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ENDS OF FUNDAMENTAL GROUPS IN SHAPE AND PROPER HOMOTOPY

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The number of topological ends of the universal cover of a finite complex, K, is either 0, 1, 2, or ∞ and only depends upon the fundamental group of K. Call this number $e(K)$. We wish to define numbers $e(X)$ for compact metric spaces analogous to $e(K)$. To accomplish this we extend the theory of ends for finitely generated groups to certain inverse sequences of finitely generated groups and their inverse limits. Classifications for these inverse sequences and their inverse limits analogous to those for finitely generated groups are derived. Whenever the fundamental pro-group of a compact metric space, X, satisfies certain properties, we obtain a shape invariant number $e(X)$ (either 0, 1, 2 or ∞) and analyze what $e(X)$ describes geometrically.

1. **Introduction.** The number of ends of a topological space was introduced by Freudenthal in [8]. Let X be a locally compact separable metric space. Let $\{C_i\}_{i=1}^{\infty}$ be a collection of compact subsets of X such that $C_i \subset \mathrm{int}(C_{i+1})$ (the interior of C_{i+1}) and $\bigcup_{i=1}^{\infty} C_i = X$. The cardinality of $\varprojlim \{\pi_0(X - C_i)\}$ (where the bonds are induced by inclusions) is the number of ends of X. This number is independent of the choice of the C_i.

In [10] Hopf proves:

(i) The universal cover, $\tilde{K}$, of a compact polyhedron has either 0, 1, 2, or ∞-ends.

Call this number $e(K)$.

(ii) If K_1 and K_2 are compact polyhedra and $\pi_1(K_1) = \pi_1(K_2)$ then $e(K_1) = e(K_2)$.

This paper is motivated by the desire to extend Hopf's theorem to compacta i.e., if X is a compact connected metric space we would like numbers $e(X)$ analogous to Hopf's $e(K)$. In § 5 we accomplish this for a large class of compacta, with $e(X)$ a shape invariant of X. Geometrically $e(X)$ is counting the number of ends of the universal covers of certain compact polyhedra associated with X. We also derive sufficient conditions to obtain a space, $\bar{X}$, associate with X (analogous to the universal cover of K) so that the number of ends of $\bar{X}$ is $e(X)$. This $\bar{X}$ will reduce to the universal cover whenever X is LC° and semi-locally 1-connected. With mild restrictions we show if X and Y are pointed homotopy equivalent then $\bar{X}$ and $\bar{Y}$ are pointed proper homotopy equivalent.

Let Y be a locally compact separable metric space with one end, and $\{C_i\}$ a collection of compact subsets such that $C_i \subset \text{int}(C_{i+1})$ and $\bigcup_{i=1}^{\infty} C_i = Y$. If the inverse sequence $\{\pi_1(Y - C_i)\}$ (with bonding homomorphisms induced by inclusions and proper attention to the base points) satisfies conditions described in § 3 then a theory for the number of ends of the fundamental group of the end of Y can be derived. It is a well-known conjecture that these conditions are satisfied whenever Y is the universal cover of a finite complex. We do not explore this avenue in this paper although the basic tools are implicitly evident. This is one reason for the words "and proper homotopy" in the title of the paper.

The results of Freudenthal [8] and Hopf [10] led to the following group theory:

If G is a finitely generated group there is a number, $e(G)$, called the number of ends of G such that:

(i) $e(G)$ is either 0, 1, 2, or ∞.

(ii) If K is a compact polyhedron and $\pi_1(K) \cong G$ then $e(G) = e(K)$.

(iii) $e(G) = 0$ if and only if G is finite.

(iv) $e(G) = 2$ if and only if G has an infinite cyclic subgroup finite index.

(v) (Stallings) $e(G) = \infty$ if and only if G is an amalgamated free product or HNN extension of a certain type (see § 3).

(vi) $e(G) = 1$ otherwise.

Our methods naturally lead us to generalize these results about ends of groups in two ways: to suitable inverse sequences of groups (objects in the category of pro-groups) in § 3, and to suitable topological groups in § 4. [All this is carefully explained in those sections, but a reader familiar with the shape theory of compacta will not be surprised that we approach the desired geometrical theorems about $e(X)$ through inverse sequences of groups and/or their topological inverse limits.] The geometrical theorems, described above, are in § 5.

Finally, there are several unexpected features, two of which are:

1. The compacta for which $e(X)$ is defined are pointed 1-movable (the fundamental pro-group is Mittag-Leffler (M-L)), and the bonding homomorphisms (of the fundamental pro-group) have finite kernel. Although the pointed 1-movable condition is a frequently used concept the finite kernel condition has not previously been used. We call this condition Mittag-Leffler finite (MLF).

2. 2-ended finitely generated groups can be classified as those with a *normal* infinite cyclic subgroup of finite index. In § 4 we describe a 2-ended topological group with no normal infinite cyclic subgroup of any sort.

3. The condition MLF is not an *ad hoc* condition. The class of compacta with MLF fundamental pro-groups appears to be the natural class for a theory of ends.

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2. Ends and homomorphisms with finite kernel.

All groups considered in § 2 will be finitely generated.

Let G be a finitely generated group with specified generators $\langle g_1, \cdots, g_n \rangle$. Construct a 1-complex $L(g_1, \cdots, g_n)$ with one vertex for each element of G and one edge joining verticies a and b if $ag_i = b$ for some $i \in \{1, 2, \cdots, n\}$. The number of ends of $L(g_1, \cdots, g_n)$ (see § 1 for the meaning of this) does not depend on the choice of $g_1, \cdots, g_n$ [10].

DEFINITION. The number of ends of G is the number of ends of $L(g_1, \cdots, g_n)$ in the topological sense.

Proofs of the following two theorems can be found in Stallings [14] (p. 54 and p. 38 respectively).

THEOREM 2.1. *A finitely generated group has either* 0, 1, 2 *or* ∞*-ends.*

THEOREM 2.2. *A finitely generated group has* 2*-ends if and only if it has an infinite cyclic subgroup of finite index.*

As a companion to these two theorems we state the trivial:

PROPOSITION 2.3. *A finitely generated group has* 0*-ends if and only if it is finite.*

The main result of this section is the following:

PROPOSITION 2.4. *If G and H are finitely generated groups and $f: G \to H$ is an epimorphism with finite kernel then G and H have the same number of ends.*

The proof will be done as a sequence of lemmas.

LEMMA 2.5. *If f is as in* 2.4 *then G is* 0*-ended if and only if H is* 0*-ended.*

This is trivial.

LEMMA 2.6. *Let G be 2-ended and let $f: G \to H$ be an epimorphism whose image H, is infinite then H is 2-ended and f has finite kernel.*

Proof. Let a be a generator of Z_a, an infinite cyclic subgroup of finite index in G. Let $[x_1], \cdots, [x_n]$ be the right cosets of G/Z_a. ($p \in [x_i]$ if and only if $p = a^k x_i$ for some integer k.) $f(a)$ has infinite order in H, since if not H would be finite, contrary to assumption. It suffices to observe that if b and c are in the same right coset of G/Z_a then $f(b)$ and $f(c)$ are in the same right coset of $H/Z_{f(a)}$ i.e., there are at most n right cosets in $H/Z_{f(a)}$.

To see f has finite kernel, one need only observe that f is 1-1 on the elements of a right coset $[x_i]$. I.e., for $k \neq m$ $f(a^k x_i) = (f(a))^k f(x_i) \neq (f(a))^m f(x_i) = f(a^m x_i)$ since $f(a)$ has infinite order in H.

LEMMA 2.7. *Let H be 2-ended and let $f: G \to H$ be an epimorphism with finite kernel then G is 2-ended.*

Proof. Say x generates Z_x an infinite cyclic subgroup of finite index in H. If $y \in f^{-1}(x)$ then y has infinite order in G and f maps each right coset of G/Z_y bijectively to one of G/Z_x. Thus if G/Z_y were infinite f would not be finite-to-one i.e., f would have infinite kernel, contrary to assumption.

LEMMA 2.8. *If $f: G \to H$ is an epimorphism with finite kernel then G is ∞-ended if and only if H is ∞-ended.*

Proof of "only if". Let $\{g_1, \cdots, g_t\}$ generate G. Then $\{f(g_1), \cdots, f(g_t)\}$ generates H. Let $L_1 = L(g_1, \cdots, g_t)$, and $L_2 = L(f(g_1), \cdots, f(g_t))$ f induces $f_1: L_1 \to L_2$ which is a covering projection with fiber of the same cardinality as $\ker(f)$. Hence f_1 is proper. Choose C a compact subset of L_1 such that $L_1 - C$ has n-infinite components, where $n > |\ker(f)|$. $f_1^{-1}f_1(C)$ is a compact set containing C and $L_1 - f_1^{-1}f_1(C)$ has m-infinite components with $m \geqq n$. The infinite components of $L_1 - f_1^{-1}f_1(C)$ cover the infinite components of $L_2 - f_1(C)$. But at most $|\ker(f)|$ infinite components of $L_1 - f_1^{-1}f_1(C)$ can cover any particular infinite component of $L_2 - f_1(C)$. Hence $L_2 - f_1(C)$ must have at least two infinite components, and L_2 has at least two ends. By Lemma 2.7, L_2 is not 2-ended, and therefore L_2 is ∞-ended.

Proof of "if". Choose D a compact subset of L_2 such that $L_2 - D$ has three or more infinite components. If D_1, D_2 and D_3 are such components, then $f_1^{-1}(D_i)$ for $i \in \{1, 2, 3\}$ contains an infinite component of $L_1 - f_1^{-1}(D)$ i.e., L_1 has at least three ends and therefore G is ∞-ended.

Lemma 2.5 is the 0-ended case of Proposition 2.4, Lemmas 2.6 and 2.7 the 2-ended case and Lemma 2.8 the ∞-ended case. Thus the only remaining case, the 1-ended case also holds, completing the proof of Proposition 2.4.

3. **Inverse sequences of groups.** Let $G_1 \overset{f_1}{\leftarrow} G_2 \overset{f_2}{\leftarrow} \cdots$ be an inverse sequence of groups and homomorphisms. The sequence is *Mittag-Leffler* (M-L) [9] if given $n > 0$, there is $m(n) \geqq n$ such that $\operatorname{im}(G_{m(n)+k} \to G_n) = \operatorname{im}(G_{m(n)} \to G_n)$ for all $k \geqq 0$. If every bond is onto or each G_n is finite, then clearly $\{G_n, f_n\}$ is M-L. The sequences $\{G_n, f_n\}$ and $\{H_n, k_n\}$ are *pro-isomorphic* if there exists cofinal subsequences $\{G_{m(j)}\}_{j=1}^{\infty}$ and $\{H_{m(j)}\}_{j=1}^{\infty}$ and homomorphisms p_j and q_j making diagram (A) commute for all j.

(A)
$$\begin{array}{ccc} G_{m(j)} & \longleftarrow & G_{m(j+1)} \\ {\scriptstyle p_j}\downarrow & \nwarrow{\scriptstyle q_j} & \downarrow{\scriptstyle p_{j+1}} \\ H_n(j) & \longleftarrow & H_{n(j+1)} \end{array}$$

where the horizontal homomorphisms are compositions of bonds. (This and related notions were introduced in [2].) Clearly pro-isomorphism is an equivalence relation. It is easy to see that any M-L system is pro-isomorphic to one with epimorphic bonds. We are interested in M-L inverse sequences of finitely generated groups in which the bonding homomorphisms have finite kernels. We call such sequences Mittag-Leffler-finite (M-L-F). By Proposition 2.4, if $\{G_i, f_i\}$ is an M-L-F sequence of finitely generated groups and each f_i is an epimorphism then the G_i all have the same number of ends.

