QUADRATIC FORMS AND POWER SERIES FIELDS

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The goal of this paper is to explore the connection between three properties of a field F of characteristic not 2. Roughly speaking, these are:

(a) the Witt ring $W(F)$ is a group ring $R[G]$ with G an Abelian group of exponent 2.

(b) the Witt ring $W(F)$ is isomorphic to $W(K)$ where K is a power series field (i.e., F is equivalent to K with respect to quadratic forms).

(c) there are "enough" rigid elements in F.

Our purpose is to show that the connection is in some sense "quantitative", by showing that (a) and (b) can be "measured" by the index of a certain subgroup $A(F)$ in $\dot{F}$.

Preliminaries. (For proofs or details see [2] and [4]).

If q is a quadratic form over F, then $D_F(q)$ consists of the elements in $\dot{F} = F - \{0\}$ which are represented by q. We denote by $\langle x_1, \cdots, x_n \rangle \cong q$ a diagonalization of an n-dimensional form q. The group $D_F(\langle 1, x \rangle)$, $x \in \dot{F}$, may be seen as the nonzero image of the norm map from $F(\sqrt{-x})$ to F. If $D_F(\langle 1, b \rangle) = \dot{F}^2 \cup b\dot{F}^2$ (as small as is possible) then we say that b is *rigid* in F. Denote by $A(F)$ the set of all elements $x \in \dot{F}$ such that x or $-x$ is nonrigid, together with the group $\dot{F}^2 \cup -\dot{F}^2$. Then $A(F)$ is a subgroup of $\dot{F}$. Note that $b \notin A(F)$ if and only if b and $-b$ are rigid nonsquares.

Recall that the Witt ring $W(F)$ is a ring whose elements are equivalence classes of nonsingular quadratic forms, where q is equivalent to q', written $q = q'$, if and only if $q \perp \langle -1 \rangle q'$ is hyperbolic. Each equivalence class is represented by a unique anisotropic form. To avoid cumbersome notation, it is customary to refer to a form $q \in W(F)$ and mean any form equivalent to q. By q_{an} we mean the unique anisotropic form $q_{an} = q$. Set

$$T(F) = \{q \in W(F) \mid D_F(q_{\mathrm{an}}) \subseteq A(F)\} .$$

Then (see [2]), $T(F)$ is an excellent subring of $W(F)$ which is additively generated over Z by the one-dimensional forms $\{\langle a \rangle \mid a \in A(F)\}$. Moreover

THEOREM 1.1 [2]. $W(F) \cong T(F)[\dot{F}/A(F)]$. *More precisely, if* $\{b_i A(F), i \in I\}$ *is a* $Z/2Z$-*basis of* $\dot{F}/A(F)$ *and* B *is the multiplicative subgroup of* $W(F)$ *generated by the unary forms* $\{\langle b_i \rangle, i \in I\}$, *then* $W(F) = T(F)[B]$.

(In general, if G is a multiplicative group, and $\{g_i, \, i \in I\}$ a subset of G, we denote by $\langle \{g_i, \, i \in I\} \rangle$ the subgroup of G generated by $\{g_i, \, i \in I\}$.)

REMARK. The definition of $A(F)$ given in [2] differs slightly from that which we give here. It is not difficult to see that this does not affect the validity of the results there cited. We prefer the above definition since for the field F_5 with five elements one obtains $A(F_5) = \dot{F}_5^2 \subsetneq \dot{F}_5$, i.e., there are "enough" rigid elements. And, in fact, $W(F_5)$ is a group ring.

Our motivation was an old result of T. A. Springer on fields of formal Laurent series [5] (see also [4, Ch. 6]). Specifically, let $F((x))$ denote such a field, with F-coefficients, which may be obtained as the quotient field of the ring of formal power series $F[[x]]$. We refer to $F((x))$ as a power series field.

THEOREM 1.2 [4]. *Let F be a field (of characteristic not 2). Then*

(i) $\ \ W(F((x))) = W(F)[\{\langle 1 \rangle, \, \langle x \rangle\}]$ *(a group ring).*

(ii) $\ \ x \notin A(F((x)))$ *(power series fields have enough rigid elements).*

In what follows we will consider fields produced by iterating the method of obtaining a power series field. One obtains thus fields of the type $F((x_1))((x_2)) \cdots ((x_n))$ which we denote $F\{((x_i)), \, i = 1, \cdots, n\}$. Let now I be any set. We define the iterated power series field $K = F\{((x_i)), \, i \in I\}$ as follows. First well-order I. Then K is the union (composite) of the iterated power series fields of the form $F\{((x_j)), \, j \in J\}$ where J ranges over all finite (ordered) subsets of I (with respect to the fixed ordering of I).

