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WIRTINGER APPROXIMATIONS AND THE KNOT GROUPS OF F^n IN S^{n+2}

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We consider the problem of deciding whether or not a given group G has a Wirtinger presentation, i.e., a presentation in which each defining relation states that two generators are conjugate or that a generator commutes with some word. This property is important because it characterizes those groups that can be realized as knot groups of closed, orientable n-manifolds in S^{n+2}. We isolate the obstruction in the form of an abelian group somewhat related to $H_2(G)$. We do this by considering Wirtinger-presented groups that are approximations to G and prove the existence of a best-approximation.

A group G can be realized as a knot group $\pi_1(S^{n+2} - F^n)$ $(n \geqq 2)$, where F^n is a closed, orientable, connected n-manifold tamely embedded in the sphere S^{n+2}, if and only if G satisfies the following:

(1) G is finitely presented.

(2) $G/G' \cong \boldsymbol{Z}$.

(3) There exists $t \in G$ such that $G/\langle\langle t\rangle\rangle = \{1\}$.

(4) G has a Wirtinger presentation (see Definition 0.1).

The necessity of the algebraic conditions may be seen as follows: (1)-(3) are well-known (see e.g., [8] or [9]). (The methods of this paper can be used to develop a theory of Wirtinger approximations for G/G' free abelian of rank m, i.e., F^n having m components, but we restrict ourselves to $m = 1$ to minimize notation and keep the proofs clear.) (4) is well-known for 1-manifolds (not necessarily connected) in S^3 and we proceed by induction on dimension, using the method of slices [4, §6] to present $\pi_1(S^{n+2} - F^n)$. The sufficiency of the algebraic conditions is established by using methods of Yajima [14] (rediscovered by D. Johnson; see [7] for nice exposition) to construct a surface F^2 in S^4 having a given group.

In this paper, we suppose we are given a group G satisfying (1)-(3) and try to decide whether or not G satisfies (4). If we replace (4) by the property $H_2(G) = 0$, we obtain Kervaire's list [8] [9] characterizing the knot groups of spheres $S^n \subset S^{n+2}$. Thus (1)-(3) plus $H_2(G) = 0$ imply (4); a purely algebraic proof of this fact is given in [15], and we shall recover this theorem as Corollary 1.8.

There was some speculation [10, Problem 4.29], [13, Conj. 4.13] that $H_2(G) = 0$ actually is necessary for G to be $\pi_1(S^{n+2} - F^n)$, but counterexamples have been found ([2], [11], Example 3.4 below). When we know $H_2(G) = 0$, G has a Wirtinger presentation in terms of conjugates of any annihilating element t. In general, however, it is possible (Example 3.5) to have a group G with annihilating elements $s, t \in G$ such that G has a Wirtinger presentation in terms of conjugates of t but none in terms of conjugates of s.

For each choice of annihilating element $t \in G$, we show (Corollary 1.4, Theorem 1.5) that the obstruction to (G, t) having a Wirtinger presentation is a finitely generated abelian group that arises as the kernel of a certain homomorphism $\varphi: W(G, t) \twoheadrightarrow G$. The group $W(G, t)$ is the (Corollary 1.9) best Wirtinger approximation (Definition 1.2) of (G, t). As we initially define it (in Theorem 1.3), $W(G, t)$ has infinitely many generators and relations. However, $W(G, t)$ is (Corollary 1.7) finitely presentable, so there is hope, in any particular situation, of actually finding a presentation that is nice enough for us to decide whether or not φ is an isomorphism.

In §2 we describe a paractical method for obtaining $W(G, t)$ as the last of three successive Wirtinger approximations of (G, t). The first is always constructable since $G/\langle\langle t\rangle\rangle = 1$; the second is automatic. Passing from the second approximation to the third, however, may be difficult as it requires knowledge of the centralizer of t in G'. One result is (Corollary 2.3) that if the centralizer of t in G' is trivial, then the Wirtinger obstruction group $\ker(\varphi)$ is precisely $H_2(G)$.

Finally, in Conjecture 3.6, we offer a strong form of the "Property R" conjecture

DEFINITION 0.1. A *Wirtinger presentation* is a presentation $\langle x_0, x_1, \cdots; r_0, r_1, \cdots\rangle$ such that each relator r is of the form $x_i^{-1}w^{-1}x_jw$ where i, j are any subscripts and w is any word in $\{x\}$.

DEFINITION 0.2. If G is a group, $t \in G$, $\alpha: \langle x_0, x_1, \cdots; r_0, r_1, \cdots\rangle \to G$ an isomorphism, $\alpha(x_0) = t$, and $\langle x_0, x_1, \cdots; r_0, r_1, \cdots\rangle$ a Wirtinger presentation, we call $\langle x_0, x_1, \cdots; r_0, r_1, \cdots\rangle$ (together with α) a *Wirtinger presentation of* (G, t).

1. The best Wirtinger approximation of (G, t).

DEFINITION 1.1. If Y is a group, $y_0 \in Y$, such that (Y, y_0) has a Wirtinger presentation, and there exists an epimorphism $\psi: Y, y_0 \to G, t$ that induces an isomorphism $Y/Y' \to G/G'$, we call Y (together with ψ) a *Wirtinger approximation of* (G, t).