PROPOSITION 3.1. *If $\{H_i, k_i\}$ and $\{G_i, f_i\}$ are pro-isomorphic* M-L-F *sequences with epimorphic bonds then all groups G_i, $i \in \{1, 2, \cdots\}$ and H_i, $i \in \{1, 2, \cdots\}$ have the same number of ends.*

Proof. By Proposition 2.4 it suffices to show the number of ends of G_i equals the number of ends of H_j for some i and j. Assume p_i and q_i are homomorphisms making diagram (B) commute for all i.

(B)
$$\begin{array}{ccc} G_{m(i)} & \overset{r_i}{\longleftarrow} & G_{m(i+1)} \\ {\scriptstyle p_i}\downarrow & \nwarrow{\scriptstyle q_i} & \downarrow{\scriptstyle p_{i+1}} \\ H_{n(i)} & \overset{s_i}{\longrightarrow} & H_{n(i+1)} \end{array}$$

r_i and s_i are the appropriate compositions of bonds, and thus are epimorphisms with finite kernels. $\ker(p_i) \subset \ker(q_{i-1}p_i) = \ker(r_{i-1})$, and $\ker(p_i)$ is finite. p_i is an epimorphism since $p_i \circ q_i = s_i$ is. By Proposition 2.4 $G_{m(i)}$ and $H_{n(i)}$ have the same number of ends.

DEFINITION. The *number of ends* of the M-L-F sequence $\{G_i, f_i\}$ of finitely generated groups is the number of ends of the G'_i where $\{G'_i, f'_i\}$ is any M-L-F sequence pro-isomorphic to $\{G_i, f_i\}$ and each f'_i is an epimorphism. This definition is unambiguous by Proposition 3.1.

PROPOSITION 3.2. *If* $\{G_i, f_i\}$ *is* M-L-F *and each* f_i *is an epimorphism then* G_1 *finite implies* $\{G_i, f_i\}$ *is* 0-*ended, and* G_1 *having an infinite cyclic subgroup of finite index implies* $\{G_i, f_i\}$ *is* 2-*ended.*

Proof. The first part is trivial and the second follows from Theorem 2.2.

Let G be a group, A and B subgroups of G and let θ: $A \to B$ be an isomorphism. The HNN *extension* of G relative to A, B and θ is the group $G \hookleftarrow_A \theta = \langle G, x \mid x^{-1}ax = \theta(a), a \in A\rangle$, i.e., the generators and relations of (some presentation of) G, together with an additional generator, x, and additional relations $x^{-1}ax = \theta(a)$. (See [12] p. 179.) An alternative description of HNN extensions can be found in [14] (3.A.5.5) in terms of pre-groups. We avoid pre-groups in this paper.

By [14] (p. 57) the finitely generated group G has an infinite number of ends if and only if either

(1) $G = G_1 *_F G_2$, a free product with finite amalgamated subgroup F, properly contained in both factors, and of index > 2 in at least one factor, or

(2) $G = H \hookleftarrow_F \theta$ an HNN extension, where F is a finite subgroup properly embedded in H. Thus,

PROPOSITION 3.3. *If* $\{G_i, f_i\}$ *is* M-L-F, *each* f_i *is an epimorphism and* G_1 *is a free product with finite amalgamated subgroup as in* (1) *or an* HNN *extension as in* (2) *then* $\{G_i, f_i\}$ *is* ∞-*ended.*

PROPOSITION 3.4. *If* $\{G_i, f_i\}$ *is* 0-*ended and each* f_i *is an epimorphism then all* G_i *are finite.*

This is trivial.

PROPOSITION 3.5. *If* $\{G_i, f_i\}$ *is 2-ended and each* f_i *is an epimorphism then there is* $x_n \in G_n$ *for each* n *such that* $f_n(x_{n+1}) = x_n$ *and* x_n *generates an infinite cyclic subgroup of finite index in* G_n.

Proof. In a group with infinite cyclic subgroup of finite index any element of infinite order generates an infinite cyclic subgroup of finite index. By Theorem 2.2, each G_n has an infinite cyclic subgroup of finite index thus if x_1 has infinite order in G_1 and x_n is selected such that $f_{n-1}(x_n) = x_{n-1}$ for each $n \geqq 2$ then x_n has infinite order in G_n and generates in infinite cyclic subgroup of finite index in G_n.

Finitely generated groups with 2-ends have *normal* infinite cyclic subgroups of finite index. The following example shows that in a 2-ended M-L-F sequence with epimorphic bounds it may not be possible to choose compatible *normal* infinite cyclic subgroups i.e., there need not exist $x_n \in G_n$ such that x_n generates a *normal* infinite cyclic subgroup in G_n and $f_n(x_{n+1}) = x_n$ for all n.

EXAMPLE 3.6. Denote the permutation group on n symbols as S_n. The following are easily checked in S_{2^n}.

(i) $[1, 2, \cdots, 2^n][a, a+1] = [1, 2, \cdots, a, a+2, \cdots, 2^n] = [a+1, a+2][1, 2, \cdots, 2^n]$ for $a \in \{1, 2, \cdots, 2^n - 2\}$

(ii) $[1, 2, \cdots, 2^n][2^n, 1] = [2, 3, \cdots, 2^n] = [1, 2][1, 2, \cdots, 2^n]$

(iii) $[1, 2, \cdots, 2^n][2^n - 1, 2^n] = [1, 2, \cdots, 2^n - 1] = [2^n, 1][1, 2, \cdots, 2^n]$

(iv) $[1, 2, \cdots, 2^n]^2[a, a+1] = [a+2, a+3][1, 2, \cdots, 2^n]^2$ for $a \in \{1, 2, \cdots, 2^{n-3}\}$

(v) $[1, 2, \cdots, 2^n]^2[2^n - 1, 2^n] = [1, 2][1, 2, \cdots, 2^n]^2$.

LEMMA 3.6.1. *If* x *generates* Z_x, *a normal infinite cyclic subgroup of* G, *then* $gxg^{-1} = x$ *or* x^{-1} *for any* $g \in G$.

Proof. Say $gxg^{-1} = x^n$ and $g^{-1}xg = x^m$ for some integers m and n. Then $x = g^{-1}gxg^{-1}g = g^{-1}x^n g = (x^n)^m = x^{nm}$. Thus $n \cdot m = 1$ and $n = m = 1$ or $n = m = -1$.

Now we define our M-L-F sequence. Let G_1 be the infinite cyclic group with generator x_1. For $n \geqq 2$, $G_n = \langle x_n; a_{1,n}; \cdots; a_{2^{n-1}_n} | x_n a_{i,n} = a_{i+1,n} x_n$ for $i \in \{1, \cdots, 2^{n-1}1\}$; $x_n a_{2^{n-1},n} = a_{1,n} x_n$; $a_{i,n}^2 = 1$ for all i; $a_{i,n} a_{j,n} = a_{j,n} a_{i,n}$ for all i and $j\rangle$. Define $f_1: G_2 \to G_1$ by $x_2 \to x_1$ and $a_{i,2} \to 1$ for $i = 1$ or 2. For $n \geqq 2$, define $f_n: G_{n+1} \to G_n$ by $x_{n+1} \to x_n$, $a_{i,n+1} \to a_{i,n}$ for $i \in \{1, 2, \cdots, 2^{n-1}\}$ and $a_{2^{n-1}+i,n+1} \to a_{i,n}$ for $i \in \{1, 2, \cdots, 2^{n-1}\}$.

(vi) Since $f_1 \circ \cdots \circ f_{n-1}(x_n) = x_1$, x_n has infinite order.

(vii) The relations of G_n easily imply any word in G_n can be written as $a_{1,n}^{t(1)} \cdots a_{2^{n-1},n}^{t(2^{n-1})} x_n^k$ where $t(i) \in \{0, 1\}$ and k is an integer.

(viii) The words of G_n that are mapped to x_1^k are exactly those of the form in (vii).

From (vii) the infinite cyclic subgroup of G_n generated by x_n has finite index in G_n, and G_n is 2-ended f_n has finite kernel by Lemma 2.6 and is an epimorphism by definition. Thus $\{G_i, f_i\}$ is M-L-F. For $n \geqq 2$, define $g_n: G_n \to S_{2^n}$ by $x_n \to [1, 2, \cdots, 2^n]^2$ and $a_{i,n} \to [2i-1, 2i]$ for $i \in \{1, 2, \cdots, 2^{n-1}\}$. By (iv) and (v) g_n is a homomorphism. In particular,

(ix) $a_{i,n} \neq a_{j,n}$ for $i \neq j$.

LEMMA 3.6.2. *If G_n has a normal infinite cyclic subgroup, then a generator of it must have the form $a_{1,n}^{t(n)} \cdots a_{2n-1,n}^{t(2^{n-1})} x_n^{K2^{n-1}}$ for K a nonzero integer and $t_i \in \{0, 1\}$.*

Proof. $x_n^{2^{n-1}} a_{i,n} = a_{i,n} x_n^{2^{n-1}}$ for all i, thus $x_n^{-2^{n-1}} a_{i,n} = a_{i,n} x_n^{-2^{n-1}}$ and for any integer K, $a_{i,n} x_n^{K2^{n-1}} = x_n^{K2^{n-2}} a_{i,n}$ for all i. If $L \in \{1, 2, \cdots, 2^{n-1} - 1\}$ then $x_n^L a_{1,n} = a_{1+L,n} x_n^L$. Thus if $W = a_{1,n}^{t(1)} \cdots a_{2^{n-1},n}^{t(2^{n-1})} x_n^{K2^{n-1}+L}$ for $t(i) \in \{0, 1\}$, $L \in \{1, 2, \cdots, 2^{n-1} - 1\}$ and K is an integer, it suffices to show $a_{1,n} W a_{1,n} \neq W$ or W^{-1}. By Lemma 3.6.1 $a_{1,n} W a_{1,n}$ is mapped to $x_1^{K2^{n-1}+L}$ under bonding homomorphisms, and $W^{-1} = x_n^{-K2^{n-1}-L} a_{1,n}^{t(1)} \cdots a_{2^{n-1},n}^{t(2^{n-1})}$ is mapped to $x_1^{-K2^{n-1}-L}$ i.e., $a_{1,n} W a_{1,n} \neq W^{-1}$. $a_{1,n} W a_{1,n} = a_{1,n} a_{1+L,n} W \neq W$ by (ix).

Assume b_n generates a normal infinite cyclic subgroup in G_n, and $f_n(b_{n+1}) = b_n$ for all n. Say $b_1 = x_1^m$, $m \neq 0$, then by (viii)$b_p = a_{1,p}^{t(1)} \cdots a_{2^{p-1},p}^{t(2^{p-1})} x_p^m$ for $t(i) \in \{0, 1\}$. But for $2^{p-1} > |m|$, m is not a multiple of 2^{p-1} and Lemma 3.6.2 implies b_p does not generate a normal infinite cyclic subgroup in G_p.