It is now possible to make precise the main results to be demonstrated in this paper, and to show what we meant above by "quantitative" equivalence:

A. THEOREM 4.1. *For any field F, if $W(F) = R[G]$ and G has exponent 2, then there is a canonical injection of G into $\dot{F}/A(F)$.*

B. COROLLARY 3.2. *For any field F, there is an extension field k of F with $W(F) \cong W(k\{((x_i)), \, i \in I\}) = W(k)[\{\langle x_i \rangle, \, i \in I\}]$, where the cardinality of I equals the $Z/2Z$-dimension of $\dot{F}/A(F)$.*

This explains the terminology "enough rigid elements" in (c) and the "quantitative equivalence" we claimed. Namely, the group $\dot{F}/A(F)$ acts as a true measure of the extent to which $W(F)$ is a group ring over a group of exponent 2, as well as a measure of

the equivalence of F to a power series field, with respect to quadratic forms.

2. **Rigid elements under quadratic extension.** In this section we discuss the relation between $A(F)$ and $A(F(\sqrt{a}))$ for general quadratic extensions $F(\sqrt{a})$ over F, and in particular, when $a \notin A(F)$. In the particular case, it will be seen that $A(F)$ remains "invariant" under quadratic extension. I.e., if $b \notin A(F)$ and $K = F(\sqrt{b})$ then $A(K) = A(F) \cdot \dot{K}^2$. We need a result found in [1]:

LEMMA 2.1. *Let* $K = F(\sqrt{a})$, $a \in \dot{F}$. *Then for any* $x \in \dot{F}$,

$$D_K(\langle 1, x \rangle) \cap \dot{F} = D_F(\langle 1, ax \rangle) \cdot D_F(\langle 1, x \rangle) \ .$$

The next result also holds for general quadratic extensions:

PROPOSITION 2.2. *Let* $K = F(\sqrt{a})$, $a \in \dot{F}$. *Then* $A(K) \cap \dot{F} \subseteq \{1, a\} A(F)$.

Proof. Let $x \in A(K) \cap \dot{F}$. Since x or $-x$ is not rigid over K and $\{\pm 1\} \subseteq A(F)$, we may assume that x is nonrigid over K. Let $z \in D_K(\langle 1, x \rangle) - \{1, x\}\dot{K}^2$. By the Norm Principle [4, p. 208], or by direct computation, if N is the norm from K to F, then $N(z) \in D_F(\langle 1, x \rangle)$. Suppose $x \notin A(F)$. We will show that $x \in a \cdot A(F)$. Since $x \notin A(F)$, x is rigid, i.e., $D_F(\langle 1, x \rangle) = \{1, x\} \cdot \dot{F}^2$, so $N(z) \in \{1, x\}\dot{F}^2$. If $N(z) \in x\dot{F}^2$, then since $N(\dot{K}) = D_F(\langle 1, -a \rangle)$, $x \in D_F(\langle 1, -a \rangle)$. So $a \in D_F(\langle 1, -x \rangle)$. But $x \notin A(F)$ implies that $-x$ is rigid also, so $a \in \{1, -x\}\dot{F}^2$. Now if $a \in \dot{F}^2$, the proposition is trivially true. And if $a \in -xF^2$, then $x \in a(-\dot{F}^2) \subseteq aA(F)$, as was desired. The only other possibility is that $N(z) \in \dot{F}^2$.

By [4, p. 202], $N(z) \in \dot{F}^2$ implies that $z \in \dot{F} \cdot \dot{K}^2$; in fact, we may assume that $z \in \dot{F} \cap D_K(\langle 1, x \rangle)$. Since $x \in \dot{F}$, the last lemma applies, and we find that $z \in D_F(\langle 1, ax \rangle) \cdot D_F(\langle 1, x \rangle)$. If ax is rigid, then $z \in \{l, a, x, ax\} \cdot \dot{F}^2$ (x is rigid). So $z \in \{1, x\} \cdot \dot{K}^2$, a contradiction. Hence, ax cannot be rigid, in other words, $x \in aA(F)$.

A much better result can be obtained under the assumption that $a \notin A(F)$:

THEOREM 2.3. *Let* $b \notin A(F)$, $K = F(\sqrt{b})$. *Then* $A(K) = A(F) \cdot \dot{K}^2$.

Proof. Let x be nonrigid in F and $z \in D_F(\langle 1, x \rangle) - \{1, x\}\dot{F}^2$. If x is rigid in K, then $z \in D_K(\langle 1, x \rangle) = \{1, x\}\dot{K}^2$. By [4, p. 202], $\dot{K}^2 \cap$

$\dot{F} = \{1, b\}\dot{F}^2$, so $z \in \{1, b, x, bx\}\dot{F}^2$. Thus, $z \in \{b, bx\}\dot{F}^2$, so $b \in D_F(\langle 1, x \rangle)$ (a group). This implies that $-x \in D_F(\langle 1, -b \rangle) = \{1, -b\}\dot{F}^2$ (as $b \notin A(F)$, $-b$ is rigid). So $x \in \pm \dot{K}^2 \subseteq A(K)$ if x is rigid in K. Hence in general, $x \in A(K)$. We have just shown that $A(F) \cdot \dot{K}^2 \subseteq A(K)$.

