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SINGLY GENERATED ANTISYMMETRIC OPERATOR ALGEBRAS

JOHN B. CONWAY AND WACŁAW SZYMANSKI

We discuss the antisymmetry of certain algebras associated with a bounded linear operator on a Hilbert space $\mathscr{H}$. An algebra of operators on $\mathscr{H}$ is said to be antisymmetric if the only self-adjoint operators it contains are multiples of the identity. If T is a bounded operator on $\mathscr{H}$, let $\mathscr{A}_u(T)$ be the norm closure of $\{p(T): p$ is a polynomial$\}$ and let $\mathscr{R}_u(T)$ be the norm closure of $\{f(T): f$ is a rational function with poles off the spectrum of $T\}$. Suppose $T = T_1 \oplus T_2 \oplus \cdots$ and $\|T_j\| \to 0$ as $j \to \infty$. For a subset J of N, the natural numbers, let $T_J = \oplus\{T_j: j \in J\}$. It is proved here that if $\mathscr{A}_u(T_j)$ is antisymmetric for each j, then $\mathscr{A}_u(T)$ is antisymmetric if and only if for each finite set J the polynomially convex hulls of $\sigma(T_J)$ and $\sigma(T_{N \setminus J})$ have nonempty intersection.

Similar results are obtained for $\mathscr{R}_u(T)$. Under the assumption that each $\mathscr{A}_u(T_j)(\mathscr{R}_u(T_j))$ is antisymmetric, the maximal antisymmetric projections for $\mathscr{A}_u(T)(\mathscr{R}_u(T))$ are characterized. These results are then applied to completely characterize the compact operators T for which $\mathscr{A}_u(T)(=\mathscr{R}_u(T))$ is antisymmetric.

The concept of antisymmetric algebras has been fruitfully used in the theory of function algebras [4] and operator theory [2], and a systematic study of this property was begun in [5] and [6].

In this paper the cases where $T = T_1 \oplus \cdots \oplus T_n$ or $T = T_1 \oplus T_2 \oplus \cdots$ and $\|T_j\| \to 0$ as $j \to \infty$, and $\mathscr{A}_u(T_j)$ (or $\mathscr{R}_u(T_j)$) is antisymmetric for each $j \geqq 1$ will be considered. The study of the case where T is an arbitrary infinite direct sum of antisymmetric operators poses some difficulties. The assumption that $\|T_j\| \to 0$ obviates these difficulties and permits a study of compact operators.

By Theorem 1.4.5 of [1], if T is a compact operator, $T = T_1 \oplus T_2 \oplus \cdots$ where each T_j is an irreducible compact operator. Because T_j is irreducible, $\mathscr{A}_u(T_j)$ is antisymmetric [5]. Since T is compact it must be that the above direct sum is either finite or $\|T_j\| \to 0$ as $j \to \infty$. Hence, the study of compact operators will be covered by the more general theorems.

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Zame observed that the infinite case could be treated provided the assumption was made that the norms converge to zero. He also made several other suggestions that have resulted in more incisive results and more lucid exposition.

1. Preliminary results. If $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on the separable Hilbert space $\mathscr{H}$, and $\mathscr{A}$ contains the identity, a projection P in $\mathscr{A}'$, the commutant of $\mathscr{A}$, is antisymmetric for $\mathscr{A}$ if $\mathscr{A}_P \equiv \{A \mid P\mathscr{H}: A \in \mathscr{A}\}$ is an antisymmetric subalgebra of $\mathscr{B}(P\mathscr{H})$. *A maximal antisymmetric projection* for $\mathscr{A}$ is an antisymmetric projection for $\mathscr{A}$ that is maximal, for the usual ordering, in the set of all antisymmetric projections. Let $\mathscr{M}(\mathscr{A})$ denote the set of all maximal antisymmetric projections for $\mathscr{A}$. $\mathscr{M}(\mathscr{A})$ is contained in the center of the von Neumann algebra generated by $\mathscr{A}$ [6]. Also, if Q is any antisymmetric projection for $\mathscr{A}$, there is a P in $\mathscr{M}(\mathscr{A})$ such that $Q \leqq P$ [6]. Also, if $P \in \mathscr{M}(\mathscr{A})$ and Q is an antisymmetric projection for $\mathscr{A}$ such that $PQ \neq 0$, then $Q \leqq P$ ([5], Proposition 2). Finally, each antisymmetric projection that belongs to $\mathscr{A}$ must be maximal.