An alternative characterization of 2-ended groups is as follows: G is 2-ended if and only if G has a finite normal subgroup F such that G/F is infinite cyclic or is the infinite dihedral group $Z_2 * Z_2$ [15] (p. 38). This motivates the following:

LEMMA 3.7. *If $\{G_i, f_i\}$ is a 2-ended M-L-F sequence, and each f_i is an epimorphism then there is a compatible sequence of finite normal subgroups $F_i \subset G_i$ (compatible in the sense $f_n^{-1}(F_n) = F_{n+1}$) such that G_i/F_i is infinite cyclic for all i, or $Z_2 * Z_2$ for all i.*

Proof. Since G_1 is 2-ended there is F_1, a normal finite subgroup of G_1, such that G_1/F_1 is the integers, Z, or $Z_2 * Z_2$. Let $f_0: G_1 \to Z$ or $Z_2 * Z_2$ be the quotient homomorphism. $f_0 \circ f_1: G_2 \to Z$ or $Z_2 * Z_2$ is an epimorphism with kernel $f_1^{-1}(F_2)$. Thus $G_2/f_1^{-1}(F_1) = Z$ or $Z_2 * Z_2$. Inductively define $F_n = f_{n-1}^{-1}(F_{n-1})$, for $n \geqq 2$. The kernel of $f_0 \circ \cdots \circ f_{n-1}$ is F_n and since each f_i has finite kernel, F_n is finite and normal and $G_n/F_n = Z$ or $Z_2 * Z_2$ for all n.

It remains to characterize ∞-ended M-L-F sequences.

A *bipolar structure* on a group G, as defined in Stallings [14] is a partition of G into six disjoint sets, termed F, S, EE, EE^*, E^*E, E^*E^* satisfying the axioms below. We let X, Y, Z be symbols standing for the letters E or E^*, and if $X = E$ or E^* then $X^* = E^*$ or E respectively.

Axioms.

1. F is a finite subgroup of G.
2. $F \cup S$ is a subgroup of G in which F has index 1 or 2.
3. If $f \in F$, $g \in XY$, then $fg \in XY$.
4. If $s \in S$, $g \in XY$ then $gs \in XY^*$.
5. If $g \in XY$, then $g^{-1} \in YX$.
6. If $g \in XY$, $h \in Y^*Z$ then $gh \in XZ$.
7. If $g \in G$, there exists a minimal integer, $N(g)$ such that whenever $g_1, \cdots, g_n \in G$ and there exists $X_0, \cdots, X_n$ such that $g_i \in X^*_{i-1}X_i$ and $g = g_1 \cdots g_n$ then $n \leqq N(g)$.
8. $EE^* \neq \varnothing$.

If G is a group with bipolar structure then $P \in G$ is *irreducible* if Axiom 7 holds with $N(P) = 1$. Thus P is irreducible if $P \in F \cup S$ or if P cannot be written as $g \cdot h$ with $g \in XY$ and $h \in Y^*Z$. If G has a bipolar structure, then $(x_1, x_2, \cdots, x_n)$ is a *reduced word* if each x_i is irreducible, $x_1 \notin F \cup S$, and $x_i \in XY$ implies $x_{i+1} \in Y^*Z$ for $i \in \{1, 2, \cdots, n-1\}$, or $x_1 \in F \cup S$ and $n = 1$.

LEMMA 3.8. *If $f: H \to G$ is an epimorphism with finite kernel and G has bipolar structure with partition F, S, EE, EE^*, E^*E, E^*E^*, then H has bipolar structure with partition $f^{-1}(F)$, $f^{-1}(S)$, $f^{-1}(EE)$, $f^{-1}(EE^*)$, $f^{-1}(E^*E)$, $f^{-1}(E^*E^*)$.*

The proof is immediate since (in Axiom 7) $N(h)$ is bounded above by $N(f(h))$ for any $h \in H$.

LEMMA 3.9. *If $f: H \to G$ is an epimorphism with finite kernel and G has bipolar structure, then $N(h) = N(f(h))$ for any $h \in H$.*

Proof. If $(x_1, \cdots, x_n)$ is reduced and $h = x_1 \cdots x_n$ then $N(h) = n$ by [14] (p. 32). To see $N(f(h)) = n$ it remains to show $(f(x_1), \cdots, f(x_n))$ is reduced. Since H obtains its bipolar structure from f^{-1}, we need only show: If $x \in H - f^{-1}(F \cup S)$ is irreducible then $f(x)$ is irreducible, and $f(x) \in G - F \cup S$. Certainly $f(x) \in G - F \cup S$. If $f(x) = ab$ with $a \in XY$ and $b \in Y^*Z$, choose $c \in f^{-1}(a)$ and $d \in f^{-1}(b)$. Then $c \in f^{-1}(XY)$, and $d \in f^{-1}(Y^*Z)$ implying cd is not irreducible.

But $f(x^{-1}cd) = 1$, so $x^{-1}cd \in f^{-1}(F)$. By [14] (3.B.2.5) $x(x^{-1}cd)$ must be irreducible giving the desired contradiction.

The following theorems, 3.10–3.14, are in [14] for G a finitely generated group. (With bipolar structure in 3.11–3.14.)

THEOREM 3.10. *If G is ∞-ended, then G has a bipolar structure* (5.A.9).

PROPOSITION 3.11. $G_1 = F \cup \{$*Irreducible elements of* $EE\}$ *and* $G_2 = F \cup \{$*Irreducible elements of* $E^*E^*\}$ *are subgroups of* G. (3.B.4.1).

THEOREM 3.12. *If $S = \emptyset$, then $G = \{F \cup S\} *_F G_1 = \{F \cup S\} *_F G_2$, the free amalgamated product,* (3.B.4.2) *and G is ∞-ended if and only if F has index >2 in G_1 or equivalently G_2* (5.A.9).

THEOREM 3.13. *If $S = \emptyset$ and there is no irreducible element of EE^*, then $G = G_1 *_F G_2$* (3.B.4.3). *F is properly contained in G_1 and G_2* (3.B.5) *and G is ∞-ended if and only if F has index >2 in either G_1 or G_2* (5.A.9).

THEOREM 3.14. *If $S = \emptyset$ and there is an irreducible element t of EE^* then $tFt^{-1} \subset G_1$ and if $\phi: F \to G_1$ is defined by $f \to tft^{-1}$, then G is the* HNN *extension $G_{1_F} \leftrightarrows_\phi$* (3.B.4.4) *and G is ∞-ended if and only if F is properly embedded in G_1* (5.A.9).

REMARK 3.15. The hypothesis $S = \emptyset$ and t irreducible in EE^* also give $G = G_{2_F} \leftrightarrows_\phi$ where $\phi: F \to G_2$ by $f \to t^{-1}ft$ and since F is finite it can easily be shown F is properly embedded in G_1 if and only if it is embedded properly in G_2 ($t^{-1}G_1t = G_2$).

THEOREM 3.16. *If G is a finitely generated ∞-ended group and $f: H \to G$ is an epimorphism with finite kernel then H is ∞-ended and naturally inherits either the free product with finite amalgamation structure of Theorem* 3.12 *or Theorem* 3.13, *or the* HNN *structure of Theorem* 3.14.

Proof. By Lemma 2.8 H is ∞-ended. By Lemma 3.8 and Theorem 3.10, f^{-1} imposes a bipolar structure on H, determined by the one on G. By Lemma 3.9 $h \in H$ is irreducible if and only if $f(h)$ is irreducible. Thus $f^{-1}(G_1) = f^{-1}(F) \cup \{$Irreducible elements of $f^{-1}(EE)\}$. If $S \neq \emptyset$, then $f^{-1}(S) \neq \emptyset$ and by Theorem 3.12 $H = (f^{-1}(F \cup S)) *_{f^{-1}(F)} f^{-1}(G_1)$ and $f^{-1}(F)$ has index >2 in $f^{-1}(G_1)$. By Theorem 3.13, if $S = \emptyset$ and there is no irreducible element of EE^*, then $H = f^{-1}(G_1) *_{f^{-1}(F)} f^{-1}(G_2)$, $f^{-1}(F)$ is properly contained in $f^{-1}(G^1)$

and $f^{-1}(G_2)$, and $f^{-1}(F)$ has index >2 in either G_1 or G_2. If $S = \varnothing$ and there is an irreducible element t of EE^*, let $u \in f^{-1}(t)$. Then by Theorem 3.14, $uf^{-1}(F)u^{-1} \subset f^{-1}(G_1)$ and if $\psi: f^{-1}(F) \to f^{-1}(G_1)$ is defined by $x \to uxu^{-1}$, then $H = f^{-1}(G_1) \Leftrightarrow_{f^{-1}(F)} \psi$ and $f^{-1}(F)$ is properly embedded in $f^{-1}(G_1)$.

COROLLARY 3.17. *If* $\{G_i, f_i\}$ *is an* ∞*-ended* M-L-F *sequence each* f_i *is an epimorphism then either all* G_i *have compatible free product with finite amalgamation structures as in Theorem* 3.12 *or Theorem* 3.13, *or compatible* HNN *extension structures as in Theorem* 3.14.

Introduction to § 4. As explained in § 1 we are building the algebraic machinery necessary to extend the theorems by Hopf and Stallings from compact polyhedra K to compacta X. If $X \equiv \lim\limits_{\longleftarrow}\{X_n, f_n\}$ is a suitable compactum (this will be made precise in § 5), we would like the Čech fundamental group $\check{\pi}_1(X, *) \equiv \lim\limits_{\longleftarrow}\{\pi_1(X_n, *), f_n\}$ to play the role of $\pi_1(K, *)$. $\check{\pi}_1(X, *)$ is not, in general, finitely generated, but in our situation $\check{\pi}_1(X, *)$ considered as a topological group will be compactly generated. When thus interpreted, it plays the correct role.

That is one reason for the theory of ends of M-L-F topological groups which follows.

Another is that an alternative theory of ends for compactly generated, locally compact topological groups exists, at least in part (C. Pugh and M. Shub: Axiom A actions. Inventions Math., 29 (1975), 18–31). Although we do not prove this alternative definition agrees with our "number of ends" for M-L-F topological groups a reader familiar with Pugh and Shub can check the details.

4. Applications to topological groups.

DEFINITION. An M-L-F *topological group* is a locally compact, compactly generated, complete, metrizable topological group with countable neighborhood-basis of the identity consisting of closed and open normal subgroups.

A group can be regarded as a discrete topological group. If $\{G_n, f_n\}$ is an inverse sequence of discrete topological groups, then $\lim\limits_{\longleftarrow}\{G_n, f_n\}$ in the category of topological groups is obtained by taking inverse limits separately in the category of groups to get the group structure, and in the category of spaces to get the topology. Throughout this section we will regard $\lim\limits_{\longleftarrow}\{G_n, f_n\}$ as a topological group.

PROPOSITION 4.1. *G is an* M-L-F *topological group if and only if* $G = \varprojlim\{G_n, f_n\}$, *where* $\{G_n, f_n\}$ *is an* M-L-F *sequence of finitely generated discrete groups.*

REMARK 4.1.1. For the analogous proposition for M-L groups see [1] p. 4 and [6] p. 117–8.

Proof of "only in". Let $\{I_n\}$ be a neighborhood-basis of the identity consisting of closed and open normal subgroups. We may assume that the I_n are compact since G is locally compact and that the I_n are nested. Let $\pi_n: G \to G/I_n$ be the quotient homomorphism taking g to its right coset $[g]$. For $U \subset G/I_n$, $\pi_n^{-1}(U) = \cup\{I_n x \mid x \in \pi_n^{-1}(U)\}$. Since $I_n x$ is open for any $x \in G$, $\pi_n^{-1}(U)$ is open in G; therefore π_n is continuous when G/I_n has the discrete topology. Since I_n is compact the obvious epimorphism $f_n: G/I_{n+1} \to G/I_n$ has kernel $\pi_{n+1}(I_n)$ a compact and therefore a finite group. If K is a compact set which generates G, then $\pi_n(K)$ is a finite generating set in G/I_n. $\{G/I_n, f_n\}$ is M-L-F and $\lim\{G_n/I_n, f_n\}$ is isomorphic to G.

