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MODULAR SUBLATTICES OF THE LATTICE OF VARIETIES OF INVERSE SEMIGROUPS

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Kleiman used the variety $\mathscr{G}\mathscr{P}$ of all groups to define two endomorphisms $\varphi^{\mathscr{G}\mathscr{P}}$ and $\varphi_{\mathscr{G}\mathscr{P}}$ of the lattice $\mathscr{L}(\mathscr{I})$ of varieties of inverse semigroups as follows: $\varphi^{\mathscr{G}\mathscr{P}}(\mathscr{V}) = \mathscr{G}\mathscr{P} \vee \mathscr{V}$ and $\varphi_{\mathscr{G}\mathscr{P}}(\mathscr{V}) = \mathscr{G}\mathscr{P} \wedge \mathscr{V}$. This introduced two congruences ν_1 and ν_2 on $\mathscr{L}(\mathscr{I})$ which have been very important in recent studies of $\mathscr{L}(\mathscr{I})$.

This paper is devoted to studying further properties of the ν_1 and $\nu_3 = \nu_1 \cap \nu_2$ congruence classes.

The first main result establishes that each ν_1-class is a complete modular sublattice of $\mathscr{L}(\mathscr{I})$, although, in some cases, the class may just consist of a single element.

It is not difficult to see that each ν_3-class has a minimum member. On the other hand, it is shown that not all ν_3-classes have maximum members. However, it is established that a large class of ν_3-classes do have maximum members. If $\mathscr{U}$ is a variety satisfying an identity of the form $x^{n+1}tt^{-1}x^{-n-1} = x^n tt^{-1}x^{-n}$ then the ν_3-class containing $\mathscr{U}$ has a maximum member. The condition that a variety satisfies this identity is equivalent to a member of conditions, one being that every member of $\mathscr{V}$ is completely semisimple and such that $\mathscr{H}$ is a congruence.

The nature of the maximum element in these cases is very interesting. If $\mathscr{U}$ satisfies the above identity, then the fundamental inverse semigroups contained in $\mathscr{U}$ constitute a variety, $\mathscr{V}$ say. Letting $\mathscr{G} = \mathscr{G}\mathscr{P} \cap \mathscr{U}$, the maximum element in the ν_3-class containing $\mathscr{U}$ is shown to be the Mal'cev product $\mathscr{G} \circ \mathscr{V}$ of the varieties $\mathscr{G}$ and $\mathscr{V}$. It is shown that this is not valid in general. Other properties of the Mal'cev product are obtained.

1. Notation and terminology. We shall adopt the basic notation and terminology for semigroups from [2] while, for basic results in the theory of varieties of groups, the reader is referred to [10].

The variety of all inverse semigroups, (groups, abelian groups) will be denoted by $\mathscr{I}(\mathscr{G}\mathscr{P}, \mathscr{A}\mathscr{G}\mathscr{P})$ and the trivial variety by $\mathscr{T}$. Throughout the paper the term variety, if unqualified, will always mean a variety of inverse semigroups.

We will denote by $F_X(G_X)$ the free inverse semigroup (group) on a countable set X.

For any semigroup S, $\mathscr{V}(S)$ will denote the variety generated by S. For any variety $\mathscr{V}$, $F(\mathscr{V})$ will denote the relatively free inverse semigroup in $\mathscr{V}$ of countable rank and $\rho(\mathscr{V})$ will denote the verbal congruence on $F_X = F(\mathscr{I})$ defining the variety $\mathscr{V}$. In addition, $\mathscr{L}(\mathscr{V})$ will denote the lattice of varieties contained in $\mathscr{V}$.

For any group variety $\mathscr{G}$, we denote by $\mathscr{U}(\mathscr{G})$ the fully invariant subgroup of G_X determining the variety $\mathscr{G}$. We shall also denote by $\mathscr{Q}(\mathscr{G})$ the class of varieties $\mathscr{V}$ such that $\mathscr{V} \cap \mathscr{G}\mathscr{P} = \mathscr{G}$. In particular, $\mathscr{Q}(\mathscr{T})$ denotes the class of those varieties that contain no nontrivial group.

For any inverse semigroup S, we will denote by $E(S)$ the semilattice of idempotents of S and by μ_S the maximum idempotent separating congruence on S. For the basic properties of μ_S, the reader is referred to [8].

For more extensive information on the background and context of this paper, the reader is referred particularly to the papers by Djadchenko [3], Kleiman [5], [6], and Reilly [12].

2. The modularity of ν_3-classes. For any inverse semigroup S, we denote by $\Lambda(S)$ the lattice of congruences on S and write

$$\theta(S) = \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) : \rho_1 \cap E_S \times E_S = \rho_2 \cap E_S \times E_S\} .$$

From Reilly and Scheiblich [13], we have the following result.

LEMMA 2.1. *For any inverse semigroup S,*
(1) *$\theta(S)$ is a congruence on $\Lambda(S)$;*
(2) *each θ-class is a complete modular sublattice of $\Lambda(S)$.*

Let $\delta \subseteq F_X \times F_X$ and, for each $(u_\lambda, v_\lambda) \in \delta$ let t_λ be an element of X which does not appear in either u_λ or v_λ. Then we write

$$\bar{\bar{\delta}} = \{(u_\lambda t_\lambda t_\lambda^{-1} u_\lambda^{-1}, v_\lambda t_\lambda t_\lambda^{-1} v_\lambda) : (u_\lambda, v_\lambda) \in \delta\} .$$

The following result is due to Kleiman [5].

