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EMBEDDING ASYMPTOTICALLY STABLE DYNAMICAL SYSTEMS INTO RADIAL FLOWS IN l_2

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A dynamical system Π on a separable metric space, which has a globally asymptotically stable critical point p, can be embedded into a radial flow ρ on l_2 if and only if p is uniformaly asymptotically stable. Moreover, if Π can be embedded into ρ, then there is a locally compact subset Y of l_2 such that Π can be embedded into ρ restricted to Y.

In [1] the author showed that a dynamical system on a locally compact phase space, which has a globally asymptically stable critical point, can be embedded into the radial flow on l_2 defined by $z\rho t = c^t z$. Here we generalize this result and show that a dynamical system Π on a separable metric space which has a globally asymptotically stable critical point p, can be embedded into the radial flow ρ on l_2 if and only if p is uniformly asymptotically stable. Moreover, if Π can be embedded into ρ, then there is a locally compact subset Y of l_2 such that Π can be embedded into ρ restricted to Y.

A dynamical system on a topological space X is a continuous mapping $\Pi: X \times R \to X$ such that (where $x\Pi t = \Pi(x, t)$)

(1) $x\Pi 0 = x$ for every $x \in X$,

(2) $(x\Pi t)\Pi s = x\Pi(t + s)$ for every $x \in X$ and $s, t \in R$.

For $A \subset X$ and $B \subset R$, $A\Pi B$ will denote the set $\{x\Pi t: x \in A, t \in B\}$. In the special case $B = R$ we will write $C(A)$ instead of $A\Pi R$. An element $x \in X$ is called a *critical point* of Π if $C(x) = \{x\}$. A subset A of X is *invariant* if $C(A) = A$. We will let R^+ denote the non-negative reals.

A compact subset M of X is called *stable* if for any neighborhood U of M there is a neighborhood V of M such that $V\Pi R^+ \subset U$. A stable subset M of X is called

(i) *asymptotically stable* if for any neighborhood U of M and any $x \in X$, there is a $T \in R$ such that $x\Pi[T, \infty) \subset U$,

(ii) *locally uniformly asymptotically stable* if for any $x \in X - M$, there is a neighborhood V of x such that for any neighborhood U of M there exists $T \in R$ such that $V\Pi[T, \infty) \subset U$.

(iii) *uniformly asymptotically stable* if there is a neighborhood U of M such that for any neighborhood $V \subset U$ of M there exists $T \in R$ such that $U\Pi[T, \infty) \subset V$.

A continuous function $L: R \to R^+$ is called a *Liapunov function* for a subset M of X if

(i) $L(x) = 0$ if and only if $x \in M$,

(ii) $L(x\Pi t) < L(x)$ for every $x \in X - M$ and $0 < t$,

(iii) for any neighborhood U of M there is an $\varepsilon > 0$ such that $\varepsilon < L(x)$ whenever $x \notin U$,

(iv) for any $\varepsilon > 0$ there is a neighborhood V of M such that $L(x) < \varepsilon$ whenever $x \in V$.

In [2] it is shown that a compact subset M of a metric space is asymptotically stable if and only if there is a Liapunov function for M.

Throughout this paper X will denote a separable metric space with metric d and we will assume that $d(x, y) \leqq 1$ for every $x, y \in X$. For $x \in X$ and $\varepsilon > 0$ the set $\{y \in X: d(x, y) \leqq \varepsilon\}$ will be denoted by $B(x, \varepsilon)$.

The set of all sequences $z = \{z_1, z_2, \cdots, z_n, \cdots\}$ of real numbers such that $\sum_{m=1}^{\infty} z_m^2$ converges is denoted by l_2. A norm can be defined on l_2 by $||z|| = (\sum_{m=1}^{\infty} z_m^2)$. The origin in l_2 will be denoted by $\bar{0}$. Let ρ denote the dynamical system on l_2 defined by $z\rho t = c^t z$, where $c \in (0, 1)$.