The proof of the following result is left to the reader

PROPOSITION 1.1. *If* $T = \bigoplus \{T_j: j \geqq 1\}$ *and* $\|T_j\| \to 0$ *as* $j \to \infty$, *then*

$$\sigma(T) = \bigcup_{j=1}^{\infty} \sigma(T_j) \cup \{0\} .$$

Throughout this paper, the following notation will be fixed. $\boldsymbol{R}$ and $\boldsymbol{C}$ stand for the real and complex numbers, and $\square$ denotes the empty set. If $\{\mathscr{H}_j\}$ is a finite or countable sequence of Hilbert spaces, $\mathscr{H} = \bigoplus_j \mathscr{H}_j$; if $T_j \in \mathscr{B}(\mathscr{H}_j)$, $T = \bigoplus_j T_j$. Let P_j denote the projection of $\mathscr{H}$ onto $\mathscr{H}_j$. If $J \subseteq \boldsymbol{N}$, $\mathscr{H}_J = \bigoplus \{\mathscr{H}_j: j \in J\}$, $P_J = \bigoplus \{P_j: j \in J\}$, $T_J = \bigoplus \{T_j: j \in J\}$. If K is a compact subset of $\boldsymbol{C}$, $K\hat{} =$ polynomially convex hull of K.

2. The antisymmetry of $\mathscr{A}_u(T)$. In this section we will study the antisymmetry of $\mathscr{A}_u(T)$ under the assumptions that T is a finite direct sum $T = T_1 \oplus \cdots \oplus T_n$ or T is an infinite direct sum $T = T_1 \oplus T_2 \oplus \cdots$ and $\|T_j\| \to 0$; and $\mathscr{A}_u(T_j)$ is antisymmetric for each $j \geqq 1$. However, the results and the proofs will only be given in the infinite case. The statements of the results for the finite case and their proofs are left to the reader.

LEMMA 2.1. *If* $J \subseteq \boldsymbol{N}$, $P_J \in \mathscr{A}_u(T)$ *iff* $\sigma(T_J)\hat{} \cap \sigma(T_{N \setminus J})\hat{} = \square$.

Proof. It is easy to see that $P_J \in \mathscr{A}_u(T)$ iff $\mathscr{A}_u(T) = \mathscr{A}_u(T_J) \oplus \mathscr{A}_u(T_{N\setminus J})$. The proof is now completed by applying Theorem 1.4 of [3].

THEOREM 2.2. *Suppose that* $T = T_1 \oplus T_2 \oplus \cdots$, $\|T_j\| \to 0$ *as* $j \to \infty$, *and* $\mathscr{A}_u(T_j)$ *is antisymmetric for each* j. *The following conditions are logically equivalent*:

(a) $\mathscr{A}_u(T)$ *is antisymmetric.*

(b) *For every finite nonempty subset* J *of* N, $P_J \notin \mathscr{A}_u(T)$.

(c) *For every finite nonempty subset* J *of* N,
$\sigma(T_J)^\wedge \cap \sigma(T_{N\setminus J})^\wedge \neq \square$.

Proof. Clearly (a) implies (b). By Lemma 2.1, (b) and (c) are equivalent.

Now assume that (b) holds, and we will show that $\mathscr{A}_u(T)$ must be antisymmetric. Suppose $A \in \mathscr{A}_u(T)$ and $A = A^*$. Clearly $A = A_1 \oplus A_2 \oplus \cdots$ and each $A_j \in \mathscr{A}_u(T_j)$. Let $\{p_n\}$ be a sequence of polynomials such that

$$0 = \lim_{n\to\infty} \|p_n(T) - A\| .$$

Because $\|p(T)\| \geqq \sup\{|p(z)|: z \in \sigma(T)\}$ for any polynomial p, $\{p_n\}$ is a uniformly Cauchy sequence on $\sigma(T)$. In particular, $p_n(0) \to \alpha$, for some scalar α.

