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ON LINEAR FORMS AND DIOPHANTINE APPROXIMATION

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Let $\vec{x}$ be a vector in $\boldsymbol{R}^K$ and let $\Lambda_j(\vec{x})$, $j=1,2,\cdots,J$ be J linear forms in K variables. We prove that there is a lattice point $\vec{u}$ in $\boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, for which $|\Lambda_j(\vec{u})|$ are all small (or zero) and the components of $\vec{u}$ are not too large. The bounds that we obtain improve several previous results on this problem.

1. Introduction. Let $\Lambda_1(\vec{x}), \Lambda_2(\vec{x}), \cdots, \Lambda_J(\vec{x})$ be J linear forms in K real variables $x_1, x_2, \cdots, x_K$. We assume that $B=(b_{jk})$ is a $J\times K$ matrix with complex entries such that

$$\Lambda_j(\vec{x}) = \sum_{k=1}^{K} b_{jk}x_k$$

for $j=1,2,\cdots,J$ and so $\vec{x}$ denotes the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_K \end{pmatrix}$. A basic problem in Diophantine approximation is to show that there exists a vector $\vec{u}=\begin{pmatrix} u_1 \\ \cdots \\ u_K \end{pmatrix}$ in the integer lattice $\boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, such that each $|\Lambda_j(\vec{u})|$ is small while the components $|u_k|$ are not too large. Quantitative results on this problem are known with various hypotheses on the Λ_j's; the usual method of proof involves an application of the pigeonhole principle (Baker [1], Lemma 1, p. 13, Gel'fond [3], Lemma 1, p. 11, Mordell [7], Theorem 3, p. 32, Siegel [8], Stolarsky [9], Chapter 2). In the present paper we make improvements on previous results of this kind by using a generalization of Minkowski's linear forms theorem which we established in [10].

In order to state our main theorem we make the following assumptions. We suppose that the forms Λ_j are real for $j=1,2,\cdots,p$ and that the remaining forms consist of q pairs of complex conjugate forms arranged so that $\Lambda_{p+2j-1}=\bar{\Lambda}_{p+2j}$ for $j=1,2,\cdots,q$. Thus $J=p+2q$. We also suppose that $\alpha_k \geqq 1$ for $k=1,2,\cdots,K$, $\beta_j>0$ for $j=1,2,\cdots,J$, and $\beta_{p+2j-1}=\beta_{p+2j}$ for $j=1,2,\cdots,q$.

THEOREM 1. *Let M be a positive integer and suppose that*

$$M^2\left\{\prod_{l=1}^{K}\alpha_l^{-2}\right\}\left\{\prod_{j=1}^{J}\left(1+\beta_j^{-2}\sum_{k=1}^{K}\alpha_k^2|b_{jk}|^2\right)\right\}\leqq 1\,. \tag{1.1}$$

Then there exist M distinct pairs of nonzero lattice points $\pm\vec{v}_m=$

$\pm\begin{pmatrix} v_{1m} \\ \cdots \\ v_{Km} \end{pmatrix}$, $m = 1, 2, \cdots, M$, *in* $\boldsymbol{Z}^K$ *each of which satisfies the following conditions*:

$$|\Lambda_j(\pm\vec{v}_m)| \leqq \beta_j\ , \qquad j = 1, 2, \cdots, p\ ,$$

$$|\Lambda_j(\pm\vec{v}_m)| \leqq \left(\frac{2}{\pi}\right)^{1/2}\beta_j\ , \qquad j = p+1, p+2, \cdots, J\ ,$$

$$|v_{km}| \leqq \alpha_k\ , \qquad k = 1, 2, \cdots, K\ .$$

Next we deduce several corollaries to Theorem 1 which are easier to use in applications. For simplicity these results are stated for the case $M = 1$.

COROLLARY 2. *Suppose that* $1 \leqq J < K$ *and that the coefficients* b_{jk} *satisfy* $|b_{jk}| \leqq T$ *for some positive* T. *Then for each* β, $0 < \beta \leqq T$, *there exists a lattice point* $\vec{u} = \begin{pmatrix} u_1 \\ \cdots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, *in* $\boldsymbol{Z}^K$ *such that*

$$|\Lambda_j(\vec{u})| \leqq \beta\ , \qquad j = 1, 2, \cdots, p\ ,$$

$$|\Lambda_j(\vec{u})| \leqq \left(\frac{2}{\pi}\right)^{1/2}\beta\ , \qquad j = p+1,\ p+2, \cdots, J\ ,$$

and

$$|u_k| \leqq (\beta^{-1}T\sqrt{K+1})^{J/(K-J)}\ , \qquad k = 1, 2, \cdots, K\ . \tag{1.2}$$