Proof of "if". First we need the following:

LEMMA 4.1.2. *If* $\{G_i, f_i\}$ *is* M-L-F *and* $G = \varprojlim\{G_i, f_i\}$ *then the projection* $\pi_n: G \to G_n$ *is proper* i.e., *if* C *is compact in* G_n *then* $\pi_n^{-1}(C)$ *is compact in* G.

Proof. Since G_n is discrete, its compact subsets are precisely its finite subsets. Discreteness also implies that G is closed in $\prod_{i=1}^{\infty} G_i$. $\pi_n(C) = (f_1 \circ \cdots \circ f_{n-1}(C) \times f_2 \circ \cdots \circ f_{n-1}(C) \times \cdots \times C \times f_n^{-1}(C) \times f_{n+1}^{-1} f_n^{-2}(C) \times \cdots) \cap G$. Since each f_i has finite kernel each term in this product is finite and $\pi_n^{-1}(C)$ is compact, proving 4.1.2.

Now assume G and $\{G_n, f_n\}$ as in Lemma 4.1.2. Since inverse limits of pro-isomorphic sequences of discrete groups are isomorphic topological groups, we may assume each f_n is onto. If K is a finite generating set of G_1, containing the identity, then by Lemma 4.1.2. $\pi_1^{-1}(K)$ is compact. We show $\pi_1^{-1}(K)$ generates G. If $(g_1, g_2, \cdots) \in G$ and $g_1 = k_1 k_2 \cdots k_n$ with each $k_i \in K$ then since each f_i is an epimorphism there is an $h_i \in G$ such that $\pi_1(h_i) = k_i$ for each i. $h_1 \cdots h_n$ is in the subgroup of G generated by $\pi_1^{-1}(K)$. Since $\pi_1((g_1, g_2, \cdots) \cdot h_n^{-1} \cdots h_1^{-1}) = \text{identity} \in K$, $(g_1, g_2, \cdots)$ is in the subgroup of G generated by $\pi_1^{-1}(K)$.

$I_n = \{(x_i) \in \varprojlim\{G_n, f_n\} \mid x_n = \text{identity}\}$ forms a countable neighborhood-basis of the identity consisting of closed and open normal subgroups. By Lemma 4.1.2 each I_n is compact and thus G is locally compact. Each G_i is discrete so G is metrizable and as an easy exercise G is complete.

DEFINITION. If G is an M-L-F group the *number of ends* of G is the number of ends of any M-L-F sequence $\{G_n, f_n\}$ where $G = \varprojlim \{G_n, f_n\}$.

This definition is unambiguous: M-L sequences which have isomorphic (topological group) inverse limits are pro-isomorphic, see [1] and [6]; combine this with 3.1.

Many of the theorems on ends for finitely generated groups can be generalized to theorems for M-L-F groups if the word finite is replaced by compact. The following is merely an exercise in the definitions.

PROPOSITION 4.2. *An* M-L-F *group is* 0-*ended if and only if it is compact.*

DEFINITION. A closed subgroup, H, of a topological group G has *compact index* in G if the space of right cosets, G/H, with quotient topology is compact.

PROPOSITION 4.3. *An* M-L-F *group G is* 2-*ended if and only if G has a closed infinite cyclic subgroup of compact index.*

The proof of "only if" will be done as a sequence of lemmas. Let $\{G_i, f_i\}$ be an M-L-F sequence with each G_i 2-ended and each f_i an epimorphism such that $G = \varprojlim \{G_i, f_i\}$. By Proposition 3.5 there are $x_n \in G_n$ generating infinite cyclic subgroups of finite index in G_n such that $f_n(x_{n+1}) = x_n \cdot x = (x_1, x_2, \cdots) \in \{G_i, f_i\} = G$.

LEMMA 4.3.1. *$Z_x = \{x_1^n, x_2^n, \cdots)|n$ is an integer$\}$ is a closed discrete subgroup of $\prod_{i=1}^{\infty} G_i$.*

Proof. Say Z_x accumulated at $y = (y_1, y_2, \cdots)$. Since G_i is discrete $\{y_1\} \times G_2 \times G_3 \times \cdots$ is open in $\prod_{i=1}^{\infty} G_i$ and thus must contain an infinite number of the x^n. Each must have first coordinate y_2, but at most one can.

A similar argument shows;

LEMMA 4.3.2. *G is closed in $\prod_{i=1}^{\infty} G_i$.*

Lemmas 4.3.1 and 4.3.2 imply

LEMMA 4.3.3. *Z_x is a closed discrete subgroup of G.*

Let Z_i be the infinite cyclic subgroup of G_i generated by x_i. Say $G_i/Z_i = [w_{i,1}], [w_{i,2}], \cdots, [w_{i,m(i)}]$. f_i maps the elements of the right coset $[w_{i+1,k}]$ bijectively onto $[f_i(w_{i+1,1})]$. Here $[w_{i+1,k}] = \{w_{i+1,k}x_{i+1}^n \mid n$ is an integer$\}$, and $w_{i+1,k}$ is callled a representative of $[w_{i+1,k}]$. By first selecting $w_{1,1}, \cdots, w_{1,m(1)}$ and then $w_{2,1}, w_{2,2}, \cdots, w_{2,m(2)}$ etc. we may assume without loss of generality that the above selected representatives of right cosets are mapped by the appropriate bonds to other selected representatives of right cosets. I.e., $f_n(w_{n+1,k}) \in \{w_{n,1}, w_{n,2}, \cdots, w_{n,m(n)}\}$ for all k. $W \equiv \prod_{k=1}^{\infty}\{w_{k,1}, w_{k,2}, \cdots, w_{k,m(k)}\}$ is compact in $\prod_{k=1}^{\infty} G_k$. By Lemma 4.3.2 $W \cap G$ is compact.

LEMMA 4.3.4. *G/Z_x is compact.*

Proof. It suffices to show $\pi(W \cap G) = G/Z_x$ where $\pi: G \to G/Z_x$ is projection. If $[a] \in G/Z_x$, $a = (a_1, a_2, \cdots) \in G$, then let $a_1 = w_{1,k(1)}x_1^s$. Because of how we selected representatives we have $a_n = w_{n,k(n)}x_n^t$ where $f_1 \circ \cdots \circ f_{n-1}(w_{n,k(n)}) = w_{1,k(1)}$. Also, since $f_1 \circ \cdots \circ f_{n-1}(a_n) = a_1$ we have $f_1 \circ \cdots \circ f_{n-1}(x_n^t) = x_1^s$ and therefore $t = s$. Thus, $\pi((w_{1,k(1)}, w_{2,k(2)}, \cdots)) = \pi((w_{1,k(1)}, w_{2,k(2)}, \cdots)\ (x_1^s, x_2^s, \cdots)) = \pi(a) = [a]$.

For the "if" part assume $G = \underleftarrow{\lim}\{G_n, f_n\}$ where $\{G_n, f_n\}$ is M-L-F and each f_n is an epimorphism. Let $x = (x_1, x_2, \cdots)$ generate Z_x a closed infinite cyclic subgroup of compact index in G. Let $\pi_i: G \to G_i$ be the projection morphism: Y_i the subgroup of G_i generated by x_i; and let $\theta_i: G/Z_x \to G_i/Y_i$ be defined by $[w] \to [w_i]$, where $w = (w_1, w_2, \cdots)$. To see θ_i is a well-defided function, let $z \in [w]$ i.e., $z = wx^n$. Then $z_i = w_ix_i^n$ and $z_i \in [w_i]$. Since each f_i is onto, π_i and θ_i are onto. Topologize G_i/Y_i and G/Z_x with the quotient topologies of the projections $\alpha_i: G_i \to G_i/Y_i$ and $\alpha_x: G \to G/Z_x$, respectively. Diagram (C) commutes on the level of functions.

(C)
$$\begin{array}{ccc} G & \xrightarrow{\pi_i} & G_i \\ \alpha_x \downarrow & & \downarrow \alpha_i \\ G/Z_x & \xrightarrow{\theta_i} & G_i/Y_i \end{array}$$

By Dugundji [4] (p. 126) θ_i is continuous. Since G_i has the discrete topology and G/Z_x is compact, G_i/Y_i has the discrete topology and is compact i.e., G_i/Y_i is finite. If x_i has finite order, then since each f_n has finite kernel each x_n would have finite order. This implies Z_x accumulates at the identity, contrary to the assumption Z_x is a closed infinite cyclic subgroup of G. Thus each x_i generates an infinite cyclic subgroup of finite index in G_i.

As we remarked in § 3, every finitely generated group with two ends has a normal infinite cyclic subgroup of finite index. One might expect a normal infinite cyclic subgroup of compact index in our 2-ended M-L-F groups. But this is not the case.

EXAMPLE 4.3. The 2-ended M-L-F group, G, determined by Example 3.6 contains no normal infinite cyclic subgroup.

Recall that G_1 was infinite cyclic with generator x_1. The closing argument of Example 3.6 shows that $(b_1, b_2, \cdots) \in G$ cannot generate a normal infinite cyclic subgroup if $b_1 = x_1^m$ for $m \neq 0$. But if $m = 0$ then $(b_1, b_2, \cdots)$ has order two. Thus G has no normal infinite cyclic subgroup.

PROPOSITION 4.4. *If G is a 2-ended* M-L-F *group, then G contains a compact open normal subgroup F such that G/F is Z or $Z_2 * Z_2$.*

Proof. By Proposition 4.1 G is the inverse limit of an M-L-F sequence $\{G_n, f_n\}$ of 2-ended groups. Choose F_1 a finite normal subgroup of G_1 such that G_1/F_1 is Z or $Z_2 * Z_2$ ([14] p. 38). If $\pi_1: G \to G_1$ and $\pi: G_1 \to G_1/F_1$ are projections then $\ker(\pi \circ \pi_1) = \pi_1^{-1}(F_1)$. By Lemma 4.1.2 $\pi_1^{-1}(F_1)$ is compact and open. Since $\pi_1^{-1}(F_1) = \ker(\pi \circ \pi_1)$, $\pi_1^{-1}(F_1)$ is normal in G. Finally $G_1/F_1 = G/\ker(\pi \circ \pi_1) = G/\pi_1^{-1}(F_1)$.

Next we classify the ∞-ended M-L-F groups.

REMARK 4.5. In a bipolar structure the requirement that F and S be finite is strictly a requirement to prove theorems about ends. Nowhere is the finiteness of F and S used in Stallings [14] p. 31–34. In particular it is not used in the first parts of our Theorems 3.12–3.14, nor is it used in Lemmas 3.8 and 3.9, and Proposition 3.11. Thus if H is a group with bipolar structure (not necessarily a finiteness condition on F and S) and $f: G \to H$ is an epimorphism then f^{-1} induces a bipolar structure (possibly without finiteness condition) on G, and thus the corresponding amalgamated free product or HNN extension structure on G.