LEMMA 2.2. *If δ is a basis for the identities of a variety $\mathscr{V}$ then $\bar{\bar{\delta}}$ is a basis for the identities of $\mathscr{U} \vee \mathscr{G}\mathscr{P}$.*

From this we immediately obtain the following corollary:

COROLLARY 2.3. *Let $\mathscr{V}$ be any variety. Then $\mathscr{V} \vee \mathscr{G}\mathscr{P} = \{S \in \mathscr{I} : S/\mu_S \in \mathscr{V}\}$.*

This leads us to the main result of this section. For varieties $\mathscr{V}$, $\mathscr{W}$ with $\mathscr{V} \subseteq \mathscr{W}$ we write $[\mathscr{V}, \mathscr{W}]$ for the set of varieties $\mathscr{U}$ with $\mathscr{V} \subseteq \mathscr{U} \subseteq \mathscr{W}$.

THEOREM 2.4. *For any variety $\mathscr{V}$ of inverse semigroups, the interval $[\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$ is a complete modular sublattice of $\mathscr{L}(\mathscr{I})$.*

Proof. Let $\rho_1 = \rho(\mathscr{V} \vee \mathscr{G}\mathscr{P})$ and $\rho_2 = \rho(\mathscr{V})$. Then $F(\mathscr{V} \vee \mathscr{G}\mathscr{P})$ is just F_X/ρ_1 and $F(\mathscr{V})$ is just F_X/ρ_2. Let $\tau = \rho_2/\rho_1$. Then $(F_X/\rho_1)/\tau$ is isomorphic to F_X/ρ_2.

Since $F_X/\rho_1 \in \mathscr{V} \vee \mathscr{G}\mathscr{P}$, it follows from Corollary 2.3, that $(F_X/\rho_1)/\mu \in \mathscr{V}$, where μ is the maximum idempotent separating congruence on F_X/ρ_1. Suppose that we denote by ρ_3 the congruence on F_X inducing μ on F_X/ρ_1, that is, such that F_X/ρ_3 is isomorphic to $(F_X/\rho_1)/\mu$. Then $F_X/\rho_3 \in \mathscr{V}$ and so $\rho_2 \subseteqq \rho_3$. Hence $\tau = \rho_2/\rho_1 \subseteqq \rho_3/\rho_1 = \mu$. Thus τ is an idempotent separating congruence on F_X/ρ_1. Hence $(\rho_1/\rho_2) \in \theta(F_X)$.

Now let $\mathscr{W} \in [\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$ and $\rho_4 = \rho(\mathscr{W})$. Then $\rho_1 \subseteqq \rho_4 \subseteqq \rho_2$. Hence, ρ_1, ρ_2 and ρ_4 are all $\theta = \theta(F_X)$ equivalent.

Let us write $\mathscr{L}(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}) = [\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$ and $\mathscr{L}'(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ for the lattice of fully invariant congruences of the form $\rho(\mathscr{W})$, $\mathscr{W} \in [\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$. Then $\mathscr{L}'(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ is the set of all fully invariant congruences between (and including) $\rho(\mathscr{V})$ and $\rho(\mathscr{V} \vee \mathscr{G}\mathscr{P})$ and is anti-isomorphic to $\mathscr{L}(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$.

From the above, we see that $\mathscr{L}'(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ is contained in a single θ-class A, say, which, by Lemma 2.1, is a modular sublattice of $\Lambda(F_X)$. Since $\mathscr{L}'(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ is a sublattice of $\Lambda(F_X)$ and so of A, it follows that $\mathscr{L}'(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ must be a modular lattice. Hence, $\mathscr{L}(\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P})$ is also modular, as required.

Since $\mathscr{L}(\mathscr{I})$ is a complete, it follows that $[\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$ is a complete sublattice.

Let two mappings $\varphi^\vee$ and $\varphi^\wedge$ be defined on $\mathscr{L}(\mathscr{I})$ as follows:

$$\varphi^\vee(\mathscr{V}) = \mathscr{V} \vee \mathscr{G}\mathscr{P}, \qquad \varphi^\wedge(\mathscr{V}) = \mathscr{V} \cap \mathscr{G}\mathscr{P}, \qquad \text{for all } \mathscr{V} \in \mathscr{L}(\mathscr{I}).$$

LEMMA 2.5. *Kleiman* [5]. *The mapping $\varphi^\vee$ is a homomorphism of $\mathscr{L}(\mathscr{I})$ onto the lattice of varieties containing $\mathscr{G}\mathscr{P}$ and $\varphi^\wedge$ is a homomorphism of $\mathscr{L}(\mathscr{I})$ onto $\mathscr{L}(\mathscr{G}\mathscr{P})$.*

This leads to certain useful partitions of $\mathscr{L}(\mathscr{I})$. Let $\nu_1(\nu_2)$ be the congruence on $\mathscr{L}(\mathscr{I})$ induced by $\varphi^\vee(\varphi^\wedge)$ and $\nu_3 = \nu_1 \cap \nu_2$. Thus,

$$(\mathscr{V}, \mathscr{W}) \in \nu_1 \text{ if and only if } \mathscr{V} \vee \mathscr{G}\mathscr{P} = \mathscr{W} \vee \mathscr{G}\mathscr{P},$$

$$(\mathscr{V}, \mathscr{W}) \in \nu_2 \text{ if and only if } \mathscr{V} \cap \mathscr{G}\mathscr{P} = \mathscr{W} \cap \mathscr{G}\mathscr{P}.$$

Of course, for any group variety $\mathscr{G}$, the ν_2-class containing $\mathscr{G}$ is just $\mathscr{Q}(\mathscr{G})$.

Recall [8] that an inverse semigroup is said to be *fundamental* if it has no nontrivial idempotent separating congruences or, equivalently, no nontrivial congruences contained in Green's relation $\mathscr{H}$. Such semigroups have also been called *antigroups* ([3] and elsewhere). We shall denote the class of the such inverse semigroups by $A\mathscr{G}\mathscr{P}$. It should be noted that this class is not itself a variety although it does have a role to play.