Proof. We apply Theorem 1 with $M = 1$, $\alpha_k = \alpha \geqq 1$, and $\beta_j = \beta \leqq T$. Then the left hand side of (1.1) is

$$\begin{aligned} \alpha^{-2K}\prod_{j=1}^{J}\left(1 + \beta^{-2}\alpha^2\sum_{k=1}^{K}|b_{jk}|^2\right) &\leqq \alpha^{2J-2K}(\alpha^{-2} + \beta^{-2}T^2K)^J \\ &\leqq \alpha^{2J-2K}(\beta^{-2}T^2(K+1))^J\ . \end{aligned} \tag{1.3}$$

If we choose

$$\alpha = (\beta^{-1}T\sqrt{K+1})^{J(/K-J)}$$

then $\alpha \geqq 1$ and the expression on the right of (1.3) is equal to 1. Hence the corollary follows from the theorem.

We note that in previous versions of Corollary 2 (see Gel'fond [3]) the bound on $|u_k|$ was

$$|u_k| \leqq 2(\beta^{-1}TK)^{J/(K-J)}\ .$$

However, in the special case $J = 1$ a bound similar to (1.2) was obtained by Mahler [6].

If the coefficients b_{jk} are integers we obtain an improvement in "Siegel's lemma" (Baker [1], Siegel [8], Stolarsky [9]).

COROLLARY 3. *Suppose that* $1 \leqq J < K$ *and that the coefficients* b_{jk} *are integers satisfying* $|b_{jk}| \leqq T$ *for some* $T \geqq 1$. *Then there exists a lattice point* $\vec{u} = \begin{pmatrix} u_1 \\ \cdots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, *in* $\boldsymbol{Z}^K$ *such that*

$$\Lambda_j(\vec{u}) = 0\,, \qquad j = 1, 2, \cdots, J\,, \tag{1.4}$$

and

$$|u_k| \leqq (T\sqrt{K+1})^{J/(K-J)}\,, \qquad k = 1, 2, \cdots, K\,.$$

Proof. We apply Corollary 1 with $0 < \beta < 1$, $p = J$ and $q = 0$. Since $\Lambda_j(\vec{u})$ is an integer whenever $\vec{u} \in \boldsymbol{Z}^K$ it follows that there exists $\vec{u} \in \boldsymbol{Z}^K$, $\vec{u} \neq \vec{0}$, such that (1.4) holds and

$$|u_k| \leqq (\beta^{-1}T\sqrt{K+1})^{J/(K-J)}\,, \qquad k = 1, 2, \cdots, K\,. \tag{1.5}$$

Now among the finitely many lattice points $\vec{u} \in \boldsymbol{Z}^K$, $\vec{u} \neq \vec{0}$, which satisfy (1.4) and (1.5) with $\beta = 1/2$ there must be at least one which satisfies (1.4) and (1.5) for values of β arbitrarily close to 1. Thus we may take $\beta = 1$ on the right of (1.5) for some $\vec{u} \in \boldsymbol{Z}^K$, $\vec{u} \neq \vec{0}$.

COROLLARY 4. *Suppose that* $1 \leqq J < K$ *and that* $H_1, H_2, \cdots, H_K$ *are positive integers. Then there exists a lattice point* $\vec{u} = \begin{pmatrix} u_1 \\ \cdots \\ u_K \end{pmatrix}$, $\vec{u} \neq \vec{0}$, *in such that*

$$|u_k| \leqq H_k\,, \qquad k = 1, 2, \cdots, K\,,$$

$$|\Lambda_j(\vec{u})| \leqq \frac{2\left(\sum_{k=1}^{K} H_k^2 |b_{jk}|^2\right)^{1/2}}{\left(\prod_{k=1}^{K} H_k\right)^{1/J}}\,, \qquad j = 1, 2, \cdots, p\,,$$

$$|\Lambda_j(\vec{u})| \leqq \frac{2\left(\frac{2}{\pi}\right)^{1/2}\left(\sum_{k=1}^{K} H_k^2 |b_{jk}|^2\right)^{1/2}}{\left(\prod_{k=1}^{K} H_k\right)^{1/J}}\,, \qquad j = p+1,\ p+2, \cdots, J\,.$$