In the light of the last proposition it suffices to show that $A(K) \subseteq F \cdot K^2$ to complete the proof. For if this holds, then $A(K) \subseteq \dot{F} \cdot \dot{K}^2 \cap A(K) = (A(K) \cap \dot{F}) \cdot \dot{K}^2 \subseteq A(F) \cdot \dot{K}^2$. Extend $b\dot{F}^2 \in \dot{F}/\dot{F}^2$ to a $\mathbf{Z}/2\mathbf{Z}$-basis of $\dot{F}/\dot{F}^2$, say $\{b\dot{F}^2\} \cup \{a_j\dot{F}^2, j \in J\}$. Since $-b$ is rigid, $D_F(\langle 1, -b \rangle) = N(\dot{K}) = \{1, -b\}\dot{F}^2$, so by [3, Theorem 8] a $\mathbf{Z}/2\mathbf{Z}$-basis of $\dot{K}/\dot{K}^2$ is given by

$$\{a_j\dot{K}^2, j \in J\} \cup \{z\dot{K}^2\}$$

where z is any element in $\dot{K} - \dot{F} \cdot \dot{K}^2$ (for such $z \notin \dot{F} \cdot \dot{K}^2$, $N(z) \notin \dot{F}^2$, in fact $N(z) \in -b\dot{F}^2$).

Suppose $z \in \dot{K} - \dot{F} \cdot \dot{K}^2$ and z is nonrigid. Then $\{1, z\}\dot{K}^2 \subsetneqq D_K(\langle 1, z \rangle)$. We have seen above that $\dot{F} \cdot \dot{K}^2$ has index 2 in $\dot{K}$, so $D_K(\langle 1, z \rangle) \cap \dot{F} \cdot \dot{K}^2 \supsetneqq \dot{K}^2$. Thus there exists some $y \in D_K(\langle 1, z \rangle) \cap \dot{F}$ with $y \notin \dot{K}^2$. So $-z \in D_K(\langle 1, -y \rangle)$. By the Norm Principle again, $N(-z) \in D_F(\langle 1, -y \rangle)$. But $-z \notin \dot{F} \cdot \dot{K}^2$, so $N(-z) \in -b\dot{F}^2$ and $-b \in D_F(\langle 1, -y \rangle)$. Finally, $y \in D_F(\langle 1, b \rangle) = \{1, b\}\dot{F}^2$ (b is rigid), i.e., $y \in \dot{K}^2$, which is a contradiction. So each $z \notin \dot{F} \cdot \dot{K}^2$ is rigid.

Since $T(F)$ is defined in terms of $A(F)$, it is not surprising to find that $T(F)$ is also "invariant" under a "rigid extension". This will be the key result in our proof of Theorem 3.1 below.

THEOREM 2.4. *Let $b \notin A(F)$, $K = F(\sqrt{b})$. If $r^*: W(F) \to W(K)$ is the map induced by the inclusion of $F \to K$, then the restriction of r^* to $T(F)$ is a canonical isomorphism onto $T(K)$.*

Proof. Note that by Theorem 2.3, $A(K) = A(F) \cdot \dot{K}^2$ so $T(K)$ is additively generated by forms $\{\langle a \rangle, a \in A(K)\} = \{r^*(\langle a \rangle), a \in A(F)\}$. Thus, it is clear that the map $r^*: W(F) \to W(K)$, induced by the inclusion $F \to K$, maps $T(F)$ onto $T(K)$, i.e., $r^*(T(F)) = T(K)$. It suffices to show that the restriction of r^* to $T(F)$ is injective. Suppose φ is an anisotropic form in $T(F)$ and $r^*(\varphi)$ is isotropic. Then by a result of Scharlau [4, p. 200], there is an isometry

$$\varphi \cong \langle 1, -b \rangle\langle x \rangle \perp \gamma ,$$

for some $\gamma \in W(F)$ and $x \in \dot{F}$. So $\{x, -bx\} \subseteq D_F(\varphi)$. But $\varphi \in T(F)$, so $\{x, -bx\} \subseteq D_F(\varphi) \subseteq A(F)$ (as φ is anisotropic). Since $A(F)$ is a group, this implies that $-b \in A(F)$ so $b \in A(F)$, which contradicts the hypothesis. Hence, if $\varphi \in T(F)$, $r^*(\varphi)$ is anisotropic if and only

if φ is anisotropic. So $\varphi = 0$ if and only if $r^*(\varphi) = 0$ for any $\varphi \in T(F)$.