PROPOSITION 2.6. *Kleiman* [5]. *Let* $\mathscr{V}, \mathscr{W} \in \mathscr{L}(\mathscr{I})$. *Then* $\mathscr{V} \vee \mathscr{G}\mathscr{P} = \mathscr{W} \vee \mathscr{G}\mathscr{P}$ *if and only if* $\mathscr{V} \cap A\mathscr{G}\mathscr{P} = \mathscr{W} \cap A\mathscr{G}\mathscr{P}$.

This result provides us with a further description of ν_1. Thus, for $\mathscr{V}, \mathscr{W} \in \mathscr{L}(\mathscr{I})$,

$$(\mathscr{V}, \mathscr{W}) \in \nu_1 \quad \text{if and only if} \quad \mathscr{V} \cap A\mathscr{G}\mathscr{P} = \mathscr{W} \cap A\mathscr{G}\mathscr{P} .$$

Let us now focus our attention on a single ν_1-class $\mathscr{N}$. Let $\mathscr{V} \in \mathscr{N}$. Then clearly $\mathscr{M}_1 = \mathscr{V} \vee \mathscr{G}\mathscr{P}$ is the maximum element of $\mathscr{N}$. On the other hand, by Proposition 2.6 $\mathscr{M}_1 \cap A\mathscr{G}\mathscr{P} \subseteq \mathscr{V}$, for all $\mathscr{V} \in \mathscr{N}$. Hence, $\mathscr{M}_0 = \mathscr{V}(\mathscr{M}_1 \cap A\mathscr{G}\mathscr{P})$ is a minimum element of $\mathscr{N}$ and $\mathscr{M}_1 = \mathscr{M}_0 \vee \mathscr{G}\mathscr{P}$. Combining this with Theorem 2.4 we have the following result.

THEOREM 2.7. *Let* $\mathscr{N}$ *be a* ν_1*-class of* $\mathscr{L}(\mathscr{I})$. *Then* $\mathscr{N}$ *has both a minimum element,* $\mathscr{M}_0$, *and a maximum element,* $\mathscr{M}_1 = \mathscr{M}_0 \vee \mathscr{G}\mathscr{P}$. *Moreover,* $\mathscr{N}$ *is a complete modular sublattice of* $\mathscr{L}(\mathscr{I})$.

Since the minimum element of any ν_1-class is generated by its fundamental elements and, conversely, any variety generated by its fundamental members is the minimum in its ν_1-class, we say that any such variety is *fundamental*.

For any fundamental variety $\mathscr{V}$, let $\varphi^{\mathscr{V}}: \mathscr{L}(\mathscr{G}\mathscr{P}) \to [\mathscr{V}, \mathscr{V} \vee \mathscr{G}\mathscr{P}]$ be defined by

$$\varphi^{\mathscr{V}}(\mathscr{G}) = \mathscr{V} \vee \mathscr{G}, \quad \text{for all} \quad \mathscr{G} \in \mathscr{L}(\mathscr{G}\mathscr{P}) .$$

Then $\varphi^{\mathscr{V}}$ maps $\mathscr{L}(\mathscr{G}\mathscr{P})$ into $(\mathscr{V})\nu_1$, the ν_1-class containing $\mathscr{V}$.

It is interesting to note some of the situations that arise in this context.

Let Y_2 denote the two-element semilattice, B_2 denote the Brandt semigroup of rank two with trivial structure group and B_2^1 denote B_2 with an identity adjoined.

Then Djadchenko [3] has shown that for $\mathscr{V}$ equal to $\mathscr{V}(Y_2)$ or $\mathscr{V}(B_2)$, $\varphi^{\mathscr{V}}$ is an isomorphism of $\mathscr{L}(\mathscr{G}\mathscr{P})$ onto $(\mathscr{V})\nu_1$. Although $\mathscr{V}(B_2^1)$ covers $\mathscr{V}(B_2)$ in $\mathscr{L}(\mathscr{I})$, an example in [12], shows that $\varphi^{\mathscr{V}}$, where $\mathscr{V} = \mathscr{V}(B_2^1)$ does not map $\mathscr{L}(\mathscr{G}\mathscr{P})$ onto $(\mathscr{V}(B_2^1))\nu_1$. However,

φ' is still one-to-one.

At the other extreme, $(\mathscr{V})\nu_1$ can be much smaller than $\mathscr{L}(\mathscr{G}\mathscr{P})$ and can even consist of a singleton, as the following example illustrates.

EXAMPLE. Let $T = \mathscr{M}^0(1; G_X, G_X; \Delta)$ and $S = T \cup G_X$. Define a multiplication on S by defining it to be the given multiplication within T and G_X, such that the zero of T is the zero of S and such that, for any $g \in G_X$, $(1; h, k) \in T$,

$$\begin{aligned} g(1; h, k) &= (1; gh, k) \\ (1; h, k)g &= (1; h, g^{-1}k) . \end{aligned}$$

It is a routine matter to check that S is an inverse semigroup with respect to this multiplication. In fact, S is a subsemigroup of the translational hull $\Omega(T)$ of T. Moreover, S is fundamental. Hence, $\mathscr{V}(S)$ is fundamental. But clearly $G_X \in \mathscr{V}(S)$. Therefore, $\mathscr{V}(S) \vee \mathscr{G}\mathscr{P} = \mathscr{V}(S)$ and the ν_1-class containing $\mathscr{V}(S)$ contains only $\mathscr{V}(S)$ itself.