DEFINITION 1.2. If $\varphi: W, s \to G, t$ is a Wirtinger approximation of (G, t) such that given any other Wirtinger approximation $\psi: Y, y_0 \to G, t$ there exists an epimorphism $\hat{\psi}: Y, y_0 \to W, s$ such that $\varphi \circ \hat{\psi} = \psi$, then W (together with φ) is called a *best Wirtinger approximation of* (G, t).

THEOREM 1.3. *Let G be a group with $t \in G$ such that $G/G' \cong \boldsymbol{Z}$ and $G/\langle\langle t\rangle\rangle = \{1\}$. Then there exists a best Wirtinger approximation of (G, t), φ: $W(G, t)$, $s \to G, t$.*

Proof. Let $F = \langle x_e, x_t, \cdots, x_g, \cdots\rangle$ be the free group generated by $\{x_g\}_{g \in G}$. Define a homomorphism $\tilde{\sigma}: F \to G$ by $\tilde{\sigma}(x_g) = g^{-1}tg$, and let $R = \ker \tilde{\sigma}$. By hypothesis, $G/\langle\langle t\rangle\rangle = \{1\}$; since the range of $\tilde{\sigma}$ includes all conjugates of t, it includes a generating set for G, and so $\tilde{\sigma}$ is an epimorphism. Now let R_0 be the normal closure in F of the set $\mathscr{S} \cap R$, where $\mathscr{S}$ is the set of all words of the form $x_h^{-1}w^{-1}x_e w$. In other words,

$$R_0 = \langle\langle\{x_h^{-1}w^{-1}x_e w \mid h \in G,\ w \in F \text{ and } x_h^{-1}w^{-1}x_e w \in R\}\rangle\rangle .$$

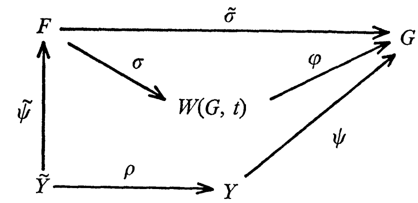
Let σ denote the projection $F \to F/R_0$. We define $W(G, t) = F/R_0$, $s = \sigma(x_e)$, and $\varphi = \tilde{\sigma} \circ \sigma^{-1}$ (which is well-defined since $R_0 \subseteqq R$).

We first claim that φ: $W(G, t) \to G$ is a Wirtinger approximation of (G, t). By definition of R_0, $(W(G, t), s)$ has a Wirtinger presentation. Since $\tilde{\sigma}$ is an epimorphism, so is φ. Also $\varphi(s) = \tilde{\sigma} \circ \sigma^{-1}(s) = \tilde{\sigma}(x_e) = ete = t$. All that remains is to check that φ induces an isomorphism of commutator quotients. By hypothesis, $G/G' \cong \boldsymbol{Z}$. On the other hand, $W(G, t)/W(G, t)'$ is free abelian of rank equal to the number of distinct conjugacy classes of the generators $\sigma(x_g)$, $g \in G$. But for each $g \in G$, if $w_g \in F$ such that $\tilde{\sigma}(w_g) = g$, then $\tilde{\sigma}(x_g^{-1}w_g^{-1}x_e w_g) = e$. Thus $x_g^{-1}w_g^{-1}x_e w_g \in R_0$ and so $\sigma(x_g)$ is conjugate to $\sigma(x_e)$ in $W(G, t)$. Therefore φ induces an epimorphism, hence isomorphism, of $\boldsymbol{Z}$ onto $\boldsymbol{Z}$.

Suppose now that ψ: $Y, y_0 \to G, t$ is another approximation of (G, t). We have $Y \cong \langle y_0, y_1, \cdots;$ relators of the form $y_j^{-1}v^{-1}y_k v\rangle$. Since $Y/Y' \cong G/G' \cong \boldsymbol{Z}$, each of the generators y_i of Y is conjugate to y_0. Thus we may assume that the defining relators for Y include a preferred one for each y_i $(i \neq 0)$ of the form $y_i^{-1}v_i^{-1}y_0 v_i$. By substituting for y_j and y_k, the remaining relators can be written in the form $y_0^{-1}u^{-1}y_0 u$. We shall show that the function $y_0 \to s = \sigma(x_e)$, $y_i \to \sigma(x_{\psi(v_i)})$ defines the desired map of Y onto $W(G, t)$.

Let $\tilde{Y}$ be the free group $\langle \tilde{y}_0, \tilde{y}_1, \cdots\rangle$ and let ρ: $\tilde{Y} \to Y$ be defined by $\rho(\tilde{y}_i) = y_i$. The function $\hat{\psi}(\tilde{y}_0) = x_e$, $\hat{\psi}(\tilde{y}_i) = x_{\psi(v_i)}$ $(i \neq 0)$ defines a homomorphism of $\tilde{Y}$ into F.

Claim (1). The diagram below is commutative.

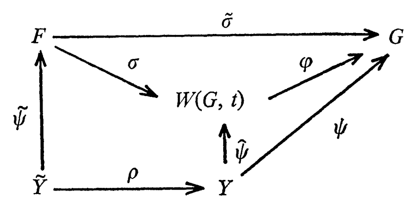


Proof of (1). We defined $\varphi = \tilde{\sigma}\circ\sigma^{-1}$, so the upper triangle commutes. For $\tilde{y}_0$, we have $\psi\rho(\tilde{y}_0) = \psi(y_0) = t$, while $\tilde{\sigma}\tilde{\psi}(\tilde{y}_0) = \tilde{\sigma}(x_e) = ete$. For a generator $\tilde{y}_i$ $(i \neq 0)$, we have $\psi\rho(\tilde{y}_i) = \psi(y_i)$, while $\tilde{\sigma}\tilde{\psi}(\tilde{y}_i) = \tilde{\sigma}(x_{\psi(v_i)}) = \psi(v_i)^{-1}t\psi(v_i)$. But since ψ is a homomorphism and $y_i^{-1}v_i^{-1}y_0v_i = 1 \in Y$, we have $\psi(y_i) = \psi(v_i)^{-1}t\psi(v_i)$ in G.