If I_j denotes the identity on the space $\mathscr{H}_j$,

$$\begin{aligned} \|A_j - \alpha I_j\| &\leqq \|A_j - p_n(T_j)\| + \|p_n(T_j) - p_n(0)I_j\| + p_n(0)I_j - \alpha I_j\| \\ &\leqq \|A - p_n(T)\| + \|p_n(T_j) - p_n(0)I_j\| \, \|p_n(0)I_j - \alpha I_j\| . \end{aligned}$$

Because $\|T_j\| \to 0$ as $j \to \infty$, $\|p(T_j) - p(0)I_j\| \to 0$ as $j \to \infty$ for any polynomial p. Thus, the above inequality implies that $\|A_j - \alpha I_j\| \to 0$ as $j \to \infty$.

Now $A_j \in \mathscr{A}_u(T_j)$ and $A_j = A_j^*$. Since $\mathscr{A}_u(T_j)$ is antisymmetric, $A_j = \alpha_j I_j$ for some α_j in $\boldsymbol{R}$. From the preceding paragraph it follows that $\alpha_j \to \alpha$ as $j \to \infty$. Suppose $\alpha_k \neq \alpha$ for some k. Put $J = \{j: \alpha_j = \alpha_k\}$. Because $\alpha_j \to \alpha$, J must be finite. Also α_k is an isolated point of the spectrum of $A = \alpha_1 I_1 \oplus \alpha_2 I_2 \oplus \cdots$. Hence $P_J \in \mathscr{A}_u(A) \subseteqq \mathscr{A}_u(T)$, contradicting (b). Thus $A = \alpha I$.

Notice that if J is an infinite subset of N such that $N\setminus J$ is also infinite, then Proposition 1.1 implies $0 \in \sigma(T_J) \cap \sigma(T_{N\setminus J})$.

The remainder of this section is devoted to the description of $\mathscr{M}(\mathscr{A}_u(T))$, the set of maximal antisymmetric projections for $\mathscr{A}_u(T)$. Let $\mathscr{P}$ denote the collection of nonempty subsets J of N such that $\mathscr{A}_u(T_J)$ is antisymmetric. It follows from Theorem 2.2 that

(2.3) $$J \in \mathscr{P} \text{ iff } \sigma(T_L)^\wedge \cap \sigma(T_{J\setminus L})^\wedge \neq \square$$

for every finite nonempty subset L of J.

It follows from (2.3) that if $\mathscr{P}$ is ordered by inclusion, each member of $\mathscr{P}$ is contained in a maximal element of $\mathscr{P}$. Let $\mathscr{K} =$ the collection of maximal elements of $\mathscr{P}$. Because each $\mathscr{A}_u(T_j)$ is assumed to be antisymmetric, each integer belongs to at least one member of $\mathscr{K}$. Moreover, the sets belonging to $\mathscr{K}$ are pairwise disjoint. Hence, $\mathscr{K}$ is a partition of N and $I = \sum \{P_J : J \in \mathscr{K}\}$.

PROPOSITION 2.4. *If $\mathscr{K}$ is the partition of N described above, then:*

(a) *For J and K in $\mathscr{K}$, $J \neq K$, $\sigma(T_J)^\wedge \cap \sigma(T_K)^\wedge = \square$.*

(b) *If $J \in \mathscr{K}$ and $0 \notin \sigma(T_J)^\wedge$, then $P_J \in \mathscr{A}_u(T)$.*

(c) *$\mathscr{K}$ contains at most one infinite set J, and this infinite set is the only one for which $0 \in \sigma(T_J)^\wedge$.*

Proof. (a) Suppose J and K belong to $\mathscr{K}$ and $\sigma(T_J)^\wedge \cap \sigma(T_K)^\wedge \neq \square$. Let L be a finite subset of $J \cup K$ and assume that $L \cap K \neq \square$. Thus $\sigma(T_{(J\cup K)\setminus L})^\wedge \cap \sigma(T_L)^\wedge \supseteq \sigma(T_{K\setminus L})^\wedge \cap \sigma(T_L)^\wedge \neq \square$. By (2.3), $J \cup K \in \mathscr{P}$. By the maximality of J and K, $J = K$.

