in K\}$ function as a local coefficient system. The s-dimensional chain group ($s \geqq 0$) is given by

$$C_s(K; H_t(F; G)) = \sum_{\sigma \in K^{(s)}} H_s(|\sigma|, |\dot\sigma|) \otimes H_t(F_\sigma; G) \tag{3.1}$$

where the right side represents a direct sum, $K^{(s)}$ denotes the set of s-simplexes in K and $|\dot\sigma| \subset |\sigma|$ the boundary. The chain boundary ∂ is defined on generators $a \otimes c$ through a formula

$$\partial(a \otimes c) = \sum_{\tau \in K^{(s-1)}} \varepsilon_\sigma^\tau(a) \otimes i_\sigma^\tau(c) \tag{3.2}$$

where

$$\varepsilon_\sigma^\tau: H_s(|\sigma|, |\dot\sigma|) \longrightarrow H_{s-1}(|\tau|, |\dot\tau|)$$

are incidence homomorphisms and

$$i_\sigma^\tau: H_t(F_\sigma; G) \longrightarrow H_t(F_\tau; G) \tag{3.3}$$

are so-called *fiber projections*, obtained as follows. For each $\sigma \in K$ we choose a point $\hat\sigma \in |\sigma|$ and identify F_σ with $F_{\hat\sigma}$ via ${}_\sigma\Phi$. For $\tau < \sigma$, one has $\hat\tau \in |\sigma|$, and by virtue of (B2) the homeomorphism ${}_\sigma\Phi_{\hat\sigma}^{\hat\tau}$ defines an injection $F_\sigma \to F_\tau$ (whose homotopy class does not depend upon $\hat\tau$ and $\hat\sigma$) which induces the fiber projection (3.3). This completes our definition of the chain complex associated with a simplicial bundle.

Now let $f: X \to Y$ be a map. A simplicial bundle $p: E \to |K|$ is said to *approximate* f if $E \subset X$ and p is the restriction of f to E. A sequence $\{{}_\alpha p: {}_\alpha E \to |{}_\alpha K|\}$ of simplicial bundles is called an *approximating sequence* for f if

(A1) each ${}_\alpha p$ approximates f and every compact subset of X is contained in some ${}_\alpha E$;

(A2) for $\alpha < \beta$, ${}_\alpha E \subset {}_\beta E$ and some subdivision of ${}_\alpha K$ is a subcomplex of ${}_\beta K$.

It has been shown in [1] that every submersion admits an approximating sequence.

Given an approximating sequence, one obtains chain projections

$$\phi_\alpha^\beta: C_s({}_\alpha K; H_t({}_\alpha F; G)) \longrightarrow C_s({}_\beta K; H_t({}_\beta F; G)) \tag{3.4}$$

for $\alpha < \beta$. These are defined on generators $a \otimes c$ by a formula of the form

$$\phi_\alpha^\beta(a \otimes c) = \sum_{\omega \in {}_\beta K^{(s)}} B_{\sigma,\omega}(a) \otimes j_{\sigma *}^{\omega}(c) \,, \tag{3.5}$$

where

$$B_{\sigma,\omega}: H_s(|\sigma|, |\dot{\sigma}|) \longrightarrow H_s(|\omega|, |\dot{\omega}|)$$

represents a subdivision operator for $|\omega| \subset |\sigma|$ and is zero otherwise, and $j_\sigma^\omega: {}_\alpha F_\sigma \to {}_\beta F_\omega$ denotes the composition

$${}_\alpha F_{\hat{\sigma}} \xrightarrow{\Phi_{\hat{\sigma}}^{\hat{\omega}}} {}_\alpha F_{\hat{\omega}} \subset {}_\beta F_{\hat{\omega}}$$

which makes sense for $|\omega| \subset |\sigma|$ by virtue of (A2).