THEOREM 4.6. *The* M-L-F *group G is ∞-ended if and only if G has subgroups A, B and C such that G is (in the obvious manner) isomorphic, in the category of groups, to $A *_C B$ where C is compact and open in G, properly contained in A and B and of index ≥ 3 in either A or B; or G is isomorphic, in the category of groups, to $H \looparrowleft_C \phi$ where C is compact and open in G and the infinite cyclic subgroup of $H \looparrowleft_C \phi$ generated by the extra generator x is closed in G.*

Proof of "only if". Assume $G = \varprojlim \{G_i, f_i\}$ where each G_i is ∞-ended and each f_i is an epimorphism. If $G_1 = A_1 *_{C_1} B_1$ as in Theorem 3.12 or 3.13 then by Remark 4.5 $G = \pi_1^{-1}(A) * \pi_1^{-1}(B) \pi_1^{-1}(C_1)$ where $\pi_1: G \to G_1$ is projection. If $G_1 = H_1 \looparrowleft_{C_1} \phi_1$ as in Theorem 3.1, with t irreducible in EE^* then by Remark 4.5 $G = \pi_1^{-1}(H_1) \looparrowleft_{\pi_1^{-1}(C_1)} \phi$ where $q \in \pi_1^{-1}(t)$ and $\phi: \pi_1^{-1}(C_1) \to \pi_1^{-1}(H_1)$ by $y \to qyq^{-1}$. By Lemma 4.1.2 $\pi_1^{-1}(C_1)$ is compact in both cases and is open since π_1 is continuous. The infinite cyclic subgroup of G generated by q is closed in G by Lemma 4.3.1.

Proof of "if".

Case 1. $G = A *_C B$ as above.

Assume $G = \varprojlim \{G_i, f_i\}$ with each f_i an epimorphism. Let $\pi_n: G \to G_n$ be projection. By Lemma 4.1.2 $\ker(\pi_n)$ is compact. $\bigcap_{i=1}^{\infty} \ker(\pi_i) = (e, e, \cdots)$, the identity of G. Since C is open and G is metrizable, $\ker(\pi_n) \subset C$ for some N. We show for this N that G_n is the free product with finite amalgamation $A_n *_{C_n} B_n$, where $A_n = \pi_n(A)$, $B_n = \pi_n(B)$ and $C_n = \pi_n(C)$. Since G_n is discrete, π_n is continuous and C is compact, C_n is finite.

Define $\phi_n: A_n *_{C_n} B_n \to G_n$ to be the homomorphism which is an inclusion on $A_n \cup B_n$. Define $\psi_n: A *_C B \to A_n *_{C_n} B_n$ to be the epimorphism which is projection into the nth coordinate on $A \cup B$.

The following diagram (D) commutes:

$$\text{(D)} \qquad \begin{array}{ccc} & A *_C B & \\ {}^{\psi_n}\swarrow & & \searrow^{\pi_n} \\ A_n *_{C_n} B_n & \xrightarrow{\ \phi_n\ } & G_n \end{array}$$

π_n is an epimorphism since each f_i is an epimorphism. Thus ϕ_n is an epimorphism. It remains to show ϕ_n is a monomorphism. If $\phi_n(x) = e$, choose $y \in \psi_n^{-1}(x)$. $y \in \ker(\pi_n) \subset C$, and thus $x \in C_n$. By the definition of ϕ_n, $x = e$.

We have proved $G_m = A_m *_{C_m} B_m$ for all $m \geqq N$. Choose $k \geqq N$ large enough to ensure C_k has index $\geqq 2$ in A_k and B_k and index $\geqq 3$ in either A_k or B_k. Then $G_k = A_k *_{C_k} B_k$ is ∞-ended and by Proposition 2.4 all G_i are ∞-ended.

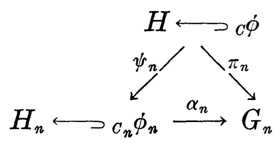
Case 2. $G = H \looparrowleft_C \phi$ as above.

Recall $H \looparrowleft_C \phi = \langle H, x \mid x^{-1}cx = \phi(c) \forall c \in C \rangle$. As in Case 1, $\ker(\pi_n) \subset C$ for some N. Let $H_n = \pi_n(H)$, $C_n = \pi_n(C)$ and $x_n = \pi_n(x)$. For a and b in C if $\pi_n(a) = \pi_n(b)$, then $\pi_n(x^{-1}ax) = \pi_n(x^{-1}bx)$ and $\pi_n(\phi(a)) = \pi_n(\phi(b))$. Thus $\phi_n: C_n \to H_n$ by $c_n \to \pi_n(\phi(c))$ where $\pi_n(c) = c_n$

is well-defined. In forming $H_n \leftarrow\!\supset_{C_n}\phi_n$ we identify $x_n^{-1}c_nx_n$ with $\phi_n(c_n)$ for all $c_n \in C_n$. If we define $\psi_n: H \cup \{x\} \to H_n \leftarrow\!\supset_{C_n}\phi_n$ by $\psi_n = \pi_n$ then ψ_n extends to an epimorphism of $H \leftarrow\!\supset_C \phi$ since $\psi_n(x^{-1}cx) = x_n^{-1}c_nx_n = \phi_n(c_n) = \pi_n(\phi(c)) = \psi_n(\phi(c))$ $(\phi(c) \in H)$. Define $\alpha_n: H_n \leftarrow\!\supset_{C_n}\phi_n \to G_n$ to be the homomorphism which extends the inclusion of $H_n \cup \{x_n\}$ into G_n.

The following diagram (E) commutes:

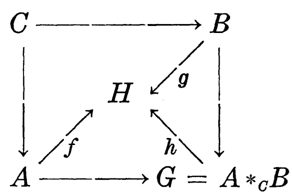
(E)



The same closing argument as that of Case 1 works here to complete the proof.

THEOREM 4.7. *If $G = A*_CB$ as in Theorem 4.6, then in the following diagram the outer square commutes; furthermore, given any topological group H and continuous homomorphisms $f: A \to H$ and $g: B \to H$ making the north-west triangle commute the resulting homomorphism of groups, h, (which exists and is unique by the universal property for amalgamated free products in groups) is continuous. (Unlabeled maps are inclusions.)*

(F)



Proof. Let W be a neighborhood of x in H. Let $y \in h^{-1}(x) \subset h^{-1}(W)\cdot h^{-1}(W) = h^{-1}(Wx^{-1})y$. Since C is open in G, A is open in G. Hence $f^{-1}(Wx^{-1})$ is a neighborhood of e lying in $h^{-1}(Wx^{-1})$. Thus $f^{-1}(Wx^{-1})y$ is a neighborhood of y lying in $h^{-1}(W)$. So $h^{-1}(W)$ is open.

REMARK 4.8. If $G = H \leftarrow\!\supset_C \phi$ as in Theorem 4.6 a similar result is true.

5. Geometric applications. As explained in § 1, we will now use the theorems of § 3 to derive a new shape invariant for a large class of compacta (those with M-L-F "fundamental pro-group"), and a natural geometric interpretation of this invariant, (namely the number of ends of universal covers of compact polyhedra in an associated inverse sequence). For a somewhat smaller class of compacta (those with stable "fundamental pro-group") an added na-

tural geometric interpretation arises. We will use [5] as our reference for shape theory.

We deal throughout with a pointed connected compactum $(X, *)$, [Convention: "compactum means compact metric space. $*$ is used for all base points.] We will call a compactum, X, an M-L-F *compactum* if X is connected and if for some (hence any) $* \in X$, pro-π_1 $(X, *)$ is M-L-F.

The shape invariant for M-L-F compacta X mentioned above can now be defined. It is $e(X) \equiv$ the number of ends of pro-π_1 $(X, *)$; see § 3 $e(X)$ is one of the numbers 0, 1, 2 or ∞.

THEOREM 5.1. *If X is an M-L-F compactum then $(X, *) = \varprojlim \{(X_n, *), f_n\}$ where each X_n is a compact polyhedron and the universal cover of X_n has $e(X)$ ends for all n.*

Proof. By a trick of Krasinciewicz [11] (or see Theorem 4 of [7]) one can arrange $(X, *) = \varprojlim \{(X_n, *), f_n\}$ where each X_n is a compact polyhedron and $f_{n\#}$ is an epimorphism. By Hopf's theorem (see § 1) and Proposition 2.4 the universal cover of X_n has $e(X)$ ends for all n.

Note that if $(Y, *)$ is an M-L-F compactum then by a theorem of Krasinciewicz (see [7]) $(Y, *)$ is pointed shape equivalent to some $(X, *)$ where X is compact connected and LC°. *We permanently assume $(X, *) = \varprojlim \{(X_n, *), f_n\}$ to have these properties and we assume each $f_{n\#}$ is an epimorphism on π_1.*

It remains to investigate when $e(X)$ can be interpreted geometrically as the number of topological ends of a locally compact space $\bar{X}$ (which reduces to the universal cover when X is LC° and semi-locally 1-connected). The $\bar{X}$ we have in mind is $\varprojlim \{\tilde{X}_n, \tilde{f}_n\}$. More precisely $\sim$ denotes the "pointed universal cover functor" and in the following commutative diagram (G) the limit p of covering projections, p_n, is a fibration with unique path lifting:

$$\text{(G)} \qquad \begin{array}{ccccc} (\tilde{X}_1, *) & \xrightarrow{\tilde{f}_1} & (\tilde{X}_2, *) & \longleftarrow \cdots & (\bar{X}, *) \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p \\ (X_1, *) & \longleftarrow & (X_2, *) & \longleftarrow \cdots & (X, *) \end{array}$$

Since X is M-L-F, $\bar{X}$ is locally compact. To prove this we need:

PROPOSITION 5.2. *Let X and Y be finite complexes. If $f: X \to Y$ induces $f_\#: \pi_1(X, *) \to \pi_1(Y, *)$ a homomorphism with finite kernel then $\tilde{f}: \tilde{X} \to \tilde{Y}$ is proper.*

Proof. Diagram (H) commutes.

$$
\text{(H)} \qquad \begin{array}{ccc} (\tilde{X}, *) & \xrightarrow{\tilde{f}} & (\tilde{Y}, *) \\ \downarrow p & & \downarrow q \\ (X, *) & \xrightarrow{f} & (Y, *) \end{array}
$$

Let $[g] \in \pi_1(X, *)$, then the covering transformation of $\tilde{X}$ determined by $[g]$ is defined as follows:

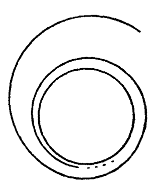
If $x \in \tilde{X}$ and λ is a path from x to $*$, then $p(\lambda)\cdot g \cdot p(\lambda^{-1})$ forms a loop at $p(x)$. $[g](x)$ is the endpoint of the lift of this loop to x. By [3] (p. 12) we have:

$$
(1) \qquad \tilde{f}([g](x)) = f_\#([g])(\tilde{f}(x)) \,.
$$

Let C be a compact subset of Y. For each cell e_i in X choose a cell $\tilde{e}_i$ in $\tilde{X}$ over e_i. Then each of the cells of $\tilde{X}$ over e_i is $h(\tilde{e}_i)$ for some $h \in \pi_1(X, *)$. It suffices to show only finitely many cells of $\tilde{X}$ over any cell, e_i, of X touch $\tilde{f}^{-1}(C)$. Suppose $\{g_1, g_2, \cdots\} \subset \pi_1(X, *)$ and $g_i(\tilde{e}_n)$ intersects $\tilde{f}^{-1}(C)$ for all i. Then by (1) $f_\#(g_i)(\tilde{f}(\tilde{e}_n))$ intersects C for each i. $\tilde{f}(\tilde{e}_n)$ meets only finitely many cells of $\tilde{Y}$ since the closure of $\tilde{e}_n$ is compact. Thus since $f_\#$ is finite-to-one, there is a cell $\tilde{e}$ of $\tilde{Y}$ and infinitely many $h \in \pi_1(Y, *)$ such that $h(\tilde{e})$ meets C, contradicting the local finiteness of $\tilde{Y}$.