Now S satisfies the identity $x^3x^{-3} = x^2x^{-2}$ and so $x^3x^{-3} = x^2x^{-2}$ is an identity that is valid for $\mathscr{V}(S)$. Consequently, any element of $\mathscr{V}(S)$ is completely semisimple (for details see [12]). Thus $\mathscr{V}(S)$ is certainly not the variety of all inverse semigroups, and, in some sense, is not far up the lattice $\mathscr{L}(\mathscr{I})$.

3. Maximum elements in ν_3-classes. Since ν_3-classes are sublattices of ν_1-classes, it follows from Theorem 2.7, that each ν_3-class is a modular sublattice of $\mathscr{L}(\mathscr{I})$. In this section we will show that each ν_3-class has a minimum and, in some cases, also a maximum member.

PROPOSITION 3.1. *Let V be a ν_2-class. Let $\mathscr{W} \in V$, $\mathscr{G} = \mathscr{W} \cap \mathscr{G}\mathscr{P}$ and $\mathscr{V} = \mathscr{V}(\mathscr{W} \cap A\mathscr{G}\mathscr{P})$. Then $\mathscr{U} = \mathscr{G} \vee \mathscr{V}$ is the minimum member of V.*

Proof. Since $\mathscr{W} \cap A\mathscr{G}\mathscr{P} \subseteq \mathscr{U} = \mathscr{G} \vee \mathscr{V} \subseteq \mathscr{W}$, it follows that $\mathscr{U} \cap A\mathscr{G}\mathscr{P} = \mathscr{W} \cap A\mathscr{G}\mathscr{P}$. Similarly,

$$\mathscr{G} \subseteq \mathscr{U} \cap \mathscr{G}\mathscr{P} \subseteq \mathscr{W} \cap \mathscr{G}\mathscr{P} = \mathscr{G} .$$

Hence $\mathscr{U} \cap \mathscr{G}\mathscr{P} = \mathscr{W} \cap \mathscr{G}\mathscr{P}$. Thus $(\mathscr{U}, \mathscr{W}) \in \nu_3$ and $\mathscr{U} \subseteq \mathscr{W}$. Since $\mathscr{G}$ and $\mathscr{V}$ are independent of the choice of $\mathscr{W} \in V$, it follows that $\mathscr{U}$ is the minimum element of V.

Although we shall see in §5 that not all ν_3-classes contain a maximum element, some do and we can identify the maximum element for a large family of ν_3-classes.

Let $\mathscr{CSH}$ denote the class of all completely semisimple inverse semigroups on which $\mathscr{H}$ is a congruence. This class is not itself a variety, however if we denote by $\mathscr{L}(\mathscr{CSH})$ the set of varieties contained in $\mathscr{CSH}$, then $\mathscr{L}(\mathscr{CSH})$ is a sublattice of the lattice $\mathscr{L}(\mathscr{I})$ of all inverse semigroup varieties.

The class $\mathscr{CSH}$ was introduced in [12] where the following result was established.

THEOREM 3.2. *Let $\mathscr{V}$ be a variety of inverse semigroups. Then the following statements are equivalent.*

(1) *For every $S \in \mathscr{V}$, S is completely semisimple and $\mathscr{H}$ is a congruence: that is, $\mathscr{V} \in \mathscr{L}(\mathscr{CSH})$;*

(2) *$\mathscr{V}$ satisfies $x^{n+1}tt^{-1}x^{-n-1} = x^n tt^{-1}x^{-n}$, for some positive integer n.*

(3) *$\mathscr{V} \cap A\mathscr{GP}$ satisfies $x^{n+1} = x^n$, for some positive integer n;*

(4) *$\mathscr{V} \cap A\mathscr{GP}$ is a variety;*

(5) *$\mathscr{V} \cap A\mathscr{GP} \in \mathscr{Q}(\mathscr{T})$.*

Further insight into the nature of the conditions in Theorem 3.2 can be obtained from the next observation due to Djadchenko [3].

LEMMA 3.3. *A variety $\mathscr{V}$ contains no nontrivial groups, that is $\mathscr{V} \in \mathscr{Q}(\mathscr{T})$, if and only if an identity of the form $x^{n+1} = x^n$ is valid in $\mathscr{V}$ for some positive integer n.*

Note that, in the light of Lemma 3.3 and Proposition 2.6, conditions (3) and (4) of Theorem 3.2 imply that $\mathscr{V}$ is ν_1-equivalent to a variety in $\mathscr{Q}(\mathscr{T})$.

We shall show that ν_3-classes containing varieties of the type identified in Theorem 3.2 or, equivalently, ν_3-classes contained in ν_1-classes that intersect nontrivially with $\mathscr{Q}(\mathscr{T})$ have maximum members. In order to identify the maximum member, we shall need the concept of the product of two classes of algebras introduced by Mal'cev [7].

DEFINITION. If $\mathscr{U}$ and $\mathscr{V}$ are subclasses of a class $\mathscr{K}$ of algebras, then the product $\mathscr{U} \circ_{\mathscr{K}} \mathscr{V}$ is defined as consisting of the algebras A from $\mathscr{K}$ such that for some congruence ρ on A, $A/\rho \in \mathscr{V}$ and each ρ-class which is a subalgebra of A is in $\mathscr{U}$.

If $\mathscr{U}$, $\mathscr{V}$ are varieties of groups and $\mathscr{K}$ is the variety of all groups, then $\mathscr{U} \circ_{\mathscr{K}} \mathscr{V}$ is just the standard product of group varieties, as studied in [10].

Although the product of two varieties of inverse semigroups has not been studied, in general, the work of Houghton [4] followed by that of Bales [1] has provided considerable information on the product of a variety of groups and a variety of inverse semigroups in the variety $\mathscr{I}$ of all inverse semigroups.

Since we shall only be considering products in the variety $\mathscr{I}$, we shall denote the product of two subvarieties $\mathscr{U}$, $\mathscr{V}$ of $\mathscr{I}$ simply by $\mathscr{U} \circ \mathscr{V}$, without any subscript.

