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COMPARISON AND OSCILLATION CRITERIA FOR SELFADJOINT VECTOR-MATRIX DIFFERENTIAL EQUATIONS

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Let

$$L(y) = \sum_{k=0}^{n} (-1)^k (P_k(x) y^{(k)}(x))^{(k)}$$

where the coefficients are real, continuous, $m \times m$, symmetric matrices, $y(x)$ is an m-dimensional vector-valued function, and $P_n(x)$ is positive definite for all $x \in I$. We consider both the case for which the singularity is at ∞, $I = [1, \infty)$, and the case for which the singularity is at 0, $I = (0, 1]$.

The main theorem is a comparison result in which the equation $L(y) = 0$ is compared with an associated scalar equation. Then, general theorems for the oscillation and nonoscillation of $L(y) = 0$ are presented which can be used when the comparison theorem does not apply. Some of the proofs indicate how scalar oscillation and nonoscillation criteria can be extended to the vector-matrix case when the associated scalar theorem has been proved using the quadratic functional criteria for oscillation. In general, proofs using the associated Riccati equation and other familiar methods do not extend as easily.

1. Introduction.

Most of the theorems contained herein appear in the recent Ph. D. dissertation of Wright [25].

For general treatments of the oscillation of $L(y) = 0$, the reader is referred to the lecture notes of Coppel [3] and Kreith [14], the book of Reid [20], the paper of Etgen and Lewis [5], and the references contained therein.

Matrix notation will be used throughout this paper. For example, if A is a matrix with elements a_{ij}, $\int A$ will be the matrix with elements $\int a_{ij}$. Differentiation is defined similarly. $A \geq B$ is valid if, and only if, $A - B$ is positive semidefinite. The letter I will denote the identity matrix. By $\|A\|$, we shall mean the operator norm of the matrix A which is induced by the Euclidean vector norm, i.e., $\|A\| = \sup \|A\xi\|$ where the supremum is taken over all vectors ξ of norm 1. The notation A^* shall denote the conjugate-transpose of the matrix A and for vectors ξ_1 and ξ_2, $(\xi_1, \xi_2) = (\xi_2^* \cdot \xi_1)^{1/2}$ is the inner product.

If there exists a number $b > a$ such that $L(y) = 0$ has a nontrivial solution satisfying

$$(1) \qquad y^{(i)}(a) = 0 = y^{(i)}(b) , \quad (0 \leqq i \leqq n-1) ,$$

then b is a *conjugate point* of a and the least such b is denoted by $\eta(a)$. The equation $L(y) = 0$ is *oscillatory on an interval* $[c, d]$ if there exists a nontrivial y satisfying (1) for numbers a and b in $[c, d]$. Otherwise, $L(y) = 0$ is *nonoscillatory* on $[c, d]$. The equation $L(y) = 0$ is *oscillatory at zero* if for each $\delta > 0$ there exists an interval $[a, b] \subset (0, \delta)$ such that $L(y) = 0$ is oscillatory on $[a, b]$. The equation $L(y) = 0$ is *oscillatory at infinity* if for each N there exists an interval $[a, b] \subset (N, \infty)$ such that the equation is oscillatory on $[a, b]$.

Let $\mathscr{A}_m^n(a, b)$ denote the set of all m-dimensional vector-valued functions, $y(x)$, that have compact support in [a, b], where the first n-1 derivatives are absolutely continuous, and the derivative of order n is essentially bounded. The set $\mathscr{A}_m^n(a, b)$ is called the set of *admissible functions*. For $y \in \mathscr{A}_m^n(a, b)$, define the *quadratic functional*

$$I(y) = \int_a^b \sum_{k=0}^{n} (P_k(x) y^{(k)}(x), y^{(k)}(x))\, dx .$$

We remind the reader that for $n > 1$ and $m > 1$ other definitions of oscillation are sometimes used. One of the primary motivations for the study of oscillation in terms of conjugate points, as above, is the connection with the spectrum of differential operators generated by L (c.f. Glazman [8, pp. 35, 95–106]). The next theorem is well-known and it is central to our study. It connects oscillation theory, the calculus of variations, and spectral theory of differential operators. In this regard, we refer the reader to the books of Reid [20] and Gelfand and Fomin [8, Ch. 5], in addition to the above reference.