The chain projections give us a direct system of chain complexes, and therefore also a direct system of homology groups. We obtain the desired homology of f as a direct limit by setting

$$H_{s,t}(f; G) \approx \varinjlim H_s({}_\alpha K; H_t({}_\alpha F; G)) \,.$$

4. Identification of the fiber homology. In this section we shall prove the following

THEOREM. *For every submersion $f: X \to Y$ and coefficient module G there is a canonical isomorphism*

$$\psi_*: H_{0,*}(f; G) \approx H_*(f_*; G) \,.$$

The proof of our Main Theorem, which will be given in the following section, depends upon this result, or more precisely, upon the surjectivity of ψ_*.

To prove the theorem at hand, we require the algorithm of § 3 for the computation of the homology groups $H_{s,t}(f; G)$. We will take an approximating sequence $\{_\alpha p: {}_\alpha E \to |{}_\alpha K|\}$ for f, and for each α we let ${}_\alpha C_{s,t}$ denote the corresponding chain group (3.1). One now observes that when σ is a vertex of ${}_\alpha K$, $H_0(|\sigma|, |\dot{\sigma}|)$ can be identified with the additive group of integers, and this permits us to identify ${}_\alpha C_{0,t}$ with the direct sum

$$\sum_{\sigma \in {}_\alpha K^{(0)}} H_t({}_\alpha F_{\hat{\sigma}}; G) . \tag{4.1}$$

Considering formula (3.2) for the chain boundary $\partial: {}_\alpha C_{1,t} \to {}_\alpha C_{0,t}$, and bearing in mind the definition of the fiber projections i_σ^{τ}, one sees that $\partial({}_\alpha C_{1,t})$ is precisely the submodule $W_{\alpha,t}$ of the direct sum (4.1) generated by the subset

$$\{[w - w']: w \underset{\sigma}{\sim} w' \bmod {}_\sigma\Phi \text{ for } \sigma \in {}_\alpha K^{(1)}\} .$$

Moreover, the inclusion induced homomorphism

$$H_t({}_\alpha F_\sigma; G) \longrightarrow H_t(f_{\hat{\sigma}}; G) ,$$

defined for every vertex σ in ${}_\alpha K$, induces a canonical homomorphism

$$\psi_\alpha: \sum_{\sigma \in {}_\alpha K^{(0)}} H_t({}_\alpha F_\sigma; G) \longrightarrow \sum_{y \in Y} H_t(f_y; G)$$

which obviously maps $W_{\alpha,t}$ into the submodule W (defined in § 1). Consequently one obtains a canonical homomorphism

$$\psi_{\alpha *}: {}_\alpha H_{0,t} \longrightarrow H_t(f_*; G) , \tag{4.2}$$

where ${}_\alpha H_{0,t}$ denotes the 0-dimensional homology of the chain complex ${}_\alpha C_{*,t}$.

We recall that the homology module $H_{0,t}(f; G)$ is given as the direct limit of a direct system

$$\{{}_\alpha H_{0,t}, \phi_{\alpha *}^{\beta}\} \tag{4.3}$$

the projections $\phi_{\alpha *}^{\beta}$ being induced by the chain projections (3.4). It is easy to verify that the homomorphisms (4.2) commute with the projections $\phi_{\alpha *}^{\beta}$. For under the representation (4.1) of the 0-dimensional chain groups, formula (3.5) assumes the form

$$\phi_\alpha^\beta(c) = \sum_{\sigma \in {}_\alpha K^{(0)}} j_{\sigma *}^{\sigma}(c) , \tag{4.4}$$

where $j_{\sigma *}^{\sigma}: {}_\alpha F_\sigma \to {}_\beta F_\sigma$ is induced by the inclusion ${}_\alpha F_{\hat{\sigma}} \subset {}_\beta F_{\hat{\sigma}}$, so that the desired commutativity holds already on the chain level (i.e., $\psi_\beta \circ \phi_\alpha^\beta = \psi_\alpha$ for $\alpha < \beta$). The homomorphisms (4.2) constitute therefore a homomorphism of the direct system (4.3), and this induces a homomorphism

$$\psi_*\colon H_{0,t}(f; G) \longrightarrow H_t(f_*; G)$$

in the direct limit.