We note in passing that the Mal'cev product of varieties of inverse semigroups is not always a variety of inverse semigroups. If we let $\mathscr{Y}$ denote the variety of semilattices, then it follows from the construction of the free inverse semigroup due to Scheiblich [14] that $F_X \in \mathscr{Y} \circ \mathscr{G}\mathscr{P}$. Hence, any variety containing $\mathscr{Y} \circ \mathscr{G}\mathscr{P}$ must contain $\mathscr{I}$. However, any element S of $\mathscr{Y} \circ \mathscr{G}\mathscr{P}$ must be E-unitary (that is, $a \in S$, e and $ea \in E(S)$ implies that $a \in E(S)$) while not every element of $\mathscr{I}$ is E-unitary.

However, Bales [1] has shown that the product $\mathscr{U} \circ \mathscr{V}$ is a variety whenever $\mathscr{U}$ is a group variety.

As observed by Bales [1], any inverse semigroup identity $u(x_1, \cdots, x_n) = v(x_1, \cdots, x_n)$ is equivalent to the following two identities:

$$\begin{aligned} &u(x_1, \cdots, x_n)v(x_1, \cdots, x_n)^{-1} \\ &\qquad = v(x_1, \cdots, x_n)v(x_1, \cdots, x_n)^{-1} \\ &u(x_1, \cdots, x_n)^{-1}u(x_1, \cdots, x_n) \\ &\qquad = v(x_1, \cdots, x_n)^{-1}v(x_1, \cdots, x_n) . \end{aligned}$$

Thus, for any variety $\mathscr{V}$, there is a basis of identities of the form

$$u(x_1, \cdots, x_n) = i(x_1, \cdots, x_n)$$

where $i(x_1, \cdots, x_n)$ is an idempotent in F_X. For convenience, we abbreviate expressions of the form $u(x_1, \cdots, x_n)$ to $u(\bar{x})$.

If we write

$$\begin{aligned} \mathrm{Idem}\,(\mathscr{V}) = \{(u(\bar{x}), i(\bar{x})) \in F_X \times F_X \colon &u(\bar{x}) = i(\bar{x}) \text{ is an} \\ &\text{identity in } \mathscr{V} \text{ and } i(\bar{x}) \text{ is an idempotent in } F_X\} , \end{aligned}$$

then as observed above, Idem $(\mathscr{V})$ provides a basis of identities for $\mathscr{V}$.

Combining Theorems 3.2 and 3.6 from Bales [1], we obtain the following important theorem.

THEOREM 3.4. *Let $\mathscr{G}$ be a variety of groups and $\mathscr{V}$ a variety of inverse semigroups. Then $\mathscr{G} \circ \mathscr{V}$ is a variety.*

Moreover, the identities

$$u(v_1(\bar{x}_1), \cdots, v_n(\bar{x}_n)) = i_1(\bar{x}_1) \cdots i_n(\bar{x}_n)$$

for all $u(\bar{x}) \in U(\mathscr{G})$ *and for all* $(v_j(\bar{x}_j), i_j(\bar{x}_j)) \in \mathrm{Idem}(\mathscr{T})$, *form a basis of identities for* $\mathscr{G} \circ \mathscr{T}$.

We now combine Theorems 3.2 and 3.4 in order to show that a large class of ν_3-classes contain maximum members which can be described by the Mal'cev product.

THEOREM 3.5. *Let* $\mathscr{U} \in \mathscr{L}(\mathscr{C}\mathscr{S}\mathscr{H})$. *Let* $\mathscr{G} = \mathscr{U} \cap \mathscr{G}\mathscr{P}$ *and* $\mathscr{T} = \mathscr{U} \cap A\mathscr{G}\mathscr{P}$. *Then* $\mathscr{G} \circ \mathscr{T}$ *is the maximum member of the* ν_3-*class containing* $\mathscr{U}$.

Proof. Let $\mathscr{W}$ be any element of the ν_3-class containing $\mathscr{U}$. Then $\mathscr{W} \cap A\mathscr{G}\mathscr{P} = \mathscr{U} \cap A\mathscr{G}\mathscr{P} = \mathscr{T}$. Therefore, for any $S \in \mathscr{W}$, $S/\mu_S \in \mathscr{T}$. Also, for any idempotent $e \in S$, since $\mathscr{W} \cap \mathscr{G}\mathscr{P} = \mathscr{U} \cap \mathscr{G}\mathscr{P} = \mathscr{G}$, we have $e\mu_S \in \mathscr{G}$. Therefore, μ_S is a congruence on S such that $S/\mu_S \in \mathscr{T}$ and each class of μ_S that is a subalgebra of S belongs to $\mathscr{G}$. Hence, $S \in \mathscr{G} \circ \mathscr{T}$. Therefore, $\mathscr{W} \subseteq \mathscr{G} \circ \mathscr{T}$.

To complete the theorem, therefore, it is only necessary to show that $(\mathscr{G} \circ \mathscr{T}, \mathscr{U}) \in \nu_3$.

Since $\mathscr{U} \in \mathscr{L}(\mathscr{C}\mathscr{S}\mathscr{H})$, it follows from Theorem 3.2(3) that an identity of the form $x^{n+1} = x^n$ is valid in $\mathscr{T} = \mathscr{U} \cap A\mathscr{G}\mathscr{P}$, for some positive integer n. Therefore, if $v(x) = x^{n+1}x^{-n}$ then $(v(x), x^n x^{-n}) \in \mathrm{Idem}(\mathscr{T})$.