THEOREM 1. *The following statements are equivalent*:
(i) *The equation* $L(y) = 0$ *is nonoscillatory on* $[a, b]$.
(ii) *If* $y \in \mathscr{A}_m^n(a, b)$ *and* $y \not\equiv 0$, *then* $I(y) > 0$.

A paper of Etgen and Pawlowski [6] introduced the use of positive functionals to establish oscillation criteria for second-order matrix differential equations. For $n = 1$, the oscillation of matrix differental equations at ∞ is equivalent to the oscillation of $L(y) = 0$ at ∞ [5, p. 254]. Subsequent uses of positive functionals in this regard can be found in [5] and [7].

A linear mapping, g, from the Banach space $\mathscr{B}_m = \{A \mid A$ is an $m \times m$ complex-valued matrix$\}$ into the set of complex numbers is said to be a *positive functional* if $g(A^*A) \geqq 0$ for all $A \in \mathscr{B}_m$ or equivalently, $g(A) \geqq 0$ whenever $A = A^*$ and $A \geqq 0$. We say that

g is *nontrivial* if $g(A) \neq 0$ for some $A \in \mathscr{B}_n$. For a more general setting the reader is referred to Rickart [21].

The following characterization, which was communicated to us by Professor Alan Hopenwasser, is known in a more general context [22, pp. 45-48].

THEOREM 2. *For every nontrivial positive linear functional g, there exist nonzero vectors $v_1, \cdots, v_s$, $s \leqq m$, such that*

$$g(A) = \sum_{i=1}^{s} (Av_i, v_i) \tag{2}$$

for every $A \in \mathscr{B}_m$.

2. A comparison theorem. We now use the characterization of Theorem 2 and the fundamental criterion for oscillation given in Theorem 1 to establish the following comparison theorem.

THEOREM 3. *The oscillation of*

$$\sum_{k=0}^{n} (-1)^k (g(P_k(x))u^{(k)}(x))^{(k)} = 0 \tag{3}$$

on $[a, b]$, for some nontrivial positive functional g, implies the oscillation of $L(y) = 0$ on $[a, b]$.

Proof. If (3) is oscillatory on $[a, b]$, then by Theorem 1 there is an admissible function $f \in \mathscr{A}_1^n(a, b)$ such that $f \not\equiv 0$ on $[a, b]$ and

$$\sum_{k=0}^{n} \int_a^b g(P_k(x))(f^{(k)}(x))^2 \, dx \leqq 0 .$$

Using the characterization of g given by Theorem 2 this inequality becomes

$$\sum_{i=1}^{s} \sum_{k=0}^{n} \int_a^b (P_k(x)v_i, v_i)(f^{(k)}(x))^2 \, dx \leqq 0$$

which implies that for some $j \in \{1, 2, \cdots, s\}$

$$\sum_{k=0}^{n} \int_a^b (P_k(x)v_j, v_j))f^{(k)}(x))^2 \, dx \leqq 0 .$$

If we let $\xi = v_j$, then the scalar equation

$$\sum_{k=0}^{n} (-1)^k ((P_k(x)\xi, \xi)u^{(k)}(x))^{(k)} = 0 \tag{4}$$

must be oscillatory on $[a, b]$ by Theorem 1. The conclusion follows from Theorem 1 since $f(x) \cdot v_j$ is an element of $\mathscr{A}_m^n(a, b)$. The proof is complete.

If $\eta_1(a)$, $\eta_2(a)$, and $\eta_3(a)$ are the least conjugate points of a with respect to equations $L(y) = 0$, (4), and (3), then Theorem 3 shows that $\eta_1(a) \leqq \eta_2(a) \leqq \eta_3(a)$.

COROLLARY 3.1. *If there is a positive functional g such that* (3) *is oscillatory at* ∞ *(at* 0), *then* $L(y) = 0$ *is oscillatory at* ∞ *(at* 0).