To establish the surjectivity of ψ_*, it will suffice to show that for any point $y \in Y$ and homology class $w \in H_t(f_y; G)$, there exists an index α and an element $c \in {}_\alpha C_{0,t}$ such that

$$[\psi_\alpha(c) - w] \in W . \tag{4.5}$$

But this is very easy to prove. Let w be represented by a singular cycle Z having its support in some compact subset $C \subset f_y$. Property (A1) of the approximating sequence permits us to choose an index α such that $C \subset {}_\alpha E$, and we let σ denote the carrier simplex of y. This means that y is an interior point of $|\sigma|$, and for an arbitrary vertex τ of σ we can define a singular cycle Z' in ${}_\alpha F_\tau$, through the formula

$$Z' = {}_\sigma\Phi^\tau_{y\#} Z .$$

This determines an element $w' \in H_t({}_\alpha F_\tau; G)$, and recalling the representation (4.1), we obtain thus a chain $c \in {}_\alpha C_{0,t}$, which obviously satisfies (4.5).

A considerably more complicated argument will be required to establish the injectivity of ψ_*. To simplify the proof, it will be convenient to assume that our approximating sequence satisfies the additional condition

$$\varinjlim \operatorname{mesh} {}_\alpha K = 0 , \tag{A3}$$

which implies that if U is any open set in Y, there exists an index α such that U contains a vertex of ${}_\alpha K$. It is quite easy to see that if one has an approximating sequence satisfying (A1) and (A2), the additional condition (A3) can be achieved by taking appropriate subdivisions. It will also be convenient to precede the proof of injectivity by three lemmas.

To begin with, it will be recalled that an element $u \in W$ admits a representation of the form

$$u = \sum_{i=1}^{r} [w_i - w_i'] \tag{4.6}$$

where $w_i \in H_*(f_{y_i}; G)$, $w_i' \in H_*(f_{y_i'}; G)$ and

$$w_i \pitchfork w_i' \bmod {}_i\Phi$$

for some tubular neighborhoods ${}_i\Phi\colon B_i \times {}_iF \to V_i$ belonging to f. If

K is a complex with $|K| \subset Y$, we will say that the representation (4.6) *belongs* to K provided the points y_i and y'_i are vertices of K for all i.

LEMMA 1. *Given $c \in {}_\alpha C_{0,t}$ such that $\psi_\alpha(c) \in W$, there exists an index $\beta > \alpha$ such that $\psi_\alpha(c)$ admits a representation* (4.6) *belonging to ${}_\beta K$.*

To prove this, let us consider an arbitrary representation (4.6) for the element $\psi_\alpha(c)$. By virtue of (A3) we can choose $\beta > \alpha$ so that each subset B_i contains a vertex of ${}_\beta K$. For each index i we may now choose points $\bar{y}_i, \bar{y}'_i \in B_i \cap {}_\beta K^{(0)}$, with the proviso that $\bar{y}_i = y_i$ when $y_i \in {}_\beta K^{(0)}$, and $\bar{y}'_i = y'_i$ when $y'_i \in {}_\beta K^{(0)}$. Moreover, (4.6) implies that we can represent w_i and w'_i by singular cycles Z_i and Z'_i, in ${}_iF_{y_i}$ and ${}_iF_{y'_i}$, respectively such that

$$Z'_i = {}_i\Phi^{y'_i}_{y_i \#} Z_i \ .$$

Setting

$$\bar{Z}_i = {}_i\Phi^{\bar{y}_i}_{y_i \#} Z_i$$

$$\bar{Z}'_i = {}_i\Phi^{\bar{y}'_i}_{y'_i \#} Z'_i$$

we obtain corresponding homology classes $\bar{w}_i \in H_*(f_{\bar{y}_i}; G)$ and $\bar{w}'_i \in H_*(f_{\bar{y}'_i}; G)$, and we can now rewrite the representation (4.6) for $\psi_\alpha(c)$ in the form

$$\psi_\alpha(c) = \sum_{i=1}^{r} \{[\bar{w}_i - \bar{w}'_i] + [w_i - \bar{w}_i] + [\bar{w}'_i - w'_i]\} \ .$$