Claim (2). $\ker\rho \subseteqq \ker(\sigma\circ\tilde{\psi})$.

Proof of (2). Consider the set $\{y_j^{-1}v^{-1}y_kv\}$ of defining relators for Y. As noted earlier, by using the preferred relators $y_i^{-1}v_i^{-1}y_0v_i$, we can rewrite all the others in the form $y_0^{-1}u^{-1}y_0u$. If we let $\tilde{v}_i$, $\tilde{u}$ denote the words obtained from v_i, u by replacing each y-symbol with $\tilde{y}$, we get a set of words $\{\tilde{y}_i^{-1}\tilde{v}_i^{-1}\tilde{y}_0\tilde{v}_i\} \cup \{\tilde{y}_0^{-1}\tilde{u}^{-1}\tilde{y}_0\tilde{u}\}$ whose normal closure in $\tilde{Y}$ is $\ker\rho$. The images of these words under $\tilde{\psi}$ are $x^{-1}_{\psi(v_i)}\tilde{\psi}(v_i)^{-1}x_e\tilde{\psi}(v_i)$ or $x_e^{-1}\tilde{\psi}(\tilde{u})^{-1}x_e\tilde{\psi}(\tilde{u})$. By Claim 1, these words are in $\ker(\varphi\circ\sigma) = R$; but these words are also of the right form to be in $\mathscr{S}$, hence in $R_0 = \ker\sigma$. We thus have $\sigma\circ\tilde{\psi}(\ker\rho) = \{1\}$, so $\ker\rho \subseteqq \ker(\sigma\circ\tilde{\psi})$.

Claim (3). The homomorphism $\tilde{\psi}$ induces a homomorphism $\hat{\psi}$: $Y \to W(G, t)$ making the following diagram commute.



Proof of (3). This follows immediately from *Claims* (1) and (2)

Claim (4). The homomorphism $\hat{\psi}$: $Y \to W(G, t)$ is onto.

Proof of (4). The images $\sigma(x_h)$, $h \in G$, generate $W(G, t)$. For each $h \in G$, since ψ: $Y \to G$ is assumed to be onto, there exists $\tilde{h} \in \tilde{Y}$ such that $\psi\circ\rho(\tilde{h}) = h$. But then $\tilde{\sigma}\circ\tilde{\psi}(\tilde{h}) = h$ and $\tilde{\sigma}\circ\tilde{\psi}(\tilde{h}^{-1}\tilde{y}_0\tilde{h}) = h^{-1}th$, so $x_h^{-1}\tilde{\psi}(\tilde{h})^{-1}\tilde{\psi}(\tilde{y}_0)\tilde{\psi}(\tilde{h}) \in R_0$. Thus $\sigma(x_h) = \hat{\psi}(\rho(\tilde{h}^{-1}\tilde{y}_0\tilde{h}))$.

This completes the proof of Theorem 1.3.

COROLLARY 1.4. *If* $\varphi: W \to G$ *is a best Wirtinger approximation of* (G, t), *then* (G, t) *has a Wirtinger presentation if and only if* φ *is an isomorphism*, i.e., $\ker \varphi = 0$.

Proof. The "if" is trivial. If (G, t) has a Wirtinger presentation, then id: $G, t \to G, t$ is a Wirtinger approximation of (G, t). But then there is an epimorphism $\hat{\psi}: G \to W$ such that $\varphi \circ \hat{\psi} = \mathrm{id}$, so φ is 1-1.

REMARK. It is tempting to claim that the universal mapping property of a best approximation guarantees that any two best approximations are isomorphic. But all we can get is homomorphisms of each onto the other. To prove uniqueness, we need to know that $\ker \varphi$ is Hopfian, so we postpone the uniqueness theorem until after Theorem 1.5.

THEOREM 1.5 (Properties of $\ker \varphi$). *Suppose* $\varphi: W(G, t) \to G$ *is the particular best approximation exhibited in Theorem* 1.3. *Then we have the following*:

(a) $\ker \varphi$ *is central in* $W(G, t)$.

(b) $\ker \varphi$ *is a homomorphic image of* $H_2(G; \boldsymbol{Z})$.

(c) *If* G *is finitely presented, then* $\ker \varphi$ *is a finitely generated abelian group.*

Proof of (a). Let $r \in \ker \varphi$. Choose $\tilde{r} \in F$ such that $\sigma(\tilde{r}) = r$. Since $\varphi(r) = e$, $\tilde{\sigma}(r) = e$, i.e., $\tilde{r} \in R$. For any generator x_g of F, we then have $x_g^{-1}\tilde{r}^{-1}x_g\tilde{r} \in R_0$. Thus r commutes with $\sigma(x_g)$ in $W(G, t) = F/R_0$, so r is central in $W(G, t)$.