Proof. Let $0 < \theta < 1$. We apply Theorem 1 with $M = 1$, $\alpha_k = H_k + \theta$ and

$$\beta_j = \psi_\theta\left(\sum_{k=1}^{K} \alpha_k^2 |b_{jk}|^2\right)^{1/2}\,,$$

where

$$\psi_\theta = \left\{\prod_{k=1}^{K}(H_k+\theta)^{2/J}-1\right\}^{-1/2}.$$

It follows that the left hand side of (1.1) is

$$\prod_{l=1}^{K}(H_l+\theta)^{-2}(1+\psi_\theta^{-2})^J = 1\,.$$

Thus there exists $\vec{u}\in \boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, such that

$$(1.6)\qquad |u_k|\leqq H_k\,,\qquad k=1,2,\cdots,K\,,$$

$$(1.7)\qquad |\Lambda_j(\vec{u})|\leqq \psi_\theta\Big(\sum_{k=1}^{K}(H_k+\theta)^2|b_{jk}|^2\Big)^{1/2}\,,\qquad j=1,2,\cdots,p\,,$$

and

$$(1.8)\qquad |\Lambda_j(\vec{u})|\leqq \Big(\frac{2}{\pi}\Big)^{1/2}\psi_\theta\Big(\sum_{k=1}^{K}(H_k+\theta)|^2b_{jk}|^2\Big)^{1/2}\,,\qquad j=p+1,\,p+2,\,\cdots,J\,.$$

Only finitely many $\vec{u}\in\boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, satisfy (1.6) and so, as in the proof of Corollary 3, at least one of these lattice points must satisfy (1.7) and (1.8) for all θ, $0<\theta<1$. Thus we may take $\theta=1$ on the right hand side of (1.7) and (1.8). Finally we observe that

$$(1.9)\qquad \Big(\sum_{k=1}^{K}(H_k+1)^2|b_{jk}|^2\Big)^{1/2}\leqq 2\Big(\sum_{k=1}^{K}H_k^2|b_{jk}|^2\Big)^{1/2}$$

and

$$(1.10)\qquad \psi_1=\Big(\prod_{k=1}^{K}H_k\Big)^{-1/J}\left\{\prod_{l=1}^{K}(1+H_l^{-1})^{2/J}-\prod_{l=1}^{K}H_l^{-2/J}\right\}^{-1/2}.$$

Since $K>J$ we have

$$(1.11)\qquad \prod_{l=1}^{K}(1+H_l^{-1})^{2/J}-\prod_{l=1}^{K}H_l^{-2/J}\geqq\prod_{l=1}^{K}(1+H_l^{-2K/J})^{1/K}-\prod_{l=1}^{K}H_l^{-2/J}$$
$$\geqq 1+\prod_{l=1}^{K}H_l^{-2/J}-\prod_{l=1}^{K}H_l^{-2/J}=1\,,$$

where we have used Theorem 27 and 10 of [5] in the first and second inequalities respectively. Putting (1.9), (1.10) and (1.11) together gives the desired result.

Our upper bound in Corollary 4 sharpens an inequality in Stolarsky [9], p. 15.

We also remark that Corollary 4 has an interesting geometrical interpretation. Let $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_J$ denote nonzero column vectors in $\boldsymbol{R}^K$ with $\vec{b}_j^T = (b_{j1}\, b_{j2} \cdots b_{jK})$. We write $\Lambda_j(\vec{x}) = \langle \vec{b}_j, \vec{x} \rangle$, $\|\vec{b}_j\| = (\sum_{k=1}^K |b_{jk}|^2)^{1/2}$ and recall that $|\langle \vec{b}_j, \vec{x}\rangle| \|\vec{b}_j\|^{-1}$ is the length of the projection of $\vec{x}$ onto the subspace spanned by the vector $\vec{b}_j$. Applying the corollary with $H_1 = H_2 = \cdots = H_K = H$ we find that there is always a nonzero lattice point $\vec{u} \in \boldsymbol{Z}^K$ with components at most H in absolute value and having a projection onto the span of each $\vec{b}_j$ of length at most $2H^{1-K/J}$.

2. Preliminary results. The remainder of our paper is devoted to a proof of Theorem 1. This is accomplished by combining the following lemmas. Here we write δ_{jk} for the Kronecker delta and B^* for the complex conjugate transpose of the matrix B.