REMARK. Although $A(F)$ and $T(F)$ are invariants with respect to "rigid extensions" there does not exist a natural isomorphism of $W(F)$ and $W(K)$ for $K = F(\sqrt{b})$, $b \notin A(F)$. From the proof of 2.3 we have that

$$\dot{F}/\{1, a\}A(F) \cong (\dot{F} \cdot \dot{K}^2)/A(K)$$

canonically. Since $(\dot{F} \cdot \dot{K}^2)/A(K)$ is a subgroup of $\dot{K}/A(K)$ of index 2, $\dot{F}/\{1, a\}A(F)$ embeds into $\dot{K}/A(K)$ with the same index. So $\dot{F}/A(F) \cong \dot{K}/A(K)$ but this is not natural, as it depends on the choice of a $\mathbf{Z}/2\mathbf{Z}$-basis of $\dot{K}/A(K)$, specifically, on the choice of the basis element $zA(K)$ where $z \in \dot{K} - \dot{F} \cdot \dot{K}^2$.

COROLLARY 2.6. *Let F be a field, $b \notin A(F)$ and $K = F(\sqrt{b})$. Then $W(F)$ and $W(K)$ are ring isomorphic (not canonically).*

Proof. We have $T(K) \cong T(F)$ and $\dot{K}/A(K) \cong \dot{F}/A(F)$. So by Theorem 1.2, $W(F) \cong W(K)$.

3. **Finding power series fields.** We are now prepared to construct a field $k \supseteq F$ such that $W(k) \cong T(F)$ for any field F. Once this is accomplished, we can adjoin to k the requisite number of iterated power series variables, to produce a field K such that $W(K) \cong W(F)$. Specifically, the number of iterated power series variables we must adjoin to k is equal to the $\mathbf{Z}/2\mathbf{Z}$-dimension of $\dot{F}/A(F)$. That is, if $\dot{F}/A(F)$ has a basis $\{b_i A(F) \mid i \in I\}$, then we will take $K = k\{((x_i)), i \in I\}$.

Now such a field K is the union of power series fields $K_J = k\{((x_j)), j \in J\}$, where J ranges over all finite ordered subsets of I (which has been well-ordered). By inductively applying Theorem 1.2, we find that

$$W(K_J) = W(k)[\langle \{\langle x_j \rangle, j \in J\} \rangle] .$$

Since any form in $W(K)$ may be represented by a diagonalization whose entries lie in some K_J, J finite,

$$W(K) = \cup (W(K_J) \underset{k}{\bigotimes} K)$$

$$J \text{ finite} \subset I .$$

In fact, we may identify $W(K_J)$ with its image $(WK_J) \bigotimes_k K$. For suppose $\varphi \in W(K_J)$ is anisotropic and $\varphi \otimes K$ is isotropic. Then also

$\varphi \otimes K_{J'}$ is isotropic, where $I \supseteq J' \supseteq J$ and J' is finite. Since $K_{J'} = K_J\{((x_j)), j \in J' - J\}$ and $J' - J$ is finite, by again applying Theorem 1.2, we have that the map $\varphi \to \varphi \otimes K_{J'}$ takes anisotropic forms to anisotropic forms. So $\varphi \otimes K$ must be anisotropic. Using this identification

$$W(K) = \cup \ W(k\{((x_j)), j \in J\})$$
$$J \text{ finite } \subseteq I$$
$$= \cup \ W(k)[\langle \{\langle x_j \rangle, j \in J\}\rangle]$$
$$J \text{ finite } \subseteq I$$
$$= W(k)[\langle \{\langle x_i \rangle, i \in I\}\rangle] \ .$$

Thus, if k is chosen so that $W(k) \cong T(F)$, then

$$W(K) \cong T(F)[\langle \{\langle x_i \rangle, i \in I\}\rangle] \cong T(F)[\dot{F}/A(F)] = W(F) \ .$$

Recall that a 2-extension of a field F is a field extension k of F which lies in a quadratic closure of F.

THEOREM 3.1. *For any field F, there exists a field k extending F (a 2-extension) such that*
  (i) $A(K) = \dot{k} = A(F) \cdot \dot{k}^2$,
  (ii) $W(k) \cong T(F)$.

COROLLARY 3.2. *Let F be any field. Then there exists a field K of iterated power series over a 2-extension k of F such that*
  (i) *the number of power series variables is equal to the cardinality of a set I indexing a $Z/2Z$-basis of $\dot{F}/A(F)$.*
  (ii) $W(F) \cong W(K) = W(k)[\langle \{\langle x_i \rangle, i \in I\}\rangle]$.