(b) If $0 \notin \sigma(T_J)^\wedge$, there is an $\varepsilon > 0$ such that for $\Delta = \{z \in C: |z| \leqq \varepsilon\}$, $\Delta \cap \sigma(T_J)^\wedge = \square$. Because $\|T_j\| \to 0$ as $j \to \infty$, Proposition 1.1 implies that there are a finite number of sets $J_0, \cdots, J_n$ in $\mathscr{K}$ such that if $K \in \mathscr{K}$ and $K \neq J_0, \cdots, J_n$, then $\sigma(T_K) \subseteq \Delta$; hence, $\sigma(T_K)^\wedge \subseteq \Delta$ for $K \neq J_0, \cdots, J_n$. Suppose $J = J_0$. By (a), $\sigma(T_{N\setminus J})^\wedge \subseteq \sigma(T_{J_1})^\wedge \cup \cdots \cup \sigma(T_{J_n})^\wedge \cup \Delta$, and so $\sigma(T_J)^\wedge \cap \sigma(T_{N\setminus J})^\wedge = \square$. Part (b) follows by Lemma 1.1. Part (c) follows from (a) and the fact that $0 \in \sigma(T_J)$ if J is infinite.

THEOREM 2.5. *If $\mathscr{A}_u(T_j)$ is antisymmetric for each $j \geqq 1$ and $\mathscr{K}$ is the partition of N described above, then*

$$\mathscr{M}(\mathscr{A}_u(T)) = \{P_J : J \in \mathscr{K}\} .$$

Proof. By the definition of $\mathscr{K}$, P_J is an antisymmetric projection for each J in $\mathscr{K}$. Also note that $\mathscr{A}_u(T)_{P_J} \subseteq \mathscr{A}_u(T_J)$. Let L be the unique element of $\mathscr{K}$ such that $0 \in \sigma(T_L)$. By 2.4 (b), $P_J \in \mathscr{A}_u(T)$ for J in $\mathscr{K}$ and $J \neq L$; since P_J is an antisymmetric projection, it follows, as noted in the remarks at the beginning of §1, that $P_J \in \mathscr{M}(\mathscr{A}_u(T))$.

Now let $E \in \mathscr{M}(\mathscr{A}_u(T))$. Because $\sum \{P_J : J \in \mathscr{K}\} = I$, there is a J in $\mathscr{K}$ such that $EP_J \neq 0$. By the remarks at the beginning of §1, $P_J \leqq E$. If $J \neq L$, then $E = P_J$ since P_J is also maximal. If

$E \neq P_J$ for all J in $\mathscr{K}$ different from L, then $EP_J = 0$ for J in $\mathscr{K}$, $J \neq L$. Hence, $I - P_L = \Sigma\{P_J : J \in \mathscr{K}, J \neq L\} \leqq I - E$, so $E \leqq P_L$. Because P_L is antisymmetric and E is maximal, $E = P_L$.

We have shown that $\mathscr{M}(\mathscr{A}_u(T)) \subseteqq \{P_J : J \in \mathscr{K}\}$. It only remains to show that P_L is a maximal antisymmetric projection. But this is easy since the fact that it is antisymmetric implies there is an E in $\mathscr{M}(\mathscr{A}_u(T))$ such that $P_L \leqq E$ ([6]). By the preceding paragraph, $P_L = E$.

Observe that if L is as in the preceding proof, then P_L is not necessarily in $\mathscr{A}_u(T)$.

If T is an irreducible operator (that is, $\{T\}'$ contains no nontrivial projections), then $\mathscr{A}_u(T)$ is antisymmetric [5]. Now if T is compact, $T = T_1 \oplus T_2 \oplus \cdots$ where each T_j is irreducible ([1], 1.4.5). A combination of this result and Theorem 2.2 yield a description of those compact operators T such that $\mathscr{A}_u(T)$ is antisymmetric.

THEOREM 2.6. *If T is a compact operator and $T = T_1 \oplus T_2 \oplus \cdots$ where each T_j is irreducible, then $\mathscr{A}_u(T)$ is antisymmetric iff for each finite subset J of N, $\sigma(T_J) \cap \sigma(T_{N \setminus J}) \neq \square$.*

Note that because the spectrum of a compact operator is polynomially convex, it is not necessary to use the polynomially convex hulls of the spectra in Theorem 2.6. Also, this same fact implies that $\mathscr{A}_u(T) = \mathscr{R}_u(T)$ for any compact operator.