It will be shown that

$$\sum_{i=1}^{r} \{[w_i - \bar{w}'_i] + [\bar{w}'_i - w'_i]\} = 0 \ . \tag{4.7}$$

To establish this, one observes that $\psi_\alpha(c)$ must be a linear combination of elements corresponding to vertices of ${}_\alpha K$. Our construction insures that $[w_i - \bar{w}_i]$ vanishes when $y_i \in {}_\beta K^{(0)}$, and similarly $[\bar{w}'_i - w'_i]$ vanishes when $y'_i \in {}_\beta K^{(0)}$. This means that the summation in (4.7) represents a linear combination of elements corresponding to vertices which do not belong to ${}_\beta K$. On the other hand, since the terms $[\bar{w}_i - \bar{w}'_i]$ involve only vertices belonging to ${}_\beta K$, one may now conclude (4.7). Consequently one has

$$\psi_\alpha(c) = \sum_{i=1}^{r} [\bar{w}_i - \bar{w}'_i] \ ,$$

which is a representation belonging to ${}_\beta K$.

LEMMA 2. *Given* $c \in {}_\alpha C_{0,t}$ *such that* $\psi_\alpha(c) = 0$, *there exists an index* $\beta > \alpha$ *such that* $\phi_\alpha^\beta(c) = 0$.

Since ${}_\alpha C_{0,t}$ is generated by elements $w \in H_t({}_\alpha F_\tau; G)$, where τ is a vertex of ${}_\alpha K$, it will suffice to establish the lemma in the case $c = w$. Let w be represented by a singular cycle Z in ${}_\alpha F_\tau$. Since ψ_α annihilates w, there exists a compact subspace $C \subset f_\tau$ such that Z bounds in C. If we now choose $\beta > \alpha$ such that $C \subset {}_\beta F_\tau$ (we are using properties (A1) and (A2)), then the inclusion induced homomorphism

$$j_{\tau*}^\tau \colon H_t({}_\alpha F_\tau; G) \longrightarrow H_t({}_\beta F_\tau; G)$$

annihilates w, which implies by (4.4) that $\phi_\alpha^\beta(w) = 0$.

LEMMA 3. *Let* $\Phi \colon B \times F \to V$ *and* $\bar{\Phi} \colon \bar{B} \times \bar{F} \to \bar{V}$ *be tubular neighborhoods belonging to* f, *such that* $\bar{\Phi}$ *cuts* V *and* $B \cap \bar{B}$ *is connected. Let* $y, y' \in B \cap \bar{B}$, *and let* Z *denote a singular cycle in* F_y. *Then*

$$\Phi_{y\#}^{y'} Z \sim \bar{\Phi}_{y\#}^{y'} Z \text{ in } \bar{F}_{y'} .$$

We may assume without loss of generality that $B = \bar{B}$ and $F \subset \bar{F}$, which implies, moreover, that $V \subset \bar{V}$ and $F_x \subset \bar{F}_x$ for every $x \in B$. Considering $\Phi_y^{y'}$ and $\bar{\Phi}_y^{y'}$ as maps from F_y into $\bar{F}_{y'}$, it will suffice to construct a homotopy $H \colon I \times F_y \to \bar{F}_{y'}$ such that

$$H \colon \bar{\Phi}_y^{y'} \simeq \Phi_y^{y'} . \tag{4.8}$$

Let $\phi \colon I \to B$ denote a path from y to y'. The requisite homotopy H may now be given by the formula

$$H(t, x) = \bar{\Phi}_y^{y'} \circ \bar{\Phi}_{\phi(t)}^{y} \circ \Phi_y^{\phi(t)}(x)$$

which obviously gives (4.8).