Proof of 3.1. Let $\mathscr{F}$ denote the set of all 2-extensions L of F in a fixed quadratic closure of F with the following properties:
  (1) $A(L) = A(F) \cdot \dot{L}^2$,
  (2) if $r^*_{L/F} \colon W(F) \to W(L)$ is the canonical map induced by the inclusion $F \to L$ then $r^*_{L/F}$ restricted to $T(F)$ is an injection, i.e., $T(F) \otimes_F L \cong T(F)$ via $r^*_{L/F}$.
Let us observe that $F \in \mathscr{F}$, so $\mathscr{F}$ is a nonempty set.
  Suppose $\{L_\beta, \beta \in B\}$ is a chain in $\mathscr{F}$ (totally ordered by inclusion) and $L = \cup \{L_\beta, \beta \in B\}$. We claim that $L \in \mathscr{F}$. First note that (2) is clear. For suppose that $\varphi \in T(F)$ is anisotropic and $\varphi \otimes_F L = 0$. Then the L-isometry of $\varphi \otimes_F L$ and a hyperbolic form is effected by invertible matrices of finite size with coefficients in L. Since $L = \cup \{L_\beta, \beta \in B\}$, these matrices actually have coefficients in some L_β, where $\beta \in B$. I.e., for that $\beta \in B$, $\varphi \otimes_F L_\beta = 0$. But $r^*_{L_\beta/F}$ injects $T(F)$ into $W(L_\beta)$, so $\varphi = 0$ and (2) holds for L.

Next, we need to show that $A(L) = A(F) \cdot \dot{L}^2$. Let $z \in A(L)$. We may assume that z is not rigid since both $A(L)$ and $A(F)$ contain -1. Take $y \in D_L(\langle 1, z \rangle)\dot{L}^2 - \{1, z\}\dot{L}^2$, and write $y = x_1^2 + x_2^2 z$ where x_1, $x_2 \in \dot{L}$. Now choose $\beta \in B$ such that $\{x_1, x_2, z\} \subseteq \dot{L}_\beta$. Clearly $y \notin \{1, z\} \cdot \dot{L}_\beta^2 \subseteq \{1, z\}\dot{L}^2$, so since $y \in D_{L_\beta}(\langle 1, z \rangle)$, $z \in A(L_\beta) = A(F) \cdot \dot{L}_\beta^2 \subseteq A(F) \cdot \dot{L}^2$. Conversely, take $x \in A(F)$, $x \notin \pm \dot{F}^2$ and choose $x' \in D_F(\langle 1, x \rangle) - \{1, x\} \cdot \dot{F}^2$ (we may assume x is nonrigid). Then $x' \in D_L(\langle 1, x \rangle \otimes_F L) - \{1, x\}\dot{L}^2$. For if $x' \in \{1, x\}\dot{L}^2$, then $\langle 1, -x' \rangle \otimes_F L$ or $\langle 1, -xx' \rangle \otimes_F L$ is hyperbolic. But $\langle 1, -x' \rangle$ and $\langle 1, -xx' \rangle$ are in $T(F)$ since x' (and x) is in $A(F)$. (Since $x \notin \pm \dot{F}^2$, $-x \in D_F(\langle 1, -x' \rangle) - \{1, -x'\}\dot{F}^2$, so $-x' \in A(F)$.) So by (2), $\langle 1, -x' \rangle$ or $\langle 1, -xx' \rangle$ is F-hyperbolic; thus $x' \in \{1, x\}\dot{F}^2$, a contradiction. Hence $x \in A(L)$, and so $A(L) = A(F) \cdot \dot{L}^2$.

By Zorn's lemma, we may pick a maximal element $k \in \mathscr{F}$. We need only show, to conclude the proof, that $r^*_{k|F}$ maps $T(F)$ onto $W(k)$. This will be clear if we prove that $A(k) = \dot{k}$. For then, $T(k) = W(k)$. Combining this with the fact that $A(k) = A(F) \cdot \dot{k}^2$ we have that $T(k) = T(F) \otimes_F k \cong T(F)$, so $W(k) \cong T(F)$ via $r^*_{k|F}$. Suppose, by way of contradiction, that $z \in \dot{k} - A(k)$. Let $k_0 = k(\sqrt{z})$. Then by Theorem 2.3 we have $A(k_0) = A(k) \cdot \dot{k}_0^2 = A(F) \cdot \dot{k}^2 \cdot \dot{k}_0^2 = A(F) \cdot \dot{k}_0^2$. Also, by Theorem 2.4, $r^*_{k_0/F}$ is an injection on $T(k)$. Since $r^*_{k|F}$ is an isomorphism of $T(F)$ onto $T(k)$, the composition $r^*_{k_0|F} = r^*_{k_0|k} \circ r^*_{k|F}$ is an injection of $T(F)$ into $W(k_0)$. I.e., $k_0 \in \mathscr{F}$, contradicting the maximality of k among all 2-extensions of F. Thus, $A(k) = \dot{k}$ and $W(k) \cong T(F)$.