COROLLARY 2.7. *If T is a compact operator and $T = T_1 \oplus T_2 \oplus \cdots$, where each T_j is irreducible and has infinite rank, then $\mathscr{A}_u(T)$ is antisymmetric.*

The following example illustrates Theorem 2.6.

EXAMPLE 2.8. Consider

$$T_n = \begin{bmatrix} \frac{1}{n} & 0 \\ \frac{1}{n} & \frac{1}{n+1} \end{bmatrix}$$

acting on C^2 for $n = 1, 2, \cdots$. Also let V be the Volterra operator on $L^2[0, 1]$:

$$Vf(x) = \int_0^x f(t)dt .$$

Then V and T_n are irreducible and $\|T_n\| \to 0$. Therefore, $T =$

$\bigoplus_{n=1}^{\infty} T_n \oplus V$ is a compact operator. Now $\sigma(T_n) = \{1/n, 1/(n+1)\}$ so if $T_0 = \bigoplus_{n=1}^{\infty} T_n$, then $\mathscr{A}_u(T_0)$ is antisymmetric by Theorem 2.6. Also, $0 \in \sigma(T_0)$ so that $\mathscr{A}_u(T)$ is antisymmetric by Theorem 2.2.

This section concludes with a word of caution to the reader. If $J \in \mathscr{K}$, the partition constructed for Theorem 2.5, then $\mathscr{A}_u(T_J)$ is antisymmetric and P_J is an antisymmetric projection for $\mathscr{A}_u(T)$. However, if J is an arbitrary subset of N and P_J is an antisymmetric projection for $\mathscr{A}_u(T)$, it does not follow that $\mathscr{A}_u(T_J)$ is antisymmetric. This is illustrated by the following example.

Let $X_1 = \{z \in C: |z| \leqq 1\} \cap \{z \in C: \operatorname{Re} z \leqq 0\}$, $X_2 = \{1/2\}$, $X_3 = \{z \in C: |z| = 1\}$. Let T_j be a normal operator with $\sigma(T_j) = X_j$, $j = 1, 2, 3$. Put $T = T_1 \oplus T_2 \oplus T_3$, $J = \{1, 2\}$. Now $\mathscr{A}_u(T_J) = \mathscr{A}_u(T_1) \oplus \mathscr{A}_u(T_2)$ since $X_1 \cap X_2 = X_1^\wedge \cap X_2^\wedge = \square$. Hence $\mathscr{A}_u(T_J)$ is not antisymmetric. On the other hand, suppose $A \in \mathscr{A}_u(T)$ and $AP_J = A^*P_J$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n(T) \to A$ as $n \to \infty$. Because T is normal, there is a continuous function f_0 on $\sigma(T) = X_1 \cup X_2 \cup X_3$, such that $p_n \to f_0$ uniformly on $\sigma(T)$ and $A = f_0(T)$. But by the Maximum Modulus Theorem, $\{p_n\}$ converges uniformly on the closed unit disk D to a function f that is continuous on D and analytic on its interior. Clearly $f = f_0$ on $\sigma(T)$. But f must be real-valued on X_1, so f is a constant function, say $f \equiv c$. Hence $A = f_0(T) = cI$, and $\mathscr{A}_u(T)_{P_J}$ is antisymmetric.

In the preceding example, $\mathscr{M}(\mathscr{A}_u(T)) = \{I\}$, since $\mathscr{A}_u(T)$ is antisymmetric.

3. The antisymmetry of $\mathscr{R}_u(T)$. Here we assume that T is a finite direct sum or an infinite direct sum of operators $\{T_j\}$ such that $\|T_j\| \to 0$ as $j \to \infty$, and $\mathscr{R}_u(T_j)$ is antisymmetric for each j. As in § 2, only the results for infinite direct sums will be given. In fact, no proofs of results analogous to results in § 2 are given here; these proofs following by analogous arguments.