Now let G be any group in $\mathscr{G} \circ \mathscr{T}$ and let $u(\bar{x}) \in U(\mathscr{G})$. Then

$$u(v(x_1), v(x_2), \cdots, v(x_r)) = x_1^n x_1^{-n} \cdots x_r^n x_r^{-n}$$

is an identity that is valid in $\mathscr{G} \circ \mathscr{T}$. But, since G is a group, for any $x \in G$, $v(x) = x$ and so the identity $u(x_1, \cdots, x_r) = 1$ is valid in G where 1 denotes the identity of G.

Since this is the case for all $u(\bar{x}) \in U(\mathscr{G})$, it follows that $G \in \mathscr{G}$. Hence, $(G \circ \mathscr{T}) \cap \mathscr{G}\mathscr{P} \subseteq \mathscr{G}$ and so, since it is clear that $\mathscr{G} \subseteq (\mathscr{G} \circ \mathscr{T}) \cap \mathscr{G}\mathscr{P}$, it follows that $(\mathscr{G} \circ \mathscr{T}) \cap \mathscr{G}\mathscr{P} = \mathscr{G}$ and so $(\mathscr{G} \circ \mathscr{T}, \mathscr{U}) \in \nu_2$.

On the other hand, if $S \in \mathscr{G} \circ \mathscr{T}$ then there exists a congruence ρ on S such that $S/\rho \in \mathscr{T}$ and each ρ-class that is an inverse subsemigroup of S lies in $\mathscr{G}$, that is, in particular, is a group. Hence, ρ must be an idempotent separating congruence, and, by Corollary 2.3, we have $S \in \mathscr{G}\mathscr{P} \vee \mathscr{T}$. Hence, $\mathscr{G} \circ \mathscr{T} \subseteq \mathscr{G}\mathscr{P} \vee \mathscr{T}$ and so

$$\mathscr{G}\mathscr{P} \vee \mathscr{T} \subseteq \mathscr{G}\mathscr{P} \vee (\mathscr{G} \circ \mathscr{T}) \subseteq \mathscr{G}\mathscr{P} \vee \mathscr{T}.$$

Therefore,

$$\mathscr{G}\mathscr{P} \vee (\mathscr{G} \vee \mathscr{V}) \subseteq \mathscr{G}\mathscr{P} \vee \mathscr{V}\,,$$

and $(\mathscr{G} \circ \mathscr{V}, \mathscr{U}) \in \nu_1$. Hence, $((\mathscr{G} \circ \mathscr{V}), \mathscr{U}) \in \nu_3$, as required.

Since each ν_3-class is determined by a fundamental variety and a group variety and vice versa, it will be convenient to denote the ν_3-class determined by a group variety $\mathscr{G}$ and a fundamental variety $\mathscr{V}$ by $\nu_3(\mathscr{G}, \mathscr{V})$.

It is interesting to note that in the proof of the second half of Theorem 3.5, it is established that, for any variety $\mathscr{V}$ in which an identity of the form $x^{n+1} = x^n$ is valid (that is, for any $\mathscr{V} \in \mathscr{Q}(\mathscr{T})$) and for any group variety $\mathscr{G}$, $\mathscr{G} \circ \mathscr{V} \in \nu_3(\mathscr{G}, \mathscr{V})$. This enables us to restate Theorem 3.5 in terms of the group variety and fundamental variety determining a ν_3-class.

COROLLARY 3.6. *Let $\mathscr{G}$ be a variety of groups and let $\mathscr{V} \in \mathscr{Q}(\mathscr{T})$. Then $\mathscr{G} \circ \mathscr{V}$ is the maximum element in $\nu_3(\mathscr{G}, \mathscr{V})$.*

4. Further results on the Mal'cev product. In this section we show that the Mal'cev product respects the lattice operations in $\mathscr{L}(\mathscr{G})$.

We recall (see [10]) that, for any subset U of G_X, and, in particular, for any subgroup U of G_X, and any group G, the *verbal subgroup* $U(G)$ of G is the subgroup generated by the set $\{\alpha(u)\colon u \in U$ and α is a homomorphism of G_X into $G\}$. A verbal subgroup is always fully invariant.

For a given variety of groups $\mathscr{G}$ corresponding to a fully invariant subgroup U of G_X, $U = U(G_X)$. Furthermore, for any group G, $U(G)$ is the smallest normal subgroup of G such that $G/U(G) \in \mathscr{G}$ and, in particular, G belongs to $\mathscr{G}$ if and only if $U(G) = \{1\}$.

LEMMA 4.1. *Let $\mathscr{V}$ be a variety and $\{\mathscr{G}_\lambda\colon \lambda \in \Lambda\}$ be a family of varieties of groups. Then*

$$(\vee \mathscr{G}_\lambda) \circ \mathscr{V} = \vee(\mathscr{G}_\lambda \circ \mathscr{V}) \quad \textit{and} \quad (\wedge \mathscr{G}_\lambda) \circ \mathscr{V} = \wedge(\mathscr{G}_\lambda \circ \mathscr{V})\,.$$

Proof. It is clearly the case that $(\mathscr{G}_\lambda \circ \mathscr{V}) \subseteq (\vee \mathscr{G}_\lambda) \circ \mathscr{V}$ and so that $\vee(\mathscr{G}_\lambda \circ \mathscr{V}) \subseteq (\mathscr{G}_\lambda) \circ \mathscr{V}$.

Let $S \in (\vee \mathscr{G}_\lambda) \circ \mathscr{V}$. Then there exists an idempotent separating congruence ρ on S such that $S/\rho \in \mathscr{V}$ and $N_e = e\rho \in \vee \mathscr{G}_\lambda$, for all $e \in E(S)$. For each λ, let $U^\lambda = U(\mathscr{G}_\lambda)$, $U = U(\vee \mathscr{G}_\lambda)$ and $N_e^\lambda = U^\lambda(N\hat{e})$. Since $\{N_e\colon e \in E(S)\}$ is the kernel normal system for an idempotent separating congruence, $N = \cup\{N_e\colon e \in E(S)\}$ is, in particular, a semilattice of groups. It can then be verified quite routinely that $\{N_e^\lambda\colon$

$e \in E(S)\}$ is a kernel normal system for each λ. Let the corresponding congruence, which is necessarily idempotent separating, be denoted by ρ_λ.