REMARK. It is possible to give a more constructive proof of (3.1). Say a finite 2-extension K/F is "admissible" if

$$K = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = F$$

are quadratic extensions, $F_{i+1} = F_i(\sqrt{a_i})$, with $a_i \in \dot{F}_i - A(F_i)$. Then by 2.3 and 2.4 we have inductively that $A(K) = A(F) \cdot \dot{K}^2$ and $T(K) = T(F) \otimes_F K \cong T(F)$.

Let k be the composite of all admissible extensions within a fixed algebraic closure. Then we claim that $A(k) = A(F) \cdot \dot{k}^2 = \dot{k}$ and $W(k) = T(k) = T(F) \otimes_F k \cong T(F)$. Observe that if $x \in \dot{k}$, then x lies in a composite $K = K_1 \cdot K_2 \cdots K_r$ where each K_i is admissible, so also K is admissible. The proofs that $A(k) = A(F) \cdot \dot{k}^2$ and that $T(F) \cong T(F) \otimes_F k = T(k)$ all follow by arguments similar to those used in (3.1). All that remains then is to show, as above, that $\dot{k} = A(k)$. But if $x \in \dot{k} - A(k)$, then $x \in \dot{K} - A(K)$ for some admissible K over F (since $A(K) = A(F) \cdot \dot{K}^2 \subseteq A(F) \cdot \dot{k}^2 = A(k)$). So $K(\sqrt{x})$ is admissible. Thus $K(\sqrt{x}) \subseteq k$, so $x \in \dot{k}^2 \subseteq A(k)$, a contradiction. Thus, $\dot{k} = A(k)$, so $T(F) \cong W(k)$.

4. Group rings and rigid elements. The point of Theorem 1.1 is that when one knows the 2-group $\dot{F}/A(F)$, one also knows that $W(F)$ is a group ring over this same group. In this section we answer the opposite question: does knowing that $W(F) = R[G]$ for a 2-group G tell us anything about $\dot{F}/A(F)$? Our main result is:

THEOREM 4.1. *For any field* F, *if* $W(F) = R[G]$ *and* G *is a 2-group, then there is a canonical injection* $\widetilde{\det}: G \to \dot{F}/A(F)$.

In the case that $|G| = 2$, this result was already demonstrated in [2]. Since we will make use of this, we produce it here:

THEOREM 4.2. *Let* F *be a field.*
  (i) *If* $W(F) = R[\langle 1 \rangle, \langle x \rangle]$ *for some* $x \in \dot{F}$, *then* $x \notin A(F)$.
  (ii) *If* $W(F) = R[G]$ *and* $|G| = 2$, *then* $W(F) = R[\langle 1 \rangle, \langle b \rangle]$ *for some* $b \in \dot{F}$. *In fact, by* (i), $b \notin A(F)$, *so* G *injects into* $\dot{F}/A(F)$.

Now it is possible, when $|G| = 2^n < \infty$ to prove 4.1 by induction on n. Namely, one writes $G = X_{i=1}^n\{\langle 1 \rangle, q_i\}$ in terms of a $Z/2Z$-basis $\{q_1, \cdots, q_n\}$. Set $H_1 = X_{i=2}^n\{\langle 1 \rangle, q_i\}$. Then by 4.2, $W(F) = (R[H_1])[\{\langle 1 \rangle, q_1\}] = (R[H_1])[\{\langle 1 \rangle, \langle b_1 \rangle\}]$ with $b_1 \notin A(F)$. Next set $H_2 =)X_{i=3}^n\{\langle 1 \rangle, q_i\}) \times \{\langle 1 \rangle, \langle b_1 \rangle\}$, so $W(F) = (R[H_2])[\{\langle 1 \rangle, q_2\}] = R[H_2][\{\langle 1 \rangle, \langle b_2 \rangle\}]$ with $b_2 \notin A(F)$. Continuing in this fashion we obtain elements $\{b_1, \cdots, b_n\}$ all not in $A(F)$ such that if $\widetilde{G} = X_{i=1}^n\{\langle 1 \rangle, \langle b_i \rangle\}$, $W(F) = R[\widetilde{G}]$. Then fact that $W(F)$ is a free R-module on $|G|$-generators implies that $|\widetilde{G}| = |G| = 2^n$, so G and $\widetilde{G}$ are isomorphic as $Z/2Z$-vector spaces. This implies that $\{b_1, \cdots, b_n\}$ represent independent cosets of $A(F)$ in $\dot{F}$ (otherwise 4.2 (i) would give a contradiction). So there is an obvious injection of $\widetilde{G}$ into $\dot{F}/A(F)$. However we have not obtained a canonical injection of G into $\dot{F}/A(F)$. We summarize this result:

COROLLARY 4.3. *Let* F *be a field and* $W(F) = R[G]$ *where* $|G| = 2^n < \infty$. *Then there exists a multiplicative subgroup* $\widetilde{G}$ *of* $W(F)$ *such that*
  (i) $|\widetilde{G}| = 2^n$.
  (ii) $W(F) = R[\widetilde{G}]$.
  (iii) $\widetilde{G}$ *is generated by certain one-dimensional forms* $\{\langle b_i \rangle\}$, $i = 1, \cdots, n\}$ *with* $\{b_1, \cdots, b_n\} \subseteq \dot{F}$ *and independent modulo* $A(F)$.
  (iv) $\widetilde{G}$ *injects naturally into* $\dot{F}/A(F)$.