At last we are ready to establish the injectivity of ψ_*. It will suffice to prove that if $\psi_\alpha(c) \in W$ for some $c \in {}_\alpha C_{0,t}$, then there exists an index $\gamma > \alpha$ such that $\phi_\alpha^\gamma(c) \in W_\gamma$. We first apply Lemma 1 to conclude that there exists a $\beta > \alpha$ such that $\psi_\alpha(c)$ admits a representation (4.6) belonging to ${}_\beta K$. By virtue of condition (A1) one may further assume that $V_i \subset {}_\beta E$ for each index i, which means that each homology class w_i and w_i' in the representation (4.6) can be represented by a singular cycle Z_i and Z_i' in ${}_\beta F_{y_i}$ and ${}_\beta F_{y_i'}$, respectively, such that

$$Z_i' = {}_i\Phi^{y_i'}_{y_i\#} Z_i \ .$$

The cycles Z_i and Z_i' induce corresponding homology classes ${}_\beta w_i \in H_t({}_\beta F_{y_i}; G)$ and ${}_\beta w_i' \in H_t({}_\beta F_{y_i'}; G)$, and this gives an element

$$c' = \sum_{i=1}^{r} [{}_\beta w_i - {}_\beta w_i']$$

in ${}_\beta C_{0,t}$ such that $\psi_\beta(c') = \psi_\alpha(c)$. Since $\psi_\alpha = \psi_\beta \circ \phi_\alpha^\beta$, it follows that ψ_β annihilates $[\phi_\alpha^\beta(c) - c']$. By Lemma 2, this element is also annihilated by ϕ_β^γ for some $\gamma > \beta$, and this leads to a formula

$$\phi_\alpha^\gamma(c) = \sum_{i=1}^{r} [{}_\gamma w_i - {}_\gamma w_i'] \ ,$$

in which the homology classes ${}_\gamma w_i$ and ${}_\gamma w_i'$ can once again be represented by the singular cycles Z_i and Z_i', respectively. It remains to be shown that

$$[{}_\gamma w_i - {}_\gamma w_i'] \in \partial_\gamma C_{1,t}$$

for all i. Since each subset B_i is open and connected, we may assume without loss of generality that the vertices y_i and y_i' can be joined by an edge path in ${}_\gamma K$ (condition (A3) guarantees that this will hold for sufficiently large values of γ). Let $\tau_0, \cdots, \tau_n$ denote successive vertices of this edge path, and let

$$Z_j = {}_i\Phi^{\tau_j}_{y\#} Z; \qquad j = 0, \cdots, n \ . \tag{4.9}$$

The cycles Z_j determine homology classes $\bar{w}_j \in H_t({}_\gamma F_{\tau_j}; G)$, and one sees that $\bar{w}_0 = {}_\gamma w_i$ and $\bar{w}_n = {}_\gamma w_i'$. It suffices therefore to show that

$$[\bar{w}_j - \bar{w}_{j-1}] \in \partial_\gamma C_{1,t}; \qquad j = 1, \cdots, n \ . \tag{4.10}$$

We note that

$$Z_j = {}_i\Phi^{\tau_j}_{\tau_{j-1}\#} Z_{j-1}$$

by virtue of (4.9). Moreover, if σ_j denotes the 1-simplex in ${}_\gamma K$ with vertices τ_j and τ_{j-1}, then ${}_{\sigma_j}\Phi$ cuts V_i. One may conclude by Lemma 3 that

$$\bar{w}_j \pitchfork \bar{w}_{j-1} \bmod {}_{\sigma_j}\Phi \ ,$$

which implies (4.10).