Let $\alpha_i\colon X \to X_i$ and $\bar{\alpha}_i\colon \bar{X} \to \tilde{X}_i$ be projections. Then $p_i \circ \bar{\alpha}_i = \alpha_i \circ p$ for all i. $\bar{X}$ is closed in $\prod_{i=1}^{\infty} \tilde{X}_i$ and since each $\tilde{f}_i$ is proper, each $\bar{\alpha}_i$ is proper (see the proof of 4.1.2) and $\bar{X}$ is locally compact.

Next we discuss when $\bar{X}$ is path connected. [Certainly $\bar{X}$ can have infinitely many path components when X is not LC°; an example is the compact spiral:



We leave it to the reader to check this.]

PROPOSITION 5.3. *$\bar{X}$ is path connected if and only if the natural map $j\colon \pi_1(X, *) \to \varprojlim\{\pi_1(X_i, *)\}$ is onto.*

Proof. We look at the last few terms of the homotopy exact sequence of the fibration $F \xrightarrow{i} \bar{X} \xrightarrow{p} X$:

$$\begin{array}{ccccccc}\pi_1(X,*) & \xrightarrow{\partial} & \pi_0(F,*) & \xrightarrow{i_\#} & \pi_0(\bar{X},*) & \xrightarrow{p_\#} & 0\\ j\downarrow\cong & \nearrow & & & & & \\ \lim\limits_{\leftarrow}\{\pi_1(X_i,*)\} . & & & & & & \end{array}$$

Since F is totally disconnected the above isomorphism of $\lim\limits_{\leftarrow}\{\pi_1(X_i,*)\}$ to $\pi_0(F,*)$ ls induced by the above homeomorphism of $\lim\limits_{\leftarrow}\{\pi_1(X_i,*)\}$ to F. It remains to observe the following are all equivalent:

(i) j is onto
(ii) ∂ is onto
(iii) $\pi_0(\bar{X},*)$ is trivial
(iv) $\bar{X}$ is path connected.

PROPOSITION 5.4. *Let X be an LC° compactum and let* pro-π_1 $(X,*)$ *be stable. Then* $j\colon \pi_1(X,*)\to \lim\limits_{\leftarrow}(\text{pro-}\pi_1(X,*))$ *is onto.*

Proof. Let $\{U_n\}_{n=1}^\infty$ be a nested sequence of compact Q-manifold neighborhoods of X with $\bigcap_{n=1}^\infty U_n = X$. Let $\{[W_n]\}$ be an element of $\lim\limits_{\leftarrow}\{\pi_1(U_n,*)\}$. (Here there is a common base point $*\in X$.) Then W_{n+1} is a loop in U_{n+1} which is homotopic rel. $\{0,1\}$ to W_n in U_n. By [5] (p. 94), for any $n>0$ there is a M such that for all $m>M$ W_m is homotopic rel. $\{0,1\}$ in U_n to a loop λ_n in X_n, and hence $\lambda_n\cong W_n$ rel. $\{0,1\}$ in U_n. Let $\{[W_n]\}\in\lim\limits_{\leftarrow}\{\pi_1(U_n,*)\}$ and λ be a loop in X, such that $\lambda\cong W_1$ rel. $\{0,1\}$ in U_1. By stability we may assume the inclusion of U_{n+1} into U_n induces an isomorphism on fundamental groups and thus $\lambda\cong W_n$ rel. $\{0,1\}$ in U_n for all n, finishing the proposition.

PROPOSITION 5.5. *If X is an LC° semi-locally* 1*-connected compactum then pro-$\pi_1(X,*)$ is stable.*

Proof. Let $\{U_n\}_{n=1}^\infty$ be a nested sequence of compact Q-manifold neighborhoods of X with $\bigcap_{n=1}^\infty U_n = X$. By [6] it suffices to show $\lim\limits_{\leftarrow}\{\pi_1(U_n,*)\}$ is discrete. Assume $\{[W_n^{(1)}]\}$, $\{[W_n^{(2)}]\},\cdots$ are elements of $\lim\limits_{\leftarrow}\{\pi_1(U_n,*)\}$ converging to $\{[W_n]\}$. By [5] (p. 94) we may assume W_n and $W_n^{(k)}$ are loops in X for all n and k. For any $N>0$ there is a $K(N)>0$ such that for all $k>K(N)$, and $n>N$ $W_n^{(k)}\cong W_n$ rel. $\{0,1\}$ in U_N. Since X is compact and semi-locally 1-connected there exist $\varepsilon>0$ such that any loop in X of diameter $<\varepsilon$ is homotopically trivial in X. By [5] (p. 94) there is a neighborhood V of X in U_1 such that any map $f\colon(L,L_0)\to(V,X)$, where L is a

1-dimensional finite polyhedron and L_0 is a subpolyhedron, is $\varepsilon/3$-homotopic rel. L_0 in U_1 to a map $g: L \to X$. Choose N such that for all $n \geqq N$ $U_n \subset V$. Let $k > K(N)$ and $n > N$. Then $W_n^{(k)} \cong W_n$ rel. $\{0, 1\}$ in U_N and thus in V. Let $H: [0, 1]\times[0, 1] \to V$ be a homotopy of $W_n^{(k)}$ to W_n rel. $\{0, 1\}$ i.e., $H|[0, 1] \times \{0\} = W_n^{(k)}$, $H|[0, 1] \times \{1\} = W_n$ and $H(\{0, 1\}\times[0, 1]) = *$. Choose $a_0 = 0 < a_1 < \cdots < a_m = 1$ and $b_0 = 0 < b_1 < \cdots < b_m = 1$ so that the diameter $H([a_i, a_{i+1}]\times[b_j, b_{j+1}])$ is $<\varepsilon/3$ for all i and j. Let $L = (\{a_0, a_1, \cdots, a_m\}\times [0, 1]) \cup ([0, 1] \times \{b_0, b_1, \cdots, b_m\})$ and $L_0 = (\{0, 1\} \times [0, 1]) \cup [0, 1] \times \{0, 1\}$. Choose $g: L \to X$ homotopic rel. L_0 to $H|L$. $g((\{a_i, a_{i+1}\}\times[b_j, b_{j+1}]) \cup ([a_i, a_{i+1}]\times\{b_j, b_{j+1}\}))$ is a loop of diameter $<\varepsilon$ in X and thus homotopically trivial in X. Hence $W_n^{(k)}$ is homotopic to W_n rel. $\{0, 1\}$ in X for all $n > N$ and $k > K(N)$ i.e., $\{[W_n^{(k)}]\} = \{[W_n]\}$ for all $k > K(N)$ and $\lim\limits_{\longleftarrow} \{\pi_1(U_n, *)\}$ is discrete.

REMARK 5.5.1. If a loop W of a semi-locally-1-connected compactum X, represents the trivial element of pro-π_1 (X) then W is trivial in X if X is LC°. Here is the main theorem of § 5.

THEOREM 5.6. *If $\bar{X}$ can be written as the union of compact sets, A_n, with A_n a subset of the interior of A_{n-1} and any two points in A_n can be joined by a path in A_{n+1} then $\bar{X}$ has the same number of ends as* $\lim\limits_{\longleftarrow} \{\pi_1(X_n, *)\}$.

The proof will be done as a sequence of lemmas. We assume $* \in A_1$.

REMARK 5.6.1. For $\bar{X}$ compact this is trivial.

LEMMA 5.6.2. *$\bar{X}$ contains a compact set C such that p maps the interior of C,* int(C), *onto X.*

Proof. By the Hahn-Mazurkiewicz theorem X is a Peano curve i.e., the continuous image of $[0, 1]$. Lifting this path to $\bar{X}$ gives a path whose image under p is X. Choose an A_n such that this path lies in int(A_n), then A_n is the desired C.

Let $F_i = p_i^{-1}(*)$ and $F = p^{-1}(*)$. The usual bijections $\pi_1(X_i, *) \xrightarrow{\cong} F_i$ induce a homeomorphism $H: \lim\limits_{\longleftarrow} \{\pi_1(X_i, *)\} \xrightarrow{\cong} F$, where $\lim\limits_{\longleftarrow} \{\pi_1(X_i, *)\}$ is topologized as in § 4. We will freely identify these fibers. Thus the left action of $\pi_1(X_n, *)$ on $\tilde{X}_n$ determined by the corresponding covering transformations determines a left action of F_n on $\tilde{X}_n$, hence F acts on the left of $\bar{X}$ as fiber preserving homeomorphisms such that for $a \in F$ and $x \in \bar{X}$ $\tilde{\alpha}_i(a(x)) = \tilde{\alpha}_i(a)\ (x)$ for all i.

For A a subset of a simplicial complex Y define $St(A)$ to be the closure of the union of all cells of Y that touch A. Inductively define $St^N(A)$ to be $St(St^{N-1}(A))$ for $N \geqq 2$. A set B in a topological space T is said to be *bounded* if B lies in a compact subset of T, otherwise B is *unbounded.*

LEMMA 5.6.3. *If C is a compact subset of $\bar{X}$, then each unbounded path component of $\bar{X} - C$ contains an unbounded subset of F.*

Proof. By Lemma 5.6.2 there is an M such that $p(A_M) = X$, and $C \subset A_M$. Choose N such that $\tilde{\alpha}_1(A_{M+1}) \subset St^N(*)$. If Q is an unbounded path component of $\bar{X} - A_M$, let $\{x_i\}_{i=1}^{\infty}$ be an unbounded collection of points in $Q - \tilde{\alpha}_1^{-1}(St^{4N}(*))$. Let $y_i \in A_M$ such that $p(y_i) = p(x_i)$ and let λ_i be a path from y_i to $*$ in A_{M+1}. The endpoint of the lift of $p \circ \lambda_i$ to x_i is in F, call it z_i, thus we have: $z_i \cdot \lambda_i$ has endpoints z_i and x_i; and $\tilde{\alpha}_1(\mathrm{x}_i) \in \tilde{\alpha}_1(z_i A_{M+1}) \subset St^N(\tilde{\alpha}_1(z_i)) = \alpha_1(z_i) \cdot St^N(*)$. By definition $\tilde{\alpha}_1(x_i) \notin St^{4N}(*)$, so $St^N(\alpha_1(z_i))$ contains a point of $\tilde{X}_1 - St^{4N}(*)$ and therefore $\tilde{\alpha}(z_i) \cdot St^N(*)$ misses $St^N(*)$. Since their images under $\tilde{\alpha}_1$ are disjoint, $z_i \cdot A_{M+1}$ and A_{M+1} are disjoint. $z_i \cdot \lambda_i \subset z_i \cdot A_{M+1} \subset \bar{X} - A_M$, therefore $z_i \in Q$ for all i. $\{z_i\}$ is unbounded, for if not, say $S_1(\{z_i\}) \subset D$, a compact set. Then $S_1(\{x_i\}) \subset St^N(D)$ a compact set but $\tilde{\alpha}_1$ is proper and $\{x_i\}$ is unbounded giving the desired contradiction.