In a sense, 4.1 is weaker than 4.3, in that one cannot in general produce a group $\widetilde{G}$ satisfying (ii) and (iii) above. Clearly, the

inductive argument fails if $|G|$ is not finite. However, by means of another approach it is possible to show that there is always a canonical injection of G into $\dot{F}/A(F)$.

In order to do this, we consider first the ordinary determinant det: $\bar{W}(F) \to \dot{F}/\dot{F}^2$. Since det $(\langle 1, -1 \rangle) = -\dot{F}^2$, the map det does not induce a group homomorphism det: $W(F) \to \dot{F}/\dot{F}^2$. But by considering a "coarser" invariant $\overline{\text{det}}$: $W(F) \to \dot{F}/\pm\dot{F}^2$, namely $\overline{\text{det}}(q) = \pm\text{det}(q)\dot{F}^2$, one obtains an additive group homomorphism on $W(F)$. An even coarser invariant $\widetilde{\text{det}}$: $W(F) \to \dot{F}/A(F)$ may be constructed by setting $\widetilde{\text{det}}(q) = \text{det}(q) \cdot A(F)$ (note that $\pm\dot{F}^2 \subseteq A(F)$).

Let us note that if q_1 and q_2 are odd-dimensional quadratic forms, then det $(q_1 \otimes q_2) = \text{det}(q_1) \cdot \text{det}(q_2)$. A similar equality holds for $\widetilde{\text{det}}$. We next consider the group G in 4.1; if $q \in G$, then $q^2 = q \otimes q = \langle 1 \rangle$, so q is odd-dimensional. Consequently, $\widetilde{\text{det}}$: $G \to \dot{F}/A(F)$ is a multiplicative homomorphism.

Proof of 4.1. By the above discussion, it suffices to show that $\widetilde{\text{det}}$: $G \to \dot{F}/A(F)$ is injective

We may clearly assume that G is a nontrivial group. For each $q \in G$, $q \neq \langle 1 \rangle$, we will define a homomorphism τ_q from $\dot{F}$ into the additive group $Z/2Z$. Fix one such $q \in G$, $q \neq \langle 1 \rangle$. Choose some subring S of $W(F)$ such that $W(F) = S[\{\langle 1 \rangle, q\}]$. If $x \in \dot{F}$ and $\langle x \rangle = s_1 + s_2 q$ with $s_1, s_2 \in S$, define $\tau_q(x) = \dim_0(s_2)$ (recall that $\dim_0(s_2) = 0$ if $\dim(s_2)$ is even and $= 1$ if $\dim(s_2)$ is odd). As in the proof of 4.2 (ii) (see [2]), if $\tau_q(x) = 1$, then s_2 is a unit in $W(F)$, and moreover $W(F) = S[\{\langle 1 \rangle, \langle x \rangle\}]$. So if $\tau_q(x) = 1$, then by 4.2 (i), $x \notin A(F)$. Thus, $A(F) \subseteq \ker(\tau_q)$. Observe that τ_q: $\dot{F} \to Z/2Z$ is a homomorphism, for if $\langle x \rangle = s_1 + s_2 q$ and $\langle x' \rangle = s_1' + s_2' q$, then $\langle xx' \rangle = (s_1 s_1' + s_2 s_2') + (s_1 s_2' + s_2 s_1')q$ (since $q^2 = \langle 1 \rangle$). And $\dim_0(s_1 s_2' + s_2 s_1') = 1$ if and only if $\dim_0(s_2) \neq \dim_0(s_2')$, so $\tau_q(xx') = \tau_q(x) + \tau_q(x')$.

We wish to show for any $q \in G$, $q \neq \langle 1 \rangle$, that det $q \notin A(F)$. Thus it will be enough to show that $\tau_q(\text{det } q) \neq 0$, as $A(F) \subseteq \ker(\tau_q)$. Consider a diagonalization $\langle x_1, \cdots, x_n \rangle \cong q$. We have, in $Z/2Z$,

$$\tau_q(\text{det } q) = \tau_q\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \tau_s(x_i) .$$

Write now, for each $i = 1, \cdots, n$,

$$\langle x_i \rangle = s_i + s_i' q, \ s_i, s_i' \in S$$

(where S is the subring of $W(F)$ used to define τ_q). Then

$$q = \sum_{i=1}^{n} \langle x_i \rangle = \left(\sum_{i=1}^{n} s_i\right) + \left(\sum_{i=1}^{n} s_i'\right)q ,$$

so by the group ring property, $\sum_{i=1}^{n} s_i' = \langle 1 \rangle$. Hence, in $\mathbf{Z}/2\mathbf{Z}$,

$$1 = \dim_0 \left(\sum_{i=1}^{n} s_i' \right) = \sum_{i=1}^{n} \dim_0 (s_i') = \sum_{i=1}^{n} \tau_q(x_i) .$$

Hence, $\tau_q(\det q) = 1$, so $\det q \notin A(F)$, and $\widetilde{\det} : G \to \dot{F}/A(F)$ is injective.