Proof. The proof follows immediately from the definitions and Theorem 3.

In the second-order case, $n = 1$, results of [5, Theorem 4.3 and p. 254] and [7, Theorem 3.2] show that (3) being oscillatory at ∞ implies that $L(y) = 0$ is oscillatory at ∞. The proof of Theorem 3 is not only valid for arbitrary n but it is considerably simpler than the proofs of the associated theorems of [5] and [7] cited above. However, the results of [5] and [7] do apply in a more general infinite-dimensional setting.

The next corollary shows that many of the results in the literature concerning the oscillation of $L(y) = 0$ are corollaries of the fact that the oscillation of (4) for some $\xi \neq 0$ implies the oscillation of $L(y) = 0$ [17, Theorem 3.2].

COROLLARY 3.2. *If the equation* (3) *is oscillatory at* ∞ *(at* 0), *then there is a nonzero constant vector* ξ, *among the vectors* v_i, $i = 1, \cdots, s$, *of* (2), *such that* (4) *is oscillatory at* ∞ *(at* 0).

Proof. If (3) is oscillatory at ∞ then there is a sequence of intervals $\{[a_j, b_j]\}_j^\infty = 1$ with $\lim_{j\to\infty} a_j = \infty$ such that (3) is oscillatory on $[a_j, b_j]$ for every j. By Theorem 3, for each j there is some $i \in \{1, 2, \cdots, s\}$ such that (4) is oscillatory on $[a_j, b_j]$ for $\xi = v_i$. This implies that there is some $i \in \{1, 2, \cdots, s\}$ such that for $\xi = v_i$, (4) is oscillatory on each interval in a subsequence of the above sequence of intervals whose end points also diverge to ∞. This implies that (4) is oscillatory at ∞ for the particular choice of ξ.

The proof of the case at 0 is similar.

Let $p_{ii}^k(x)$ denote the diagonal element of $P_k(x)$ in row i and column i. Let "tr" denote the "trace" functional. The oscillation at ∞ (at 0) of

$$\sum_{k=0}^{n} (-1)^k (\operatorname{tr}(P_k(x)) u^{(k)}(x))^{(k)} = 0 \tag{5}$$

implies, by Corollary 3.2, the oscillation at ∞ (at 0) of

$$\sum_{k=0}^{n} (-1)^k (p_{ii}^k(x) u^{(k)}(x))^{(k)} = 0 \tag{6}$$

for some i, $1 \leqq i \leqq m$. The oscillation of (6) at ∞ (at 0) implies the oscillation of $L(y) = 0$ at ∞ (at 0) by Theorem 3. Hence, as an application of Corollary 3.2, we have that *the oscillation of* (5) *at* ∞ (*at* 0) *implies the oscillation of* $L(y) = 0$ *at* ∞ (*at* 0).

Next, we give some examples to show how known oscillation criteria for scalar equations can be used with Theorem 3 in order to establish criteria for the oscillation of $L(y) = 0$.

EXAMPLE 1. If there is a positive functional g such that

$$\int_1^\infty g(P(x))^{-1}\, dx = -\int_1^\infty x^{2(n-1)} g(Q(x))\, dx = \infty$$

then

$$(-1)^n (P(x) y^{(n)}(x))^{(n)} + Q(x) y(x) = 0 \tag{7}$$

is oscillatory at ∞.

Proof. This follows by using the corresponding scalar criterion recently established by Müller-Pfeiffer [18].

EXAMPLE 2. If there is a positive functional g such that

$$\int_1^\infty x^\alpha g(Q(x))\, dx = -\infty$$

for some $\alpha < 2n - 1$, then

$$(-1)^n y^{(2n)} + Q(x) y(x) = 0 \tag{8}$$

is oscillatory at ∞.

Proof. Apply the result of Lewis [16, corollary to Theorem 2].

Finally, we give a generalization of the well-known oscillation criterion of Glazman and Hille using the scalar result established in [9, p. 100].