LEMMA 5.6.4. *If $p(\mathrm{int}(A_M)) = X$, then there is a bounded neighborhood of A_M containing all but finitely many of the path components of $\bar{X} - A_M$.*

Proof. Choose N such that $\tilde{\alpha}_1(A_{M+1}) \supset St^N(*)$. We show $\tilde{\alpha}_1^{-1}(St^{4N}(*)) \equiv W$ is the desired neighborhood of A_M. W is bounded since $\tilde{\alpha}_1$ is proper. Assume an infinite number of path components of $\bar{X} - A_M$, say $C_1, C_2, \cdots$, are not contained in W. Since $\bar{X}$ is path connected each of these path components meet $bd(W)$, the boundary of W. Choose $x_i \in C_i \cap bd(W)$. Say the x_i accumulate at $x \in bd(W)$. Let $y \in \mathrm{int}(A_M)$ such that $p(y) = p(x)$. If λ is a path from y to $*$ in A_{M+1}, then lifting $p \circ \lambda$ to x is a path from x to $z \in F$ and z. A_{M+1} is a neighborhood of x containing $z \cdot \lambda$, a path from x to z. $\tilde{\alpha}_1(x) \in bd(St^{4N}(*))$ and $\tilde{\alpha}_1(z \cdot A_{M+1}) \subset St^N(\tilde{\alpha}_1(z)) = \tilde{\alpha}_1(z) \cdot St^N(*)$. Thus $bd(St^{4N}(*)) \cup St^N(\alpha_1(z))$ is a nonempty subcomplex of $\tilde{X}_1$, and must contain a vertex u. If $St^N(\alpha(z)) \cap St^N(*) \neq \varnothing$ there would be an edge path from $*$ to u of length $\leqq 3N$ contradicting the fact $u \in bd(St^{4N}(*))$. Hence $\tilde{\alpha}_1(z \cdot A_{M+1}) \cap \tilde{\alpha}_1(A_{M+1}) = \varnothing$, and $A_M \cap z \cdot A_{M+1} = \varnothing$. Since $z \cdot A_{M+1} \subset \bar{X} - M$ and $z \cdot A_M$ is a neighborhoood of x, the x_i are not all from different path components of $\bar{X} - A_M$ giving the desired contradiction.

COROLLARY 5.6.5. *If $\tilde{X}_1$ is N-ended then $\bar{X}$ has at least N ends. In particular if $\tilde{X}_1$ is ∞-ended then $\bar{X}$ is ∞-ended.*

Proof. If C is a compact subset of $\tilde{X}_1$ and $\tilde{X}_1 - C$ has M unbounded path components, then an argument analogous to the proof of Lemma 5.6.3 shows each unbounded path component, C_i, of $\tilde{X}_1 - C$ contains an unbounded subset, S_i, of F_1. By Lemma 5.6.4 $\bar{X} - \tilde{\alpha}_1^{-1}(C)$ has only a finite number of unbounded path components and $\tilde{\alpha}_1^{-1}(S_i)$ is an unbounded subset of F since each $f_{i\#}$ is an epimorphism. One of the unbounded path components of $\bar{X} - \tilde{\alpha}_1^{-1}(C)$ must contain an unbounded subset of $\tilde{\alpha}_1^{-1}(S_i)$; but $\tilde{\alpha}_1^{-1}(S_i)$ does not meet a path component of $\bar{X} - \alpha_1^{-1}(C)$ which contains a point of $\tilde{\alpha}_1^{-1}(S_j)$ for $i \neq j$. Thus $\bar{X} - \tilde{\alpha}_1^{-1}(C)$ must have at least M unbounded path components.

LEMMA 5.6.6. *If $\varprojlim \{\pi_1(X_i, *)\}$ is 1-ended, then $\bar{X}$ is 1-ended.*

Proof. By Corollary 5.6.5 we need to see $\bar{X}$ has at most one end. Assume $\bar{X} - A^V$ has unbounded path components C_1 and C_2. Let K_1 be a finite set of generators of $\pi_1(X_1, *)$ and assume $1 \in K_1$, then $\alpha_1^{-1}(K)$ is a compact generating set for $\varprojlim \{\pi_1(X_n, *)\}$ (see the proof of 4.1). Choose W such that $K \cup K^{-1} \subset A_W$. Let M and N be such that $\tilde{\alpha}_1^{-1}(A_{V+1}) \subset St^N(*)$ and $\tilde{\alpha}_1(A_{W+1}) \subset St^M(*)$, and hence $K_1 \subset St^M(*)$.

Claim. It suffices to show there are elements $v_1 \in C_1 \cap F$ and $v_2 \in C_2 \cap F$ such that $v_2 = v_1 k_1 k_2 \cdots k_n$ where $k_i \in K \cup K^{-1}$ and $v_1 k_1 \cdots k_m \; A_{W+1} \subset \bar{X} - A_V$ for all $m \leqq n$.

Proof. $v_1 k_1 \cdots k_m \in v_1 k_1 \cdots k_m A_W \cap v_1 k_1 \cdots k_{m+1} A_W$; so $v_1 k_1 \cdots k_m$ can be joined by a path to $v_1 k_1 \cdots k_{m+1}$ in $v_1 k_1 \cdots k_{m+1} A_{W+1} \subset \bar{X} - A_V$. Thus v_1 and v_2 are in the same path component of $\bar{X} - A_V$ giving the desired contradiction.

Let $\tilde{\alpha}_1(A_V) \subset St^N(*)$ and $\tilde{\alpha}_1(A_{W+1}) \subset St^M(*)$. Choose R such that any element of $F \cap St^{M+N}(*)$ is an R-fold product with factors in $K_1 \cup K_2$. In § 2 we define $L(K_1)$ which by definition has 1-end. The verticies of $L(K_1)$ are identified with $\pi_1(K_1, *)$ and thus with the elements of F_1. If u_1 and u_2 are verticies of the unbounded path component, C, of $L(K_1) - St^R(1)$, then an edge path in C from u_1 to u_2 gives $u_2 = u_1 k_1 \cdots k_n$ with $k_i \in K_1 \cup K_1^{-1}$, and for all $m \leqq n$ $u_1 k_1 \cdots k_m \in \tilde{X}_1 - St^{M+N}(*)$. Since $\tilde{\alpha}_1(A_{W+1}) \subset St^M(*)$, $u_1 k_1 \cdots k_m \cdot \tilde{\alpha}_1(A_{W+1}) \subset \tilde{X}_1 - St^N(*)$ for all $m \leqq n$. Since $\tilde{\alpha}_1$ is proper, Lemma 5.6.3 implies C_1 and C_2 contain points of F, v_1 and v_2 respectively such that $\tilde{\alpha}_1(v_1)$ and $\tilde{\alpha}_1(v_2)$ are verticies of C. Assume $\tilde{\alpha}_1(v_1) = u_1$

and $\tilde{\alpha}_1(v_2) = u_2$ as above. We can choose $h_j \in \tilde{\alpha}_1^{-1}(k_j) \subset K \cup K^{-1}$ since $f_{i\#}$ is an epimorphism for all i. Since $\tilde{\alpha}_1(A_V) \subset St^N(*)$, $v_1 h_1 \cdots h_m \cdot A_{W+1} \subset \bar{X} - A_V$ for all $m \leqq n$. Let $h_{n+1} = (v_1 h_1 \cdots h_n)^{-1} v_2$, then $\alpha_1(h_{n+1}) = * \equiv 1 \in K_1$, and $\alpha_1(v_1 h_1 \cdots h_{n+1} \cdot A_{W+1}) = \alpha_1(v_1 h_1 \cdots h_n \cdot A_{W+1}) \subset \tilde{X}_1 - \mathrm{St}^N(*)$. Thus $v_2 \cdot A_{W+1} \subset \bar{X} - A_V$ and by the above claim the proof is finished.

LEMMA 5.6.7. *If* $\underleftarrow{\lim}\{\pi_1(X_n, *)\}$ *is* 2-*ended then* $\bar{X}$ *is* 2-*ended.*

Proof. By Corollary 5.6.5 we need to prove $\bar{X}$ has at most 2-ends. Let $G = \underleftarrow{\lim}\{\pi_1(X_i, *)\}$, then G has a closed infinite cyclic subgroup, Z_x, with generator x, and a compact subset K such that if β is the quotient map of G to G/Z_x then $\beta(K) = G/Z_x$ (see 4.3.4). Hence $G = \cup\{x^n \cdot K \mid n$ is an integer$\}$, $\tilde{\alpha}_1(K)$ is a compact subset of the discrete fiber F_1, i.e., $\tilde{\alpha}_1(K)$ is finite. Thus (1) If V is compact in $\bar{X}$ then $\tilde{\alpha}_1(\bigcup_{k \in K} k \cdot V)$ is finite.

Let A_W be such that $p(A_W) = X$ and $x \cdot K \cup K \subset A_W$. It suffices to show for $Q \geqq W$, $\bar{X} - A_Q$ has at most two unbounded path components. By Lemma 5.6.3 it suffices to show there exists $M > 0$ and $N < 0$ such that for all $m > M$ all points of $x^m \cdot K$ are in the same path component of $\bar{X} - A_Q$ and for $n < N$ all points of x^n. K are in the same path component of $\bar{X} - A_Q$. $\tilde{\alpha}_1(x)$, $\tilde{\alpha}_1(x^2)$, $\cdots$ is a closed discrete subset of $\tilde{X}_1$. Choose K and L such that $\tilde{\alpha}_1(A_Q) \subset St^L(*)$ and $\tilde{\alpha}_1(A_{W+1}) \subset St^K(*)$. Let M be such that for all $m > M$ $\tilde{\alpha}_1(x^m)$ misses $St^{L+M}(*)$. Then $\tilde{\alpha}_1(x^m) \cdot \tilde{\alpha}_1(A_{W+1})$ misses $\tilde{\alpha}_1(A_Q)$ for all $m > M$. Thus $x^m \cdot A_{W+1}$ misses A_Q for all $m > M$. $(x^m \cdot K) \cup (x^{m+1} \cdot K) \subset x^m A^W$ so all points of $(x^m \cdot K) \cup (x^{m+1} \cdot K)$ can be joined by paths in $x^m A_{W+1} \subset \bar{X} - A_Q$ for all $m > M$. A similar argument holds for negative powers of x.

Combining Remark 5.6.1, Corollary 5.6.5 and Lemmas 5.6.6 and 5.6.7 proves Theorem 5.6.

THEOREM 5.7. *Let* Y *be a connected compactum with pro-*$\pi_1(Y, *)$ *stable. Then any* LC° *compactum* $X \equiv \underleftarrow{\lim}\{X_n\}$ *in the shape class of* Y *has the property that* $e(X)$ *is the number of topological ends of the path connected, locally compact space* $\bar{X} \equiv \underleftarrow{\lim}\{\tilde{X}_n\}$. *Moreover when* X *is semi-locally* 1-*connected,* X *is the universal cover of* X.

REMARK 5.8. By [11] there exist LC° compacta in the shape class of Y.

Proof. We prove any compact subset of $\bar{X}$ is contained in a path connected compact subset of $\bar{X}$. By Propositions 5.3 and 5.4 and Theorem 5.6 this will prove the first part of our theorem.

Let β be a path in X whose image covers X (Hahn-Mazurkiewicz theorem), and let C be a compact subset of $\bar{X}$. If β' is a lift of β to $\bar{X}$. Choose N such that for any x in the image of $\tilde{\alpha}_1 \circ \beta'$, the image of $\tilde{\alpha} \circ \beta'$ is a subset of $St^N(x)$. Let $D = C \cup (\bigcup_{x \in C} \beta_x)$ where β_x is the image of a lift of β containing x. If $\tilde{\alpha}_1(C) \subset St^M(*)$ then $\alpha_1(D) \subset \mathrm{St}^{M+N}(*)$ and D is bounded. Since pro-$\pi_1(X, *)$ is stable F is discrete and $F \cap D$ is finite. By Propositions 5.3 and 5.4 $\bar{X}$ is path connected. Let E be D union a path containing each point of $F \cap D$. Then E is connected and contains C.