Now $\cap U^\lambda = U$. Hence, since $N_e \in \vee \mathscr{G}_\lambda$, for each $e \in E(S)$, we must have $\cap N_e^\lambda$ equal to the trivial subgroup of N_e. Therefore, $\cap \rho_\lambda$ is the identity congruence. Now, for each λ, ρ/ρ_λ is an idempotent separating congruence on S/ρ_λ and $(S/\rho_\lambda)/(\rho/\rho_\lambda)$ is isomorphic to S/ρ and so lies in $\mathscr{V}$. Let $e\rho_\lambda$ be any idempotent of S/ρ_λ. Then the ρ/ρ_λ-class containing $e\rho_\lambda$ is isomorphic to N_e/N_e^λ and so, by the definition of N_e^λ, lies in $\mathscr{G}_\lambda$. Hence, for each λ, $S/\rho_\lambda \in \mathscr{G}_\lambda \circ \mathscr{V}$ and $\cap \rho_\lambda$ is the identity congruence. Therefore, $S \in \vee(\mathscr{G}_\lambda \circ \mathscr{V})$ and the first half of the lemma is established.

With regard to the second assertion, it is again the case that the inclusion one way is trivial, namely, $(\wedge \mathscr{G}_\lambda) \circ \mathscr{V} \subseteqq \wedge(\mathscr{G}_\lambda \circ \mathscr{V})$.

Let $S \in \wedge(\mathscr{G}_\lambda \circ \mathscr{V})$. Then $S \in \mathscr{G}_\lambda \circ \mathscr{V}$, for all λ. Hence, for each λ, there exists an idempotent separating congruence ρ_λ on S such that $e\rho_\lambda \in \mathscr{G}_\lambda$, for all $e \in E(S)$, and $S/\rho_\lambda \in \mathscr{V}$.

Let $\rho = \cap \rho_\lambda$. For each $e \in E(S)$, $e\rho = \cap e\rho_\lambda \in \wedge \mathscr{G}_\lambda$ while S/ρ, as a subdirect product of the S/ρ_λ, lies in $\mathscr{V}$. Hence, $S \in (\wedge \mathscr{G}_\lambda) \circ \mathscr{V}$ and the second part of the lemma is established.

From Corollary 3.3 of [1] and Corollary 2.3, we have the following result.

LEMMA 4.2. *If $\mathscr{V}$ is a variety such that $\mathscr{G}\mathscr{P} \nsubseteq \mathscr{V}$, then the mapping $\mathscr{G} \rightarrow \mathscr{G} \circ \mathscr{V}$ is a one-to-one order isomorphism of $\mathscr{L}(\mathscr{G}\mathscr{P})$ into $[\mathscr{V}, \mathscr{G}\mathscr{P} \vee \mathscr{V}]$.*

Combining Lemmas 4.1 and 4.2 we then have the following.

THEOREM 4.3. *If $\mathscr{V}$ is a variety such that $\mathscr{G}\mathscr{P} \nsubseteq \mathscr{V}$ then the mapping $\mathscr{G} \rightarrow \mathscr{G} \circ \mathscr{V}$ is a complete lattice isomorphism of $\mathscr{L}(\mathscr{G}\mathscr{P})$ into $[\mathscr{V}, \mathscr{G}\mathscr{P} \vee \mathscr{V}]$.*

5. A detailed study of a ν_2-class. In this section it will be shown that there exist ν_3-classes that have no maximum elements. Moreover, the class used to demonstrate this will contain a variety of the form $\mathscr{G} \circ \mathscr{V}$, where $\mathscr{G}$ is a group variety and $\mathscr{V}$ is a fundamental variety, thus showing, in addition, that such varieties are not always maximum in their classes.

We denote by $\mathscr{C}$ the bicyclic semigroup.

Let G be a group, α an endomorphism of G, and N the set of nonnegative integers. Then we denote by $B(G, \alpha)$ the set $N \times G \times N$ under the multiplication

$$(m; g; n)(p; h; q) = (m + p - r; \alpha^{p-r}(g)\alpha^{n-r}(h); n + q - r) ,$$

where $r = \min(n, p)$.

With respect to this multiplication $B(G, \alpha)$ is a bisimple inverse semigroup. For the basic properties of such semigroups the reader is referred to [9] and [11].

For any elements x, y of an inverse semigroup S, we denote by $[x, y]$ the "commutator" element $x^{-1}y^{-1}xy$. We require some elementary facts about elements of this form.

LEMMA 5.1. *Let S be an inverse semigroup, G be a group and α an endomorphism of G.*

(1) *If either x or y belongs to $E(S)$ then so does $[x, y]$.*

(2) *For any $x, y \in \mathscr{C}$, $[x, y] \in E(\mathscr{C})$.*

(3) *For any $x, y \in B(G, \alpha)$, $[x, y]$ belongs to a subgroup of $B(G, \alpha)$.*

Proof. The observations can be verified by straightforward computations.

We recall the definition of the wreath product of two groups G and H. We denote by G^H the group of functions from H to G where the group operation is defined componentwise. Then, by the wreath product G Wr H of G and H we mean $H \times G^H$ with multiplication defined by

$$(b, f)(c, g) = (bc, f^c g)$$

where $f^c(y) = f(yc^{-1})$, for all $y \in H$. With respect to this operation G Wr H is a group [10]. In particular, $(b, f)^{-1} = (b^{-1}, (f^d)^{-1})$, where $d = b^{-1}$.