EXAMPLE 3. If $g(Q(x)) \leqq 0$ for large x and

$$\limsup_{x\to\infty} x^{2n-1} \int_x^\infty |g(Q(x))|\, dx > A_n^2 ,$$

where

$$A_n^{-1} = (\sqrt{2n-1}/(n-1)!) \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \Big\backslash (2n-k) ,$$

then equation (8) is oscillatory at ∞.

It can be shown, [10], that

$$A_n^2 = [(2n-1)(2n-2)\cdots(n)]^2/(2n-1) .$$

Using these examples it is not difficult to see how to extend to $L(y) = 0$ the oscillation criteria of Allegretto and Erbe [2], Etgen [4], Hinton [10], Hinton and Lewis [11], Howard [13], Noussair and Swanson [19], Swanson [23], and Tomastik [24].

3. Oscillation and nonoscillation at infinity. We present here theorems which require conditions on the minimum (or maximum) eigenvalues of the coefficient matrices of $L(y) = 0$ of the associated two-term equation $K(y) = (-1)^n(P(x)y^{(n)}(x))^{(n)} + Q(x)y(x) = 0$. These results cannot be obtained by applying Theorem 3.

When $m = 1$, the next theorem is known [18] and we state it without proof. It follows by adopting the proof in the scalar case [25].

THEOREM 4. *Suppose $P_i(x) \leqq 0$ for each $i = 0, 1, \cdots, n-1$. If*

$$\int_1^\infty ||P_n(x)||^{-1} dx = \infty \tag{9}$$

and

$$\lim_{x\to\infty} \mu \int_1^x x^{2(n-1-i)} P_i(x)\, dx = -\infty \tag{10}$$

for some $i = 0, 1, \cdots, n-1$, then $L(y) = 0$ is oscillatory at ∞.

In the scalar case ($m = 1$), the original theorem of Müller-Pfeiffer [18] does not require the sign restriction $P_0(x) \leqq 0$, when (9) holds and (10) is true for $i = 0$.

Define

$$f(\alpha, n) = [(2n-1)/(2n-1+|\alpha|)]^{2n-1}[|\alpha|/(2n-1+|\alpha|)]^\alpha$$

for $\alpha < 0$ and $f(\alpha, n) \equiv 1$ for $\alpha \geqq 0$. Note that for each fixed n

$$\lim_{\alpha\to 0} f(\alpha, n) = 1 .$$

Let

$$B(\alpha, n) = f(\alpha, n)A_n^2$$

for A_n^2 defined in Example 3.

THEOREM 5. *Suppose $P(x) = x^{\alpha} I$ for some $\alpha < 2n-1$, $Q(x) \leqq 0$, and the integral $\int_1^{\infty} Q(x)\,dx$ exists. If*

$$\limsup_{x\to\infty} x^{2n-1-\alpha} \left\| \int_x^{\infty} Q(s)\,ds \right\| > B(\alpha, n)$$

then $K(y) = 0$ is oscillatory at ∞.

Proof. When $m = 1$ (the scalar case) and $0 \leqq \alpha < 2n-1$ this theorem is proved in [15, Theorem 3.1]. The proof for general m follows readily. (In Theorem 3.1 of [15], the author erroneously claims that for $\alpha < 0$ the proof follows from a theorem of Glazman.) Consequently, we consider here only the case when $\alpha < 0$.

Let $p(x)$ be a polynomial satisfying $p^{(0)} = p^{(k)}(0) = p^{(k)}(1) = 0$ for $k = 1, 2, \cdots, n-1$, and $p(1) = 1$. For $\mu \in (0, 1)$ and $1 < \rho < R$, define

$$\Phi(x) = \begin{cases} p\left(\dfrac{x - \mu\rho}{\rho(1-\mu)}\right), & x \in [\mu\rho, \rho) \\ 1 & x \in [\rho, R) \\ p\left(\dfrac{2R - x}{R}\right), & x \in [R, 2R] \end{cases}$$

and $\Phi(x) \equiv 0$ for $x \notin [\mu\rho, 2R]$.