Proof of (b). Consider the following exact sequence [12].

$$H_2(W(G, t)) \longrightarrow H_2(G) \longrightarrow \frac{\ker \varphi}{[W(G, t), \ker \varphi]} \longrightarrow H_1(W(G, t)) \longrightarrow H_1(G) \longrightarrow 0\ .$$

The last epimorphism is an isomorphism. Since $\ker \varphi$ is central in $W(G, t)$, the sequence then becomes $H_2(W(G, t)) \to H_2(G) \to \ker \varphi \to 0$.

Proof of (c). This follows immediately from (b), since $H_2(G)$ can be computed as $H_2(X)/\pi_2(X)$, where X is a finite CW-complex having fundamental group G.

REMARKS. The proof of (b) above shows that $\varphi: W(G, t) \to G$ is an isomorphism if and only if the homomorphism $\varphi_*: H_2(W(G, t)) \to H_2(G)$, induced by φ, is surjective. Thus when G is finitely presented, the problem of deciding whether or not (G, t) has a Wirtinger presentation reduces to a problem about finitely generated abelian groups. But the act of reduction may involve an unsolvable problem.

Question 1.6. Is there an algorithm for computing a presentation of $H_2(W(G, t))$ from a presentation of G?

COROLLARY 1.7. *If G is finitely presented, then $W(G, t)$ is finitely presented.*

Proof. By part (c) of Theorem 1.5, $W(G, t)$ is an extension of a finitely presented group by a finitely presented group.

COROLLARY 1.8. *If $G/G' = \boldsymbol{Z}$, $G/\langle\langle t\rangle\rangle = \{1\}$, and $H_2(G) = 0$ then (G, t) has a Wirtinger presentation.*

COROLLARY 1.9. (The uniqueness of best approximation). *If G is finitely presented, $G/G' \cong \boldsymbol{Z}$, and $G/\langle\langle t\rangle\rangle = \{1\}$, then any two best Wirtinger approximations of (G, t), $\varphi_i: W_i \to G$, $(i = 1, 2)$ are isomorphic by an isomorphism $\psi_{12}: W_1 \to W_2$ such that $\varphi_2 \circ \psi_{12} = \varphi_1$.*

Proof. Without loss of generality, let $W_2 = W(G, t)$, $\varphi_2 = \varphi$. This guarantees that $\ker \varphi_2$ is, by Theorem 1.5(c), a Hopfian group. We have epimorphisms $\psi_{ij}: W_i \to W_j$ such that $\varphi_j \circ \psi_{ij} = \varphi_i$ $(i, j = 1, 2)$. A little diagram chasing reveals that $\psi_{ij} \circ \psi_{ji}$ maps $\ker \varphi_j$ onto $\ker \varphi_j$, and that $\ker (\psi_{ij} \circ \psi_{ji}) \subseteqq \ker \varphi_j$. Since $\ker \varphi_2$ is Hopfian, the epimorphism $\psi_{12} \circ \psi_{21} | \ker \varphi_2$ is 1-1, and so the unrestricted map $\psi_{12} \circ \psi_{21}$ is 1-1. Thus ψ_{12} is the desired isomorphism.

REMARK. It is possible to extend the sequence used in the proof of Theorem 1.5. According to [5], and using the fact that $H_1(W(G, t)) \cong \boldsymbol{Z}$, there is a nonnatural homomorphism $\ker \varphi \to H_2(W(G, t))$ making the following sequence exact.

$$\ker \varphi \longrightarrow H_2(W(G, t)) \longrightarrow H_2(G) \longrightarrow \ker \varphi \longrightarrow 0 \,. \tag{1.10}$$

2. Computing $W(G, t)$. In this section, we describe a method for obtaining a presentation of $W(G, t)$ by three successive Wirtinger approximations of (G, t), $H \to C \to W(G, t) \to G$. The letters H, C are mnemonics for "homology" and "central". We denote the maps by $\varphi_1: H \to G$, $\varphi_2: C \to G$, and, as before, $\varphi: W(G, t) \to G$.

To illustrate the various steps, we shall carry along one example (a certain extension of the alternating group A_5); other examples are given in §3.

EXAMPLE 2.1. $G = \langle a, b, t; a^3 = b^5 = (ab)^2 = 1, t^{-1}at = a^{-1}, t^{-1}bt = ab^3ab^{-2}\rangle$.

Step 1. The group H is any group such that: H has an annihilating element $\hat{t}$, $H_1(H) \cong \boldsymbol{Z}$, $H_2(H) = 0$, and there exists an epimorphism φ_1: $H\hat{t} \to G, t$.

Any group H having a killer $\hat{t}$, $H_1(H) \cong \boldsymbol{Z}$ and $H_2(H) = 0$ has (by Corollary 1.8) a Wirtinger presentation on conjugates of $\hat{t}$. Thus any epimorphism φ_1: $H, \hat{t} \to G, t$ is a Wirtinger approximation. The assumption $G/\langle\!\langle t\rangle\!\rangle = \{1\}$ guarantees that if we start with any presentation of G, and are given t as a word in the generators, then we can compute a suitable group H. To avoid such impractical tactics as "enumerate all finite presentations of G", it is hoped that our assumption that $G/\langle\!\langle t\rangle\!\rangle = 1$ is accompanied by a proof. We then proceed as follows: Since G is generated by conjugates of t, G can be presented in the form $\langle t, s_1, \cdots, s_n; \{s_i = w_i^{-1}tw_i\}_{i=1,\cdots,n}$, other relators$\rangle$. Let $H = \langle t, s_1, \cdots, s_n; \{s_i = w_i^{-1}tw_i\}_{i=1,\cdots,n}\rangle$. Alternatively, for each original generator g_i of G, there is a relation R_i expressing g_i in terms of conjugates of t, and we can use these relations to define H. These methods for finding a suitable H are based on the proof given in [7] of González-Acuña's theorem [6] which states that each group of weight 1 is a homomorphic image of a knot group (of $S^1 \subset S^3$). According to that theorem, there also is a classical knot group that we could use for H. Actually, we could start with any Wirtinger approximation of (G, t) and Steps 2 and 3 below would carry us to $W(G, t)$. But by asking that $H_2(H) = 0$, we get the nice property of C that $\ker(\varphi_2: C \to G)$ is precisely $H_2(G)$ (see comments after Step 2).