Note. The definition of τ_q in the above proof is not independent of the choice of a subring $S \subseteq W(F)$ satisfying $W(F) = S[\{\langle 1 \rangle, q\}]$. For example, let $F = C((x))((y))$ and take $S = W(C((x)))$, $S' = W(C((xy)))$. Then

$$W(F) = S[\{\langle 1 \rangle, \langle y \rangle\}] = S'[\{\langle 1 \rangle, \langle y \rangle\}] .$$

So on the one hand, with respect to S, we have $\tau_{\langle y \rangle}(x) = 0$ as $\langle x \rangle \in S$. And on the other hand, with respect to S', we have $\tau'_{\langle y \rangle}(x) = 1$ as $\langle x \rangle = \langle xy \rangle \cdot \langle y \rangle$ and $\langle xy \rangle \in S'$.

If $2 \neq 0$ in $W(F)$ (i.e., $-1 \notin \dot{F}^2$) then (4.3) may be put in a simpler form:

COROLLARY 4.4. *Let F be a field with $-1 \notin \dot{F}^2$. If $W(F) = R[G]$ where G is a finite group of exponent 2, then $W(F) = R[\tilde{G}]$ where $\tilde{G}$ consists of unary forms $\{\langle b_q \rangle, q \in G\}$ with $\widetilde{\det} \, q = b_q A(F)$.*

Proof. For a given $q \in G - \langle 1 \rangle$, we may decompose, via Theorem 1.1:

$$q = \sum_{i=1}^{k} \langle b_i \rangle \varphi_i ,$$

where each $\varphi_i \in T(F)$ and $\{b_1, \cdots, b_n\}$ represent distinct cosets of $A(F)$ in $\dot{F}$. As $q^2 = \langle 1 \rangle$, whenever $1 \leq i < j \leq k$, $2\varphi_i\varphi_j = 0$. Since q is odd-dimensional, some φ_i, say φ_1, is odd-dimensional, so is not a zero-divisor [4, p. 250]. Hence, $2\varphi_i = 0$ for $i = 2, \cdots, k$. So since $2 \neq 0$, each φ_i, $i > 1$, is even-dimensional. Now each φ_i may be diagonalized by elements of $A(F)$, so for each i, $\widetilde{\det} (\langle b_i \rangle \varphi_i) = b_i^{d_i}$. $A(F)$, where $d_i = \dim_0 (\varphi_i)$. Hence $\widetilde{\det} (q) = b_1 A(F)$. Set $b_q = b_1$. Observe that $b_q \notin A(F)$. In the proof given above of 4.3 if we replace each generator q of G with the unary form $\langle b_q \rangle$, then we obtain our result.

For suppose that $G = H \times \{1, q\}$ where H is a subgroup of index 2 of G. Then $W(F) = S[\{\langle 1 \rangle, q\}]$ with $S = R[H]$. We claim that $W(F) = S[\{\langle 1 \rangle, \langle b_q \rangle\}]$, which is what is necessary to complete the proof as in 4.3. Recalling the proof of 4.1, it suffices to show that

$\langle b_q \rangle = r + sq$ with $r, s \in S$ and s odd-dimensional, i.e., to show that $\tau_q(b_q) = 1$.

Now for each $i = 1, \cdots, k$, write $\langle b_i \rangle \varphi_i = r_i + s_i q$ with $r_i, s_i \in S$. Then for all i,

$$\dim_0 (s_i) \equiv (\dim \varphi_i)\tau_q(b_i) \pmod 2 \, ,$$

since $\varphi_i \in T(F)$ may be diagonalized over $A(F)$ and $\tau_q(ab_i) = \tau_q(b_i)$ for any $a \in A(F) \subseteq \ker \tau_q$. But for $i \geq 2$, $\dim \varphi_i$ is even, so for such i, $\dim_0 (s_i) = 0$. Finally,

$$q = \sum_{i=1}^{k} r_i + q \sum_{i=1}^{k} s_i \, ,$$

so $\sum_{i=1}^{k} s_i = 1$. Hence $\dim_0 (s_1)$ must be odd, so not only is $\dim \varphi_1$ odd, but also $\tau_q(b_q) = 1$.

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