LEMMA 5. *Let $B = (b_{jk})$ be a $J \times K$ matrix with complex entries and let $D = (d_k \delta_{jk})$ be a diagonal matrix with $d_k > 0$ for $k = 1, 2, \cdots, K$. Then*

$$\det(D + B^*B) \leqq \left(\prod_{l=1}^K d_l\right) \prod_{j=1}^J \left(1 + \sum_{k=1}^K d_k^{-1} |b_{jk}|^2\right). \tag{2.1}$$

It is possible to bound $\det(D + B^*B)$ by using Hadamard's inequality (Bellman [2], Gantmacher [4], p. 252). But the result we obtain is

$$\det(D + B^*B) \leqq \prod_{k=1}^K \left(d_k + \sum_{j=1}^J |b_{jk}|^2\right),$$

and this is generally weaker than (2.1) if $1 \leqq J < K$.

Proof of Lemma 5. Let I_K denote the $K \times K$ identity matrix. We will begin by proving that

$$\det(I_K + B^*B) \leqq \prod_{j=1}^J \left(1 + \sum_{k=1}^K |b_{jk}|^2\right). \tag{2.2}$$

If Q is a $K \times K$ unitary matrix, that is if $Q^*Q = QQ^* = I_K$, then the left and right hand sides of (2.2) are unchanged when B is replaced by BQ. Since B^*B is a positive semi-definite Hermitian matrix we may choose the unitary matrix Q so that Q^*B^*BQ is a diagonal matrix. In particular we may choose Q (see Gantmacher [4], p. 274) so that

$$Q^*B^*BQ = (BQ)^*(BQ) = (\lambda_k \delta_{jk})$$

where

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_M > 0 = \lambda_{M+1} = \lambda_{M+2} = \cdots = \lambda_K .$$

Thus $\operatorname{rank}(B) = \operatorname{rank}(B^*B) = M \leqq K$. (Of course if $\operatorname{rank}(B) = 0$ then (2.2) is trivial so we may suppose that $1 \leqq M$.) By replacing B by BQ it follows that we may assume without loss of generality that $B^*B = (\lambda_k \delta_{jk})$, or equivalently that

$$\sum_{l=1}^{J} \bar{b}_{lj} b_{lk} = \lambda_k \delta_{jk} . \tag{2.3}$$

Taking $j = k \geqq M + 1$ in (2.3) we find that $b_{jk} = 0$ if $k = M + 1, M + 2, \cdots, K$.

Next we define $w_{jk} = \lambda_k^{-1/2} b_{jk}$ so that by (2.3) the $J \times M$ matrix $W = (w_{jk})$ has M orthonormal columns (and so $M \leqq J$). It follows from Bessel's inequality that

$$\sum_{k=1}^{M} |w_{jk}|^2 \leqq 1 , \tag{2.4}$$

for $j = 1, 2, \cdots, J$. Since $I_K + B^*B = (\{1 + \lambda_k\}\delta_{jk})$ we have

$$\begin{aligned} \det(I_K + B^*B) &= \prod_{k=1}^{M} (1 + \lambda_k) = \prod_{k=1}^{M} (1 + \lambda_k)^{\sum_{j=1}^{J} |w_{jk}|^2} \\ &= \prod_{j=1}^{J} \left\{ \prod_{k=1}^{M} (1 + \lambda_k)^{|w_{jk}|^2} \right\} . \end{aligned}$$

Thus to establish (2.2) it suffices to show that

$$\prod_{k=1}^{M} (1 + \lambda_k)^{|w_{jk}|^2} \leqq 1 + \sum_{k=1}^{K} |b_{jk}|^2 \tag{2.5}$$

for each $j = 1, 2, \cdots, J$. If $\sum_{k=1}^{M} |w_{jk}|^2 = 0$ then (2.5) is trivial since the left hand side is one. If $\sum_{k=1}^{M} |w_{jk}|^2 > 0$ then by the arithmetic-geometric mean inequality (see [5], Theorem 9) we have

$$\begin{aligned} \prod_{k=1}^{M} (1 + \lambda_k)^{|w_{jk}|^2} &\leqq \left(\frac{\sum_{k=1}^{M} |w_{jk}|^2 (1 + \lambda_k)}{\sum_{k=1}^{M} |w_{jk}|^2} \right)^{\sum_{k=1}^{M} |w_{jk}|^2} = \left(1 + \frac{\sum_{k=1}^{M} |b_{jk}|^2}{\sum_{k=1}^{M} |w_{jk}|^2} \right)^{\sum_{k=1}^{M} |w_{jk}|^2} \\ &\leqq \left(1 + \sum_{k=1}^{M} |b_{jk}|^2 \right) = \left(1 + \sum_{k=1}^{K} |b_{jk}|^2 \right) . \end{aligned}$$

In the last inequality we have used (2.4) together with the observation that $(1 + (c/x))^x$ is an increasing function of x for $x > 0$ and any fixed $c \geqq 0$. This proves (2.2).