EXAMPLE 2.1 (cont'd). We wish to find a set of conjugates of t that generate G. Since $t^{-1}at = a^{-1}$ and $a^3 = 1$, we have $a = (a^{-1}ta)^{-1}(t)$. Since $(ab)^2 = b^5 = 1$, $b = [(b^{-2}ab^2)(b^{-1}ab)]^2$. Thus G is generated by t, $s_1 = a^{-1}ta$, $s_2 = b^{-1}tb$, $s_3 = b^{-1}a^{-1}tab$, $s_4 = b^{-2}tb^2$, and $s_5 = b^{-2}a^{-1}tab^2$. If we replace a by $s_1^{-1}t$ and b by $(s_5^{-1}s_4s_3^{-1}s_2)^2$ in the last five equations, we get defining relations for a suitable group H. However, this is not the most useful form. Since the second step in approximating G will require knowing the kernel of φ_1: $H \to G$, it is useful to have a presentation of H in which the original generators of G appear. We are thus led to the following choice.

$$H = \langle a, b, t, s_1, \cdots, s_5; s_1 = a^{-1}ta, s_2 = b^{-1}tb, s_3 = b^{-1}a^{-1}tab,$$

$$s_4 = b^{-2}tb^2,\ s_5 = b^{-2}a^{-1}tab^2,\ a = s_1^{-1}t,\ b = (s_5^{-1}s_4s_3^{-1}s_2)^2\rangle$$
$$= \langle a, b, t; a = a^{-1}t^{-1}at,\ b = [(b^{-2}ab^2)(b^{-1}ab)]^2\rangle\ .$$

In §3 we shall give a different treatment of Example 2.1, involving a choice of initial approximation H that is harder to find but easier to use later.

Step 2. Centralize the kernel of map from H to G. That is, $C = H/[H, \ker \varphi_1]$.

This step is automatic if the presentation one has for H involves the generators from a presentation of G. It is useful to note that the kernel of the map $\varphi_2 \colon C \to G$ is precisely $H_2(G)$. This may be seen by considering the exact sequence [12]

$$0 = H_2(H) \longrightarrow H_2(G) \longrightarrow \ker \varphi_1/[H, \ker \varphi_1] \longrightarrow H_1(H) \longrightarrow H_1(G) \longrightarrow 0\ .$$

If we do not know $H_2(H) = 0$ then we still have that $\ker \varphi_2$ is a homomorphic image of $H_2(G)$. Since we can centralize an element of H by declaring that $\hat{t}$ commutes with various conjugates of that element, $\varphi_2 \colon C \to G$ is a Wirtinger approximation.

EXAMPLE 2.1 (cont'd). $C = \langle a, b, t; a = a^{-1}t^{-1}at,\ b = (b^{-1}aba)^2,\ a^3$ central, b^5 central, $(ab)^2$ central, $t^{-1}ata$ central, $t^{-1}btb^2a^{-1}b^{-3}a^{-1}$ central$\rangle$.

Step 3. Adjoin enough relations to C to describe the centralizer of t in G. That is, make the sets $\varphi_2^{-1}(t)$ and φ_2^{-1} (centralizer of t in G) commute elementwise.

This is the step that distinguishes t from other killers of G and that requires a fairly complete knowledge of the internal structure of G.

Let Z_t denote the centralizer of t in G. The crudest way to perform Step 3 would be to adjoin all relations of the form $[\tilde{t}, \tilde{x}] = 1$ where $\tilde{t}$ ranges over $\varphi_2^{-1}(t)$ and $\tilde{x}$ ranges over φ_2^{-1} (Z_t). The first obvious simplication is that we only need to consider one antecedent $\hat{t}$ for t and one representative $\hat{x}$ for each x. If $\tilde{t} = \hat{t}q_1$ and $\tilde{x} = \hat{x}q_2$, where $q_1, q_2 \in \ker \varphi_2$, then in C, $[\tilde{t}, \tilde{x}] = [\hat{t}, \hat{x}]$, since

$$\ker \varphi_2 (= \ker \varphi_1/[H, \ker \varphi_1])$$

is central in C. This makes it clear that we are only adding Wirtinger type relations, so Step 3 does yield a Wirtinger approximation of G.