Let ξ be a constant unit vector and define $y(x) = \Phi(x)\cdot\xi$. Calculations show that

$$\begin{aligned} \int_{\mu\rho}^{2R} (x^{\alpha} y^{(n)}(x), y^{(n)}(x))dx &= [\rho(1-\mu)]^{-2n} \int_{\mu\rho}^{\rho} x^{\alpha}\left(p^{(n)}\left(\frac{x-\mu\rho}{\rho(1-\mu)}\right)\right)^2 dx \\ &\quad + R^{-2n} \int_R^{2R} x^{\alpha}\left(p^{(n)}\left(\frac{2R-x}{R}\right)\right)^2 dx \\ &\leqq \frac{A_n^2}{\rho^{2n-1-\alpha}}\left[\frac{\mu^{\alpha}}{(1-\mu)^{2n-1}} + (\rho/R)^{2n-1-\alpha}\right]. \end{aligned}$$

The function of μ, $\mu^{\alpha}/(1-\mu)^{2n-1}$, assumes its minimum,

$$\left(\frac{2n-1}{2n-1+|\alpha|}\right)^{2n-1}\left(\frac{|\alpha|}{2n-1+|\alpha|}\right)^{\alpha},$$

on $(0, 1)$ at $\mu = |\alpha|/(2n-1+|\alpha|)$. With this substitution for μ, we have the inequality

$$\begin{aligned} (11) \qquad & \int_{\mu\rho}^{2R} (x^{\alpha} y^{(n)}(x), y^{(n)}(x))\,dx \\ &\leqq \frac{A_n^2}{\rho^{2n-1-\alpha}}\left[\left(\frac{2n-1}{2n-1+|\alpha|}\right)^{2n-1}\left(\frac{|\alpha|}{2n-1+|\alpha|}\right)^{\alpha} + \left(\frac{\rho}{R}\right)^{2n-1-\alpha}\right]. \end{aligned}$$

The limit in the hypothesis is equivalent to the limit obtained by replacing the norm with the absolute value of the minimum eigenvalue.

Let $\langle\rho_k\rangle \to \infty$ as $k \to \infty$ be a sequence satisfying

$$\lim_{k\to\infty} \rho_k^{2n-1-\alpha}\mu \int_{\rho_k}^{\infty} Q(s)\, ds = -B(\alpha, n) - \sigma$$

for some $\sigma > 0$, and choose N such that for $\rho = \rho_N$

$$\rho^{2n-1-\alpha}\mu \int_{\rho}^{\infty} Q(s)\, ds \geqq -B(\alpha, n) - \sigma/2 .$$

Choose R so large that

$$\rho^{2n-1-\alpha}\mu \int_{\rho}^{R} Q(s)\, ds \geqq -B(\alpha, n) - \sigma/4$$

and $(\rho/R)^{2n-1-\alpha} \leqq \sigma/4$.

If ξ is chosen to be an eigenvector corresponding to

$$\mu \int_{\rho}^{R} Q(s)\, ds$$

then by (11)

$$\int_{\mu\rho}^{2R} [(x^\alpha y^{(n)}(x), y^{(n)}(x)) - (Q(x)y(x), y(x))]\, dx \leqq 0 ,$$

and the proof is complete.

In contrast, we state the next theorem.

THEOREM 6. *Let $P(x) = x^\alpha I$ for some constant $\alpha \notin \{1, 3, \cdots, 2n-1\}$ and which satisfies $\alpha < 2n - 1$. If*

$$\int_1^\infty Q(s)\, ds$$

exists and

$$\limsup_{x\to\infty} x^{2n-1-\alpha} \left\| \int_x^\infty Q(s)\, ds \right\| \leqq [(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)]^2/(4^n|\alpha-2n+1|)$$

then $K(y) = 0$ is nonoscillatory at ∞.

The proof follows by adapting the proof of its scalar version [12, Theorem 2.2].

In [12], a criterion is established which applies when $\alpha \geqq 2n - 1$.

Added in Proof. The authors wish to bring to the readers attention a related paper by Philip Hartman, "Oscillation criteria for self-adjoint second-order differential systems and "Principal Sectional Curvatures," J. Differential Equations, 34 (1979), 326-338.

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