LEMMA 3.1. *If $J \subset N$, $P_J \in \mathscr{R}_u(T)$ iff $\sigma(T_J) \cap \sigma(T_{N \setminus J}) = \square$.*

This is proved like Lemma 2.1, but Theorem 2.2 of [3] is invoked. The proof of the next theorem is similar to the proof of Theorem 2.2.

THEOREM 3.2. *Suppose that $T = T_1 \oplus T_2 \oplus \cdots$, $\|T_j\| \to 0$ as $j \to \infty$, and $\mathscr{R}_u(T_j)$ is antisymmetric for each j. The following conditions are logically equivalent:*

(a) *$\mathscr{R}_u(T)$ is antisymmetric.*

(b) *For every finite nonempty subset J of N, $P_J \notin \mathscr{R}_u(T)$.*

(c) *For every finite nonempty subset J of N,*

$\sigma(T_J) \cap \sigma(T_{N \setminus J}) \neq \square$.

Because $\mathscr{R}_u(T) \supset \mathscr{A}_u(T)$, if $\mathscr{R}_u(T)$ is antisymmetric, $\mathscr{A}_u(T)$ is antisymmetric. However, the converse is false as the next example illustrates.

EXAMPLE 3.3. Consider the following three compact subsets of the plane: $K_1 = \{z: |z| \leqq 1\}$, $K_2 = \{z: 9 \leqq |z - 10| \leqq 10\} \setminus \{z: |\mathrm{Im}\, z| < 1/2$ and $|z| < 2\}$, and $K_3 = \{z: |z - 5| \leqq 1\}$. Let $U_i = \mathrm{int}\, K_i$ and let $\lambda_i =$ planar Lebesgue measure restricted to U_i. Let $A^2(U_i) = \left\{f: f \text{ is an analytic function on } U_i \text{ with } \int |f|^2 d\lambda_i < \infty\right\}$ and define T_i on $A^2(U_i)$ by $T_i f = zf$. It follows that $\mathscr{A}_u(T_i) = \mathscr{R}_u(T_i) = \{M_\phi: \phi$ is a continuous function on K_i that is analytic on $U_i\}$, where $M_\phi f = \phi f$. Moreover $\sigma(T_i) = K_i = K_i^\wedge$. Let $T = T_1 \oplus T_2 \oplus T_3$. By Theorem 2.2 $\mathscr{A}_u(T)$ is antisymmetric. However $\sigma(T_3) \cap [\sigma(T_2) \cup \sigma(T_1)] = \square$ so that $\mathscr{R}_u(T)$ is not antisymmetric by Theorem 3.2.

To describe $\mathscr{M}(\mathscr{R}_u(T))$, let $\mathscr{P} = \{J \subseteq N: \mathscr{R}_u(T_J)$ is antisymmetric$\}$ and let $\mathscr{K}$ be the maximal elements of $\mathscr{P}$. The next result is proved like Theorem 2.5.

THEOREM 3.4. *If $\mathscr{R}_u(T_j)$ is antisymmetric for each $j \geqq 1$ and $\mathscr{K}$ is the partition of N described above,*

$$\mathscr{M}(\mathscr{R}_u(T)) = \{P_J: J \in \mathscr{K}\} .$$

4. Final remarks. There are some situations in which $\mathscr{A}_u(T \oplus S)$ or $\mathscr{R}_u(T \oplus S)$ is antisymmetric even though $\mathscr{A}_u(T)$ and $\mathscr{A}_u(S)$ or $\mathscr{R}_u(T)$ and $\mathscr{R}_u(S)$ are not.

PROPOSITION 4.1. *Let $T \in \mathscr{L}(\mathscr{H}_T)$ and $S \in \mathscr{L}(\mathscr{H}_S)$.*

(a) *If $\mathscr{A}_u(S)$ is antisymmetric, $\sigma(T)^\wedge \subseteq \sigma(S)^\wedge$, and $\sigma(T)^\wedge$ is a spectral set for T, then $\mathscr{A}_u(T \oplus S)$ is antisymmetric.*

(b) *If $\mathscr{R}_u(S)$ is antisymmetric, $\sigma(T) \subseteq \sigma(S)$, and $\sigma(T)$ is a spectral set for T, then $\mathscr{R}_u(T \oplus S)$ is antisymmetric.*