The next simplification is that we only need to consider $Z_t \cap G'$

rather than all of Z_t. Each $x \in G$ can be expressed in the form $t^n x_0$ where $x_0 \in G'$. In G, $[t, x] = [t, x_0]$, so $x \in Z_t$ if and only if $x_0 \in Z_t \cap G'$. When we adjoin to C the relation $[\hat{t}, \hat{x}_0] = 1$, we can deduce relations of the form$[\hat{t}, \hat{t}^n \hat{x}_0] = 1$. The third simplification is that we do not need to add a relation for each element of $Z_t \cap G'$, but only for a set of generators of that group. For suppose $x_1, x_2, \cdots$ generate $Z_t \cap G'$ and suppose we have added to C relations $[\hat{t}, \hat{x}_i] = 1$. For each $x \in Z_t \cap G'$ there is some word in the $\hat{x}_i$ that we could choose for $\hat{x}$. Thus adding a relation $[\hat{t}, \hat{x}] = 1$ would be redundant.

Finally we note that we need to add only finitely many new relators $[\hat{t}, \hat{x}]$, regardless of the possibility (???) that $Z_t \cap G'$ is infinitely generated. The relators we are adding generate a subgroup of $\ker \varphi_2$. When G is finitely presented, $H_2(G)$ is finitely generated, and so the homomorphic image, $\ker \varphi_2$, also is finitely generated.

EXAMPLE 2.1 (cont'd). It is not hard to check that in G, t commutes with b^2a and with ab^3ab^2. It is much harder to show that these elements generate all of $Z_t \cap G'$. The mapping $a \to (153)$, $b \to (12345)$, $t \to (35)$ is a homomorphism of G onto the symmetric group S_5 that faithfully maps G', that is the subgroup generated by a and b, onto the alternating group A_5 [cf. 3, §6.4]. The centralizer of (35) in A_5 is just the subgroup generated by (124) and (14)(35), that is, the images of b^2a and ab^3ab^2. Modulo Theorem 2.2 below, we then have the following.

$$\begin{aligned} W(G, t) \cong \langle a, b, t;\ & a = a^{-1}t^{-1}at,\ b = (b^{-1}aba)^2, \\ & \{a^3, b^5, (ab)^2, t^{-1}ata, t^{-1}btb^2a^{-1}b^{-3}a^{-1}\} \text{ central}, \\ & [t, b^2a] = [t, ab^3ab^2] = 1\rangle\ . \end{aligned}$$

THEOREM 2.2. *The group produced by Step 3 is a (hence "the") best Wirtinger approximation. That is, if $\varphi_1\colon H, \hat{t} \to G, t$ is a Wirtinger approximation of (G, t), $C = H/[H, \ker \varphi_1]$, $W_0 = C/[\varphi_2^{-1}(t), \varphi_2^{-1}(Z_t)]$, and $\varphi_2\colon C \to G$, $\varphi_0\colon W_0 \to G$ are the induced homomorphisms, then $\varphi_0\colon W_0, \hat{t} \to G, t$ is a best Wirtinger approximation.*

Proof. By definition, the group W_0 has a presentation $W_0 \cong \langle \hat{t}, s_1, \cdots, s_n; \mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3 \rangle$, where $\mathscr{R}_1$ consists of n relations $\{s_i = w_i^{-}\hat{t}w_i\}$ and perhaps some others $\{\hat{t}^{-1}v_j^{-1}\hat{t}v_j\}$; $\mathscr{R}_2 = \{\hat{t}^{-1}c^{-1}\hat{t}c \mid c$ is a word defining an element of $\ker \varphi_1\}$; and $\mathscr{R}_3 = \{\tilde{t}^{-1}\tilde{x}^{-1}\tilde{t}\tilde{x} \mid \tilde{t}, \tilde{x}$ are words such that $\varphi_2(\tilde{t}) = t$ and $\varphi_2(\tilde{x}) \in Z_t\}$.

Any Wirtinger type word in $\hat{t}, s_1, \cdots, s_n$ that is in $\ker \varphi_0$ is already trivial in W_0. For if $s_j^{-1}v^{-1}s_kv$ is such a word (denote $\hat{t}$ by s_0 here) $(0 \leqq j, k \leqq n)$, then using $\{s_i = w_i^{-1}\hat{t}w_i\}_{i=1,\cdots,n}$, the word can be rewritten in W_0 as a conjugate of $[\hat{t}, w_k v w_j^{-1}]$. Since $[\hat{t}, w_k v w_j^{-1}] \in \ker \varphi_0$, $\varphi_0(w_k v w_j^{-1}) \in Z_t$, and so $[\hat{t}, w_k v w_j^{-1}] \in \mathscr{R}_3$.

We now exhibit an isomorphism between W_0, $\hat{t}$ and the group $W(G, t)$, s of Theorem 1.3. For $g \neq e$, $\varphi_0(w_1), \cdots, \varphi_0(w_n)$ in G, introduce the symbol x_g as an additionanal generator of W_0 and set $x_g = \hat{g}^{-1}\hat{t}\hat{g}$, where $\hat{g}$ is any word in $\hat{t}, s_1, \cdots, s_n$ for which $\varphi_0(\hat{g}) = g$. By substituting $\hat{g}^{-1}\hat{t}\hat{g}$ for x_g, and recalling the above paragraph, we see that any Wirtinger type word in the generators $\{\hat{t}, s_1, \cdots, s_n, \{x_g\}\}$ that is mapped by φ_0 to 1 in G is in the consequence of the relators defining W_0. If we identify $\hat{t}, s_1, \cdots, s_n$ with $x_e, x_{\varphi_0(w_1)}, \cdots, x_{\varphi_0(w_n)}$, we obtain the desired isomorphism between W_0, $\hat{t}$ and $W(G, t)$, s.