5. Proof of the main theorem. Let $f \colon X \to Y$ be a submersion and G a coefficient module. According to the main result in [1], there exists a convergent E^2 spectral sequence with

$$E^2_{s,t} \approx H_{s,t}(f; G) \tag{5.1}$$

and E^∞ isomorphic to the bigraded module associated with a filtration of $H_*(X; G)$. By the result of § 4, (5.1) implies

$$E^2_{0,t} \approx H_t(f_*; G) \; . \tag{5.2}$$

Since evidently $E^2_{0,0} \approx E^\infty_{0,0} \approx H_0(X; G)$, one obtains immediately the desired equality $R_0(f_*; G) = R_0(X; G)$.

Under the assumption that the fibers f_y are connected, the homology groups $H_{s,0}(f; G)$ evidently reduce to $H_s(Y; G)$, so that

$$E^2_{s,0} \approx H_s(Y; G) \; . \tag{5.3}$$

If one assumes further that the fibers f_y are t-acyclic over G for $q - m < t < q$, one may conclude (by Theorem 4.1 of [1] and (5.1)) that

$$E^2_{s,t} = 0 \text{ for all } s \text{ and } q - m < t < q \; . \tag{5.4}$$

Moreover, since the chain groups entering into the definition of $H_{s,t}(f; G)$ are trivial for $s > m = \dim Y$, one also obtains

$$E^2_{s,t} = 0 \text{ for } s > m \text{ and all } t \; . \tag{5.5}$$

The desired inequality

$$R_q(f_*; G) \leqq R_q(X; G) + R_{q+1}(Y; G) \tag{5.6}$$

follows from conditions (5.2) through (5.5) by a simple spectral sequence argument, which runs as follows.

We may assume $q \geqq 1$. The exact sequence

$$E^{q+1}_{q+1,0} \xrightarrow{d^{q+1}} E^{q+1}_{0,q} \longrightarrow \operatorname{coker} d^{q+1} \longrightarrow 0 \tag{5.7}$$

gives the rank inequality

$$\operatorname{rank} E^{q+1}_{0,q} \leqq \operatorname{rank} \operatorname{coker} d^{q+1} + \operatorname{rank} E^{q+1}_{q+1,0} \; . \tag{5.8}$$

Since all differentials into the $(q + 1, 0)$-position are trivial, one has $E^{q+1}_{q+1,0} \subset E^2_{q+1,0}$, and this implies by (5.3) that

$$\operatorname{rank} E^{q+1}_{q+1,0} \leqq R_{q+1}(Y; G) \; . \tag{5.9}$$

Moreover, one observes by (5.4) and (5.5) that d^{q+1} is the only non-trivial differential touching the $(0, q)$-position. This gives $E^2_{0,q} \approx E^{q+1}_{0,q}$, which together with (5.2) implies

$$\operatorname{rank} E^{q+1}_{0,q} = R_q(f_*; G) \; . \tag{5.10}$$

The preceding observation also gives

$$\operatorname{coker} d^{q+1} \approx E_{0,q}^{q+2} \approx E_{0,q}^{\infty} .$$

But since E^{∞} is isomorphic to the bigraded module associated to a filtration of $H_*(X; G)$, this implies

$$\text{rank coker } d^{q+1} \leqq R_q(X; G) . \tag{5.11}$$

Finally, (5.8) through (5.11) yields (5.6).

If, in addition, $H_q(X; G)$ and $H_{q+1}(Y; G)$ vanish, then one sees by the same considerations that the exact sequence (5.7) reduces to

$$0 \longrightarrow H_q(f_*; G) \longrightarrow 0 \longrightarrow 0 ,$$

and this completes our proof.