Proof. (a) If $A = A^* \in \mathscr{A}_u(T \oplus S)$, then there is a sequence $\{p_n\}$ of polynomials such that $p_n(T \oplus S) = p_n(T) \oplus p_n(S) \to A$. Hence $A = A_1 \oplus A_2$ where A_1 and A_2 are self-adjoint, $p_n(T) \to A_1$, and $p_n(S) \to A_2$. Since $\mathscr{A}_u(S)$ is antisymmetric, $A_2 = cI$. It follows that $p_n(z) \to c$ uniformly on $\sigma(T)^\wedge$. But $\sigma(T)^\wedge$ is a spectral set for T, so $\|p_n(T) - cI\| \leqq \sup\{|p_n(z) - c|: z \in \sigma(T)^\wedge\}$. Thus $p_n(T) \to cI$ and $A_1 = cI$.

(b) The proof of (b) is similar.

COROLLARY 4.2. *If $\mathscr{A}_u(S)$ [resp., $\mathscr{R}_u(S)$] is antisymmetric and*

T is a subnormal operator with $\sigma(T)^\wedge \subset \sigma(S)^\wedge$ [resp., $\sigma(T) \subset \sigma(S)$], then $\mathscr{A}_u(T \oplus S)$ [resp., $\mathscr{R}_u(T \oplus S)$] is antisymmetric.

COROLLARY 4.3. *If $\mathscr{A}_u(S)$ [resp., $\mathscr{R}_u(S)$] is antisymmetric, T is a contraction, and $\{z: |z| \leqq 1\} \subset \sigma(S)^\wedge$ [resp., $\{z: |z| \leqq 1\} \subset \sigma(S)$], then $\mathscr{A}_u(T \oplus S)$ [resp., $\mathscr{R}_u(T \oplus S)$] is antisymmetric.*

These ideas have a nice application to compact operators

COROLLARY 4.4. *If T and N are compact operators and N is normal, then $\mathscr{A}_u(T \oplus N)$ is antisymmetric if and only if $\mathscr{A}_u(T)$ is antisymmetric and $\sigma(N) \subset \sigma(T)$.*

Proof. If $\mathscr{A}_u(T)$ is antisymmetric and $\sigma(N) \subset \sigma(T)$, then $\mathscr{A}_u(T \oplus N)$ is antisymmetric by Corollary 4.2. Now assume that $\mathscr{A}_u(T \oplus N)$ is antisymmetric. Let $T = \bigoplus_{n=1}^\infty T_n$ where each T_n is irreducible and let $N = \bigoplus_{n=1}^\infty \lambda_n$ where $\{\lambda_1, \lambda_2, \cdots\}$ are the eigenvalues of N, each repeated according to its multiplicity; each λ_n is considered as acting on a one dimensional Hilbert space. Then

$$T \oplus N = \bigoplus_{n=1}^{\infty} T_n \oplus \bigoplus_{n=1}^{\infty} \lambda_n$$

is the decomposition of the compact operator $T \oplus N$ as a direct sum of irreducible compact operators. Fix an integer k and let $J = \{j \in N; \lambda_j = \lambda_k\}$: Applying (2.6) to this J, it follows that $\lambda_k \in \sigma\,(T \oplus \bigoplus_{j \notin J} \lambda_j) = \sigma(T) \cup \{\lambda_j : j \notin J\} = \sigma(T) \cup \{\lambda_j : j \notin J\} \cup \{0\}$. Clearly if $\lambda_k \neq 0$ then $\lambda_k \in \sigma(T)$. So $\sigma(N)\backslash\{0\} \subseteqq \sigma(T)$. If $\sigma(N)$ is infinite, then $\sigma(T)$ must be infinite and so $0 \in \sigma(T)$. If $\sigma(N)$ is finite and $\lambda_k = 0$, then (2.6) gives that $0 \in \sigma(T) \cup \{\lambda_j : j \neq J\}$, so $0 \in \sigma(T)$.

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INDIANA UNIVERSITY
BLOOMINGTON, IN 47401
AND
POLISH ACADEMY OF SCIENCES
KRAKOW BRANCH
AND
CENTRO DE INVESTIGACION DEL I. P. N.
MEXICO 14, D. F.