EXAMPLE 2.1 (concluded). To decide whether or not (G, t) has a Wirtinger presentation, we must decide whether or not the defining relations of G can be deduced from the relations defining $W(G, t)$. The answer is "yes," but rather than go through the derivations here, we shall defer to the next section.

COROLLARY 2.3. *If $G/G' = \boldsymbol{Z}$, $G/\langle\langle t\rangle\rangle = \{1\}$, and the centralizer of t in G is just $\langle t\rangle$, then the Wirtinger obstruction group,* $\ker \varphi$, *is precisely $H_2(G)$.*

Proof. By Theorem 2.2, Step 3 yields $W(G, t)$. But if $Z_t \cap G' = \{1\}$, then there are no relations to be added in Step 3. Thus $\ker \varphi = \ker \varphi_2$, which, as noted after Step 2, is isomorphic to $H_2(G)$.

3. Examples.

The previous section concluded with the need to decide if a certain messy-looking group is isomorphic to a given extension G of A_5. We shall give a different analysis of the same group G that makes the final calculation easier. The first two of the following lemmas are useful for several examples.

LEMMA 3.1. *The group $\mathscr{D} = \langle a, b; a^3 = b^5 = (ab)^2\rangle$ has $H_1(\mathscr{D}) = H_2(\mathscr{D}) = 0$.*

Proof. The quotient $\mathscr{D}/\mathscr{D}'$ clearly is trivial. Since $\mathscr{D}$ has the same number of generators as relations, it follows that $H_2(\mathscr{D}) = 0$.

LEMMA 3.2. (*Special case of HNN extensions.*) *Suppose D is a group with $H_1(D) = H_2(D) = 0$. If α is an automorphism of D and K is the extension $\langle D, t; t^{-1}at = \alpha(a)$, all $a \in D\rangle$, then $H_1(K) \cong \langle t\rangle$ and $H_2(K) = 0$.*

Proof. This follows immediately from the Mayer-Vietoris sequence for HNN extensions [1].

LEMMA 3.3. *The function* $a \to a^{-1}$, $b \to ab^3ab^{-2}$ *defines an automorphism of* $\mathscr{D} = \langle a, b; a^3 = b^5 = (ab)^2\rangle$.

Proof. It is useful to rewrite the relations for $\mathscr{D}$ as $aba = b^4$ and $bab = a^2$. In addition, it can be shown [3, §6.5] that $(a^3)^2 = 1$ and that $\mathscr{D}$ is finite.

Let $\alpha: \langle a, b; -\rangle \to \mathscr{D}$ be defined by $\alpha(a) = a^{-1}$, $\alpha(b) = ab^3ab^{-2}$. Then α induces a homomorphism of $\mathscr{D}$ into $\mathscr{D}$.

Suppose $x \in \mathscr{D}$ and $\alpha(x) = 1$. Consider the projection of $\mathscr{D}$ onto the alternating group $A_5 \cong \langle a, b; a^3 = b^5 = (ab)^2 = 1\rangle$. Since, as noted in §2, the function $a \to a^{-1}$, $b \to ab^3ab^{-2}$ in 1-1 on A_5, $x \in \ker(\mathscr{D} \to A_5)$. But [3, §6.5] this kernel is just the group of order 2 generated by a^3, and $\alpha(a^3) = a^3 \neq 1$. Thus $x = 1$, α is 1-1, and so α defines an automorphism of $\mathscr{D}$.

EXAMPLE 3.4. (Example 2.1, different analysis.) $G = \langle a, b, t; a^3 = b^5 = (ab)^2 = 1, t^{-1}at = a^{-1}, t^{-1}bt = ab^3ab^{-2}\rangle$.

By Lemmas 3.1-3.3, we can use $H = \langle a, b, t; a^3a^5 = (ab)^2, t^{-1}at = a^{-1}, t^{-1}bt = ab^3ab^{-2}\rangle$ for our first Wirtinger approximation of (G, t). Since the kernel of $\varphi_1: H \to G$ is just the central (remember $a^6 = 1$) subgroup of order 2 generated by a^3, we have $C = H$ (and also $H_2(G) \cong \ker\varphi_2 \cong \ker\varphi_1 \cong \boldsymbol{Z}_2$). As in §2, $Z_t \cap G'$ is generated by b^2a and ab^3ab^2. We thus have $W(G, t) \cong \langle a, b, t; a^3 = b^5 = (ab)^2, t^{-1}at = a^{-1}, t^{-1}bt = ab^3ab^{-2}, [t, b^2a] = [t, ab^3ab^2] = 1\rangle$.

We deduce $W(G,t) \cong G$ from the last relation: $1 = t^{-1}ab^3ab^2tb^{-2}a^{-1}b^{-3}a^{-1} = a^{-1}(ab^3ab^{-2})^3a^{-1}(ab^3ab^{-2})^2b^{-2}a^{-1}b^{-3}a^{-1} =$ (freely reduce, then multiply each b^{-2} or b^{-3} by b^5, each a^{-1} by a^3, and use $a^6 = b^{10} = 1$) $(b^3a)^5bab^3ab^3ab^6a^2b^2a^2 = (b^3a)^3b^3a^2b^2ab^2a^2 =$ (since $bab = a^2$)$b^2a^2ba^2ba^2b^2a^2ba^2ba^2 =$ (since $aba = b^4$) $b^2ab^3ab^2ab^3a =$ (since $bab = a^2$)$ba^2bab^2a(a^3) =$ (extract $abab$)$baba$.