For the second half of 5.7, let $q: (\tilde{X}, *) \to (X, *)$ be the universal covering projection. There are unique maps $\beta_n: (\tilde{X}, *) \to (\tilde{X}_n, *)$ such that $p_n \circ \beta_n = \alpha_n \circ q \cdot \alpha_{n-1} \circ q = f_n \circ \alpha_n \circ q = f_n \circ p_n \circ \beta_n = p_{n-1} \circ \tilde{f}_{n-1} \circ \beta_n$. By the uniqueness of pointed lifts, $\beta_{n-1} = \tilde{f}_{n-1} \circ \beta_n$ and there is a map $\beta: \tilde{X} \to \bar{X}$ by $x \to (\beta_1(x), \beta_2(x), \cdots)$ we prove β is a fiber preserving homeomorphism. Since $\alpha_n \circ q = p_n \circ \beta_n = p_n \circ \tilde{\alpha}_n \circ \beta = \alpha_n \circ p \circ \beta$ for all n, $p \circ \beta = q$ and β is fiber preserving. Thus to see that β is onto and one-to-one it suffices to show β is onto and one-to-one on fibers. Let $x \in F \subset \bar{X}$. Since $j: \pi_1(X, *) \to \varprojlim \{\pi_1(X_n, *)\} \equiv F$ is onto (see Propositions 5.4 and 5.5) there is a loop λ in X such that the lift of λ to $* \in \bar{X}$, call it λ_1, has endpoint x. I.e., $\lambda_1(0) = *$ and $\lambda_1(1) = x$. Call the lift of λ to $* \in \tilde{X}$, λ_2. Since $p \circ \beta = q$, unique path lifting in $\bar{X}$ implies $\beta \circ \lambda_2 = \lambda_1$ and thus $\beta(\lambda_2(1)) = x$ so β is onto.

If $\beta(x) = \beta(y)$ let γ be a path from x to y. $\beta_n(x) = \beta_n(y)$ for all n, thus $p_n \circ \beta_n \circ \gamma$ is a trivial loop in X_n for all n. By Remark 5.5.1 $p \circ \beta \circ \gamma$ is a trivial loop in X and hence $q\gamma$ is a trivial loop in X implying $x = y$. I.e., β is one-to-one.

THEOREM 5.9. *Let X and Y be* M-L-F *compacta such that $\bar{X}$ is path connected and X is LC°. Then for any map $f: (X, *) \to (Y, *)$ there is a unique map $\bar{f}: (\bar{X}, *) \to (\bar{Y}, *)$ such that $p \circ f = \bar{f} \circ q$. (Here p and q are projections.)*

Proof. Let $j: \pi_1(X, *) \to \varprojlim$ pro-$\pi_1(X, *)$ and $j': \pi_1(Y, *) \to \varprojlim$ pro-$\pi_1(Y, *)$ be the natural maps. Let $\tilde{\alpha}_n: (\bar{X}, *) \to (\tilde{X}_n, *)$ be projection. If λ is a loop at $*$ in X then $\tilde{\alpha}_n \circ \lambda$ is a trivial loop in $(\tilde{X}_n, *)$ and thus $p_n \circ \tilde{\alpha}_n \circ \lambda = \alpha_n \circ p \circ \lambda$ is a trivial loop in $(X_n, *$. Hence $p_\#(\pi_1(\bar{X}, *)) \subset \ker(j)$. A similar argument shows $\ker(j) \subset p_\#(\pi_1(\bar{X}, *))$ and since $p_\#$ is a monomorphism (see [13] p. 72), we have:

REMARK 5.9.1. $\pi_1(\bar{X}, *)$ is isomorphic to $\ker(j)$.

Claim 1. $f_\#(\ker(j)) \subset \ker(j')$.

Proof. Assume X and Y are embedded in the Hilbert Cube, Q. Since Q is an absolute retract f can be extended to a map $f'\colon Q \to Q$. If λ is a loop in X at $*$, representing an element of $\ker(j)$ then for any neighborhood V of X, (in Q) λ is trivial rel$\{0, 1\}$ in V. Thus by the uniform continuity of f', $f \circ \lambda$ is trivial rel$\{0, 1\}$ in any neighborhood of Y.

Let $x \in \bar{X}$ and λ a path from x to $*$. Define $\bar{f}(x)$ to be the end point of the lift of $f \circ p \circ \lambda$ to $* \in Y$. This is a well-defined function (see [13] p. 76), and it suffices to show $\bar{f}$ is continuous.

Claim 2. $\bar{f}|p^{-1}(x)$ is continuous for any $x \in X$.

Proof. Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that if $d(x_1, x_2) < \delta$ (x_1 and x_2 in $p^{-1}(x)$) and λ is a path from x_1 to x_2 then $p \circ \lambda$ is homotopically trivial rel$\{0, 1\}$ in an ε-neighborhood of $X \subset Q$. (Assume X and Y are embedded in Q and f' is as in Claim 1.) By the uniform continuity of f', $\bar{f}|_{p-1(x)}$ is continuous.

Claim 3. Let $\bar{x} \in \bar{X}$ and $\varepsilon > 0$ be given. There is a $\delta > 0$ such that if λ is a path at $x \equiv p(\bar{x})$ and diam.(λ) (The diameter of the image of λ) is less than δ then the lift of δ to $\bar{x}$ has diameter less than ε.

Proof. Define a metric, d, on X by: $d(a, b) \equiv \sum_{i=1}^{\infty} (1/2)^i (d_i(a_i, b_i)/(1 + d_i(a_i, b_i)))$, where d_i is a metric on X_i and $a_i = \alpha_i(a)$. Similarly define a bounded metric on $\bar{X}$ (also denoted by d). For an evenly covered compact neighborhood U_n of x_n there is a compact neighborhood $\tilde{U}_n$ of $\bar{x}_n$ such that $p_n|\tilde{U}_n$ is a uniformly continuous homeomorphism, as is its inverse. Thus for any n, paths of "small" diameter at x_n lift (in $\tilde{X}_n$) to paths of "small" diameter at $\bar{x}_n$. By the uniform continuity of $\alpha_i\colon X \to X_i$, for any $N > 0$ and $\delta_i > 0$ there is a $\delta_2(\delta_1, N) > 0$ such that if a path λ at x has diameter less than $\delta_2(\delta_1, N)$ then diam. $(\alpha_n \circ \lambda) < \delta_1$ for all $n \leqq N$. Choose N such that $(1/2)^N < \varepsilon/2$ and δ_1 such that if β is a path at x_n of diameter less than δ_1 then the lift of β to $\bar{x}_n$ is of diameter less than $\varepsilon/2$. By our choice of metric if λ is a path at x of diameter less than $\delta_2(\delta_1, N)$ then the lift of λ to $\bar{x}$ has diameter less than ε.

Now we show $\bar{f}$ is continuous at $x \in \bar{X}$. Let $\varepsilon > 0$ be given. By the uniform continuity of f and Claim 3 applied to $\bar{Y}$ there is a $\delta_1 > 0$ such that if λ is a path at $p(x)$ of diameter less than δ_1 then the lift of $f \circ \lambda$ to $\bar{f}(x)$ has diameter less than $\varepsilon/2$. By Claim 2 there is a $\delta_2 > 0$ such that if $d(a, x) < \delta_2$ where $a \in p^{-1}(p(x))$ then $d(\bar{f}(a), \bar{f}(x)) < \varepsilon/2$. By Claim 3 there exists $\delta_3 < \delta_1$ such that $\delta_3 > 0$

and if λ is a path at $p(x)$ of diameter less than δ_3 then the lift of λ to x has diameter less than $\delta_2/2$. Since X is LC° there is a $\delta_4 > 0$ such that if $b \in X$ and $d(b, p(x)) < \delta_4$ then there is a path of diameter less than δ_3 from b to $p(x)$. By the continuity of p there is a $\delta_5 > \delta_2/2$ such that $\delta_5 > 0$ and if $d(a, x) < \delta_5$ then $d(p(a), p(x)) < \delta_4$. We show if $d(a, x) < \delta_5$ then $d(\bar{f}(a), \bar{f}(x)) < \varepsilon$. $d(a, x) < \delta_5$ implies there is a path λ from $p(a)$ to $p(x)$ of diameter less than δ_3. Let $z \in p^{-1}(p(x))$ be the end point of λ lifted to a, $d(z, a) < \delta_2/2$ and $d(a, x) < \delta_5 < \delta_2/2$ so $d(z, x) > \delta_2$. Hence $d(\bar{f}(z), \bar{f}(x)) > \varepsilon/2$. Since diam. $(\lambda) < \delta_3 < \delta_1$, the lift of $f \circ \lambda$ to $\bar{f}(a)$ has diameter less than $\varepsilon/2$ and has end point $\bar{f}(z)$. Hence $d(\bar{f}(x), \bar{f}(a)) < \varepsilon$ and δ_5 is the desired bound.

COROLLARY 5.9.2. *If $f: (X, *) \to (Y, *)$ is a homeomorphism of LC° M-L-F compacta and $\bar{X}$ and $\bar{Y}$ are path connected then the induced map $\bar{f}: (\bar{X}, *) \to (\bar{Y}, *)$ is a homeomorphism.*

THEOREM 5.10. *Let $f: (X, *) \to (Y, *)$ be a pointed homotopy equivalence of LC° M-L-F compacta. If $\bar{X}$ and $\bar{Y}$ are path connected then the induced map $\bar{f}: (\bar{X}, *) \to (\bar{Y}, *)$ is a pointed proper homotopy equivalence.*

Proof. By Theorem 5.9 and the homotopy lifting property for fibrations with unique path lifting $\bar{f}$ is a pointed homotopy equivalence. We prove $\bar{f}$ is proper and a similar argument shows that the compositions of $\bar{f}$ with its pointed homotopy inverse are pointedly proper homotopic to the appropriate identity map. Let $U_1, U_2, \cdots, U_n$ be evenly covered compact neighborhoods covering X_1, and let $\tilde{U}_i$ be a homeomorphic copy of U_i in $p_1^{-1}(U_i)$. $\{x \cdot \tilde{U}_i | x \in F_1$ and $i \in \{1, 2, \cdots, n\}\}$ is a locally finite cover of $\tilde{X}_1$. Recall $\tilde{\alpha}_1: (\bar{X}, *) \to (\tilde{X}_1, *)$, is proper. Choose $w_x \in \tilde{\alpha}_1^{-1}(x)$ for each $x \in F_1$ and let $V_i = \tilde{\alpha}_1^{-1}(\tilde{U}_i)$, $\{w_x | x \in F_1\}$ is a discrete subset of F and $\{w_x \cdot V_i | x \in F_1$ and $i \in \{1, 2, \cdots, n\}\}$ is a locally finite cover of $\bar{X}$ by compact neighborhoods. Since $\bar{f}$ is a pointed homotopy equivalence, $\bar{f}$ maps $p^{-1}(*)$ homeomorphically onto $q^{-1}(*)$. The rest of the proof is the same as that of Proposition 5.2 with $\tilde{e}_i$ replaced by V_i and g_i replaced by w_{x_i} where $x_i \in F_1$.

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