6. Orientability of real-valued submersions.

THEOREM. *A submersion $f: X \to R$ is orientable if and only if it does not admit separatrices.*

In this section we shall prove the nontrivial half of this theorem (the "if" part). This will involve looking carefully at representations (4.6) for elements $u \in W$ in the special case $Y = R$. Such a representation will be called *simple* if

$$y_1 < y_1' \leqq y_2 < y_2' \leqq y_3 < \cdots < y_r' .$$

LEMMA 1. *Every element of W admits a simple representation.*

To begin with, one may assume without loss of generality that one has a representation with $y_i < y_i'$ for $i \leqq r$ and $y_i \leqq y_{i+1}$ for $i < r$ (this can be achieved by proper "labeling"). We also note that if $y \in (y_i, y_i')$, one may evidently "subdivide" the given representation by setting

$$[w_i - w_i'] = [w_i - w] + [w - w_i'] ,$$

where w denotes the (uniquely determined) element in $H_*(f_y; G)$ such that

$$w_i \pitchfork w \bmod {}_i\Phi \text{ and } w \pitchfork w' \bmod {}_i\Phi .$$

By the use of such subdivisions one can eliminate "partial intersections" between open intervals (y_i, y_i'), which is to say that one can achieve the following condition: two intervals (y_i, y_i') and (y_j, y_j') are either disjoint or identical. The resulting representation need not yet be simple, precisely on account of the second alternative. Let us therefore suppose that for all values of i in some subset J one has $y_i = y$ and $y'_j = y'$. It remains to be shown that one can construct a simple representation for the element

$$\sum_{i \in J} [w_i - w'_i] . \tag{6.1}$$

To this end let $C \subset X$ denote a compact subset containing all images V_i corresponding to tubular neighborhoods ${}_i\Phi$ with $i \in J$. By virtue of (1.4) there exists a partition

$$y = y^0 < y^1 < \cdots < y^s = y'$$

and tubular neighborhoods $\Phi^j \colon B^j \times F^j \to V^j$ such that $[y^{j-1}, y^j] \subset B^j$ and Φ^j cuts C for $j = 1, \cdots, s$. For each $i \in J$ the homology class w_i can be represented by a singular cycle Z_i in ${}_iF_y$, and this gives rise to cycles

$$Z_i^j = {}_i\Phi_{y\#}^{y^j} Z_i \, ; \qquad\qquad j = 0, \cdots, s .$$

We let $w_i^j \in H_*(f_{y_j}; G)$ denote the corresponding homology classes. It follows now by Lemma 3 of § 4 that

$$w_i^{j-1} \pitchfork w_i^j \bmod \Phi^j \qquad\qquad \text{for } j = 1, \cdots, s .$$

Setting

$$w^i = \sum_{i \in J} w_i^j$$

one obtains a simple representation

$$\sum_{j=1}^{s} [w^{j-1} - w^j]$$

for the element (6.1).

LEMMA 2. *Let $w \in H_t(f_y; G)$ be nonzero, and let*

$$w \pitchfork 0_{y'} \bmod \Phi$$

where $O_{y'}$ denotes the zero in $H_t(f_{y'}; G)$. Then f admits a separatrix f_{y^*} for some point y^* between y and y'.

The proof of this lemma is very simple. We may assume $y < y'$, and we let S denote the set of all points $\bar{y} \in [y, y']$ such that

$$w \pitchfork O_{\bar{y}} \bmod \Phi .$$

It is an easy consequence of (1.4) that S is open. Since $y' \in S$ and $y \notin S$, there must be a point $y^* \in [y, y')$ belonging to the boundary of S, and this means that f_{y^*} will be a separatrix.

The proof of our theorem can now proceed as follows. Let

$f: X \to R$ be a submersion without separatrices. To show that f is orientable, one must prove that a nonzero element $w \in H_t(f_y; G)$ cannot belong to W. But if $w \in W$, one concludes by Lemma 1 that it admits a simple representation

$$\sum_{i=1}^{r} [w_i - w'_i]$$

and it may be supposed on the strength of Lemma 2 that the elements w_i and w'_i are all nonzero (for if one element is zero, the other member of the pair must be zero as well). Since w'_r is the only element in the given representation which corresponds to the point y'_r, w'_r can be nonzero only if $y = y'_r$ and $w = w'_r$. But this would imply by the same token that $w_1 = 0$, and this furnishes the desired contradiction.

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