To complete the proof of the lemma we note that

$$\begin{aligned} \det(D + B^*B) &= \det(D^{1/2}) \det(I_K + D^{-1/2} B^* B D^{-1/2}) \det(D^{1/2}) \\ &= \left(\prod_{k=1}^{K} d_k \right) \det(I_K + (BD^{-1/2})^* (BD^{-1/2})) \end{aligned}$$

$$\leqq \left(\prod_{k=1}^{K} d_k\right) \prod_{j=1}^{J} \left(1 + \sum_{k=1}^{K} d_k^{-1} |b_{jk}|^2\right) .$$

Next we suppose that $L_j(\vec{x})$, $j = 1, 2, \cdots, N$ are N linear forms in K variables,

$$L_j(\vec{x}) = \sum_{k=1}^{K} a_{jk} x_k ,$$

so that $A = (a_{jk})$ is an $N \times K$ matrix. We assume that the forms L_j are real for $j = 1, 2, \cdots, r$ and that the remaining forms consist of s pairs of complex conjugate forms arranged so that $L_{r+2j-1} = \bar{L}_{r+2j}$ for $j = 1, 2, \cdots, s$. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N$ be positive with $\varepsilon_{r+2j-1} = \varepsilon_{r+2j}$ for $j = 1, 2, \cdots, s$. We define the $N \times N$ diagonal matrix E by $E = (c_j \delta_{jk})$ where $c_j = \varepsilon_j^{-1}$ if $j = 1, 2, \cdots, r$ and $c_j = (2/\pi)^{1/2} \varepsilon_j^{-1}$ if $j = r + 1, r + 2, \cdots, N$.

LEMMA 6. *Let M be a positive integer and suppose that*

$$M |\det A^* E^2 A|^{1/2} \leqq 1 .$$

Then there exist at least M distinct pairs of nonzero lattice points $\pm\vec{v}_m$, $m = 1, 2, \cdots, M$, in $\boldsymbol{Z}^K$ such that

$$|L_j(\pm\vec{v}_m)| \leqq \varepsilon_j$$

for each $j = 1, 2, \cdots, N$ and each $m = 1, 2, \cdots, M$.

For a proof of Lemma 6 we refer to [10].

3. Proof of Theorem 1. Let $N = J + K$. We apply Lemma 6 with

$$L_j(\vec{x}) = x_j , \qquad j = 1, 2, \cdots, K ,$$
$$L_{K+j}(\vec{x}) = \Lambda_j(\vec{x}) , \qquad j = 1, 2, \cdots, J .$$

Thus $r = K + p$ and $s = q$. The matrix A can then be partitioned as

$$A = \begin{pmatrix} I_K \\ B \end{pmatrix} . \tag{3.1}$$

We also let

$$\varepsilon_j = \alpha_j , \qquad j = 1, 2, \cdots, K ,$$
$$\varepsilon_{K+j} = \beta_j , \qquad j = 1, 2, \cdots, p ,$$
$$\varepsilon_{K+j} = \left(\frac{2}{\pi}\right)^{1/2} \beta_j , \qquad j = p + 1, p + 2, \cdots, J .$$

Using (3.1) it follows that

$$A^*E^2A = D + (GB)^*(GB) \tag{3.2}$$

where $D = (\alpha_k^{-2}\delta_{jk})$ is a $K \times K$ diagonal matrix and $G = (\beta_j^{-1}\delta_{jk})$ is a $J \times J$ diagonal matrix. Combining (1.1), (3.2) and Lemma 5 we find that

$$M^2 \det(AE^2A^*) \leqq 1 .$$

Thus the conclusion of Theorem 1 follows as an application of Lemma 6.

REFERENCES

1. A. Baker, *Transcendental Number Theory*, Cambridge University Press, 1975.
2. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, 1970.
3. A. O. Gel'fond, *Transcendental and Algebraic Numbers*, Dover, 1960.
4. F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, 1959.
5. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd. ed., Cambridge University Press, 1952.
6. K. Mahler, *On a problem in Diophantine approximation*, Arch. Math., **6** (1955), 208-214.
7. L. J. Mordell, *Diophantine Equations*, Academic Press, 1969.
8. C. L. Siegel, *Transcendental Numbers*, Princeton, 1949.
9. K. B. Stolarsky, *Algebraic Numbers and Diophantine Approximation*, Dekker, 1974.
10. J. D. Vaaler, *A geometric inequality with applications to linear forms*, Pacific J. Math., **83** (1979), 543-553.

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