EXAMPLE 3.5. Let $G = A_5 \oplus \boldsymbol{Z} = \langle a, b, \tau; a^3 = b^5 = (ab)^2 = 1, \tau^{-1}a\tau = a, \tau^{-1}b\tau = b\rangle$ and let $t = \tau x$, where x denotes one of a, b, or $b^2ab^4ab^3a$.

We shall show that in the third case, (G, t) has a Wirtinger presentation, while in the first two cases it does not. The third case, where x is an element of order 2, has been studied in [11], and represents, along with Example 3.4 and [2], one of the few known groups having a Wirtinger presentation and nontrivial second homology. We shall consider the three cases simultaneously.

By Lemmas 3.1, 3.2, we may take $H = \langle a, b, \tau; a^3 = b^5 = (ab)^2, \tau^{-1}a\tau = a, \tau^{-1}b\tau = b\rangle$. Since the center of $\mathscr{D} = \langle a, b; a^3 = b^5 = (ab)^2\rangle$

is contained in $\ker(\mathscr{D} \to A_5)$, and A_5 is simple, it follows that for any word x in a and b that defines a nontrivial element of A_5, $H/\tau x = \{1\}$. Since $\ker(H \to G) \subseteq Z(H)$, we have $C = H$. To pass from C to $W(G, t)$, we must distinguish the choice of t, i.e., of x.

Case 1. $x = a$ or $x = b$.

In this case, $Z_t \cap G'$ is just the cyclic subgroup of A_5 generated by x. In terms of a, b, and τ, $[t, x] = [\tau x, x] = 1$ in C. Thus we need add no relations to get from H to C to $W(G, t)$; that is, $W(G, t) \cong \langle a, b; a^3 = b^5 = (ab)^2 \rangle \oplus \langle \tau \rangle$. Since [3, §6.5] the central element a^3 of $\mathscr{D}$ has order exactly 2 in $\mathscr{D}$, we conclude that (G, t) does *not* have a Wirtinger presentation, and the obstruction group $\ker(\varphi)$ is $\boldsymbol{Z}_2$.

Case 2. $x = b^2ab^4ab^3a$.

We could use any involution for x, but this choice has the convenient property that $x^{-1}ax = a^{-1}$, $x^{-1}bx = b^{-1}$ in A_5 and in fact in $\mathscr{D}$. Thus, if $t = \tau x$, $t^{-1}at = a^{-1}$, $t^{-1}bt = b^{-1}$. The group $Z_t \cap G'$ is precisely the subgroup of A_5 consisting of elements that commute with $x = b^2ab^4ab^3a$. By identifying a with (153) and b with (12345), we see that the centralizer of x in A_5 is generated by b^3ab^3 and bab^3ab. We thus have $W(G, t) = C/\{[t, b^3ab^3], [t, bab^3ab]\} \cong \langle a, b, \tau; a^3 = b^5 = (ab)^2, \tau^{-1}a\tau = a, \tau^{-1}b\tau = b, [\tau x, b^3ab^3] = [\tau x, bab^3ab] = 1\rangle$, where $x = b^2ab^4ab^3a$. The next-to-last relation says $x^{-1}b^3ab^3x = b^3ab^3$. But since $x^{-1}ax = a^{-1}$, $x^{-1}bx = b^{-1}$ in $\mathscr{D}$, we have in $W(G, t)$: $b^{-3}a^{-1}b^{-3} = b^3ab^3$, i.e., $1 = b^3ab^6ab^3 = b^3abab^3(b^5) = b^3b^2(b^5)(abab) = b^5$. Thus the relation $b^5 = 1$ holds in $W(G, t)$, so $W(G, t) \cong G$.

CONJECTURE 3.6. If G is the group of a tame knot in the 3-sphere with meridian t and longitude λ, then $(G/\lambda, t)$ does not have a Wirtinger presentation.

Considerable progress has made (see e.g., "Property R" in [10]) on the conjecture that G/λ cannot be a high dimensional knot group of $S^n \subset S^{n+2}$, i.e., that $H_2(G/\lambda)$ must always be nontrivial, and on the special case $G/\lambda \neq \boldsymbol{Z}$. If G is the group of a knot for which the conjecture $H_2(G/\lambda) \neq 0$ has been verified (e.g., fibered knots, other knots with nontrivial Alexander polynomials) then we know that the second Wirtinger approximation $\varphi_2\colon C \to G/\lambda$ has nontrivial kernel (in fact, $\boldsymbol{Z}$). To show $(G/\lambda, t)$ does not have a Wirtinger presentation, we need to know that $[\varphi_2^{-1}(t), \varphi_2^{-1}(Z_t \cap G/\lambda)']$ is strictly smaller than $\ker \varphi_2$. While this seems likely, there are *very* few cases in which we can actually verify this. The following example has also been noted by Maeda [11].

EXAMPLE 3.7. Let G be the trefoil knot group $\langle a, b, t; t^{-1}at = b, t^{-1}bt = a^{-1}b\rangle$, $\lambda = ab^{-1}a^{-1}b$. In G/λ $(=G/G'')$, $Z_t \cap (G/\lambda)' = \{1\}$. Thus $W(G/\lambda, t)$ is just the second approximation C, and the obstruction, $\ker \varphi$, is $\boldsymbol{Z}$.

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Note. C. McGordon recently has obtained Wirtinger groups G with $H_2(G)$ infinite.

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