For each positive integer i, let G_i denote the product of i copies of Z Wr Z. We denote the identities of Z Wr Z and G_i by 1. For $k, b \in Z$ let $f_{k,b} \in Z^Z$ be defined as follows:

$$f_{k,b}(z) = \begin{cases} b & \text{if} \quad z = k \\ 0 & \text{if} \quad z \neq k . \end{cases}$$

Let $\alpha_i \colon G_i \to G_i$ be defined as follows: for all $F \in G_i$,

$$\alpha_i(F)(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ F(n-1) & \text{if} \quad 1 < n \leqq i . \end{cases}$$

It is straightforward to verify that α_i is an endomorphism of G_i such that α_i^i is the zero endomorphism and, for all $F \in G_i$,

$$\alpha_i^{i-1}(F)(n) = \begin{cases} 1 & \text{if} \quad 1 \leqq n < i \\ F(1) & \text{if} \quad n = i . \end{cases}$$

Since the group operation in Z is addition, we denote the group operation in Z^Z by addition also.

Let $B_i = B(G_i, \alpha_i)$.

Since G_i is a product of wreath products of the integers it follows that $G_i \in \mathscr{AGP} \circ \mathscr{AGP}$, for all i. Hence, the identity

$$[[x, y], [u, v]]^2 = [[x, y], [u, v]]$$

is valid in the group $\mathscr{H}$-classes of B_i.

LEMMA 5.2. *Let e denote the product of the words of the form $w^i w^{-i} w^{-i} w^i$ where $w = x, y, u, v, xy, yx, uv, vu$. Then the identity*

$$(*) \qquad [[exe, eye], [eue, eve]]^2 = [[exe, eye], [eue, eve]]$$

is valid in B_i.

Proof. We omit the details of the proof which consists of straightforward computations.

COROLLARY 5.3. $\mathscr{V}(B_i) \in \nu_3(\mathscr{AGP} \circ \mathscr{AGP}, \mathscr{V}(\mathscr{C}))$.

Proof. Since B_i/μ_{B_i} is isomorphic to $\mathscr{C}$, we have from Corollary 2.3 that $B_i \in [\mathscr{V}(\mathscr{C}), \mathscr{V}(\mathscr{C}) \vee \mathscr{GP}]$. By Lemma 5.2, (*) is valid in $\mathscr{V}(B_i)$ and so the identity

$$[[x, y], [u, v]]^2 = [[x, y], [u, v]]$$

holds in $\mathscr{V}(B_i) \cap \mathscr{GP}$. Thus $\mathscr{V}(B_i) \cap \mathscr{GP} \subset \mathscr{AGP} \circ \mathscr{AGP}$. Since $Z \ Wr \ Z \in \mathscr{V}(B_i)$ it follows that $\mathscr{V}(B_i) \cap \mathscr{GP} = \mathscr{AGP} \circ \mathscr{AGP}$.

A simple argument using the fact that, for any group variety $\mathscr{G}$ and any inverse semigroup variety $\mathscr{V}$, $\mathscr{G} \circ \mathscr{V} \cap \mathscr{GP} = \mathscr{G} \circ (\mathscr{V} \cap \mathscr{GP})$ will now establish that the varieties $\mathscr{V}(B_i)$ also contain $\mathscr{AGP} \circ \mathscr{V}(\mathscr{C})$. Hence, we have the following proposition.

PROPOSITION 5.4. *For any positive integer i,*

$$(\mathscr{V}(B_i))\nu_3 = (\mathscr{AGP} \circ \mathscr{V}(\mathscr{C}))\nu_3 = \nu_3(\mathscr{AGP} \circ \mathscr{AGP}, \mathscr{V}(\mathscr{C})) .$$

We shall now show that even although the ν_3-class $(\mathscr{AGP} \circ \mathscr{V}(\mathscr{C}))\nu_3$ contains a Mal'cev product of varieties, it has no maximum member.

We proceed by contradiction. Suppose that there is a maximum member $\mathscr{V}$. Then $\mathscr{V}(B_i) \subseteq \mathscr{V}$, for all i, and so $B_i \in \mathscr{V}$, for all i. Also, $\mathscr{V} \wedge \mathscr{GP} = \mathscr{AGP} \circ \mathscr{AGP}$ and so must satisfy the identity

$$(**) \qquad [[x, y], [u, v]]^2 = [[x, y], [u, v]] .$$

Let S denote the inverse subsemigroup of the product $\prod\{B_i; i =$

$1, 2, 3, \cdots\}$ consisting of those elements F such that there exist some fixed integers m and n (dependent on F) such that $F(i) = (m; A_i; n)$, for some $A_i \in G_i$, for all $i = 1, 2, 3, \cdots$.

Let X, Y, U, V be defined as follows: for all $i = 1, 2, 3, \cdots$, $X(i) = U(i) = (1; A_i; 0)$, $Y(i) = (1; B_i; 1)$, $V(i) = (1; C_i; 1)$ where, for $1 \leqq n \leqq i$,

$$A_i(n) = 1 \;, \quad \text{the identity of } Z \text{ Wr } Z \;,$$
$$B_i(n) = (1, f_{0,1}) \;, \qquad C_i(n) = (2, f_{0,1}) \;.$$

It can now be checked that there is no idempotent e in S for which $e[[X, Y], [U, V]]$ is an idempotent. Therefore S/σ, where σ is the minimum group congruence on S, does not satisfy (**). Therefore, $S \notin \mathscr{V}$ and we have the following result.

THEOREM 5.5. *The ν_3-class containing $\mathscr{AGP} \circ \mathscr{V}(\mathscr{C})$ has no maximum member.*

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