Let p be a uniformly asymptotically stable critical point of a dynamical system Π on a separable metric space X. Let U be a neighborhood of p such that for any neighborhood $V \subset U$ of p there is a $T > 0$ so that $U\Pi[T, \infty) \subset V$.

LEMMA 1. *$C(x) \cap (X - U) \neq \emptyset$ for every $x \in X - \{p\}$.*

Proof. Let $x \in X - \{p\}$. Since $C(x) \cap U = \emptyset$, we may assume that $x \in U$. Let V be a positively invariant neighborhood of p such that $x \notin V$ and $V \subset U$. Then $(x\Pi(-\infty, 0]) \cap V = \emptyset$. Let $T > 0$ be such that $U\Pi[T, \infty) \subset V$. If $C(x) \cap (X - U) = \emptyset$, then $x\Pi(-\infty, T) = (x\Pi(-\infty, 0])\Pi T \subset V$ which is impossible since $(x\Pi(-\infty, 0]) \cap V = \emptyset$. It follows that $C(x) \cap (X - U) \neq \emptyset$ for every $x \in X - \{p\}$.

It is known that there is a Liapunov function L for the uniformly aymptotically stable critical point p, [2]. Let λ be a number in the range of L such that $L^{-1}(\lambda) \subset U$ and set $S = L^{-1}(\lambda)$. It is easy to verify that S is a section for Π restricted to $X - \{p\}$. Since X is separable there is a countable dense subset $\{x_n\}$ of S. Define a countable number of continuous functions $f_n: S \to R^n$ by

$$f_n(x) = d(x, x_n)$$

where d is a metric on X such that $d(x, y) \leqq 1$ for all $x, y \in X$.

LEMMA 2. *If $f_n(x) \leqq f_n(y)$ for every n, then $x = y$.*

Proof. Suppose that $x \neq y$. Set $r = d(x, y)$ and $B = \{z: d(z, y) \leqq r/4\}$. Since $\{x_n\}$ is dense in S there is a k such that $x_k \in B$. Then $f_k(y) = d(y, x_k) \leqq r/4 \leqq 3d(x, x_k)/4 < f_k(x)$. A similar argument shows that there is a j such that $f_j(x) < f_j(y)$. The desired result follows directly.

LEMMA 3. *The mapping $h: S \to l_2$ defined by*

$$h(x) = \left(f_1(x), \frac{1}{2}f_2(x), \cdots, \frac{1}{n}f_n(x), \cdots\right)$$

is a homeomorphism of S onto $h(S)$. Moreover, $||h(x)||^2 \leqq \Pi^2/6$ for every $x \in S$.

Proof. Let $x \in S$ and $\varepsilon > 0$. For any $y \in B(x, \varepsilon)$, we have $d(x, x_n) - \varepsilon \leqq d(y, x_n) \leqq d(x, x_n) + \varepsilon$. Hence, for every n we have $|f_n(x) - f_n(y)| \leqq \varepsilon$ whenever $y \in B(x, \varepsilon)$. This shows that $\{f_n\}$ is uniformly equicontinuous. It is now easy to show that h is continuous. By Lemma 2 the mapping h is one-to-one. Suppose there is a sequence $\{z_i\}$ in S such that $h(z_i) \to h(z)$ for some $z \in S$. Then $f_n(z_i) \to f_n(z)$ for every n, i.e., $d(z_i, x_n) \to d(z, x_n)$ for every n. Let $\delta > 0$ and choose j so that $d(z, x_j) < \delta/4$. Since $d(z_i, x_j) \to d(z, x_j)$ we have $d(z_i, z) \leqq d(z_i, x_j) + d(z, x_j) < \delta$ for all i sufficiently large. It follows that $z_i \to z$ so that h^{-1} is continuous. Thus, h is a homeomorphism of S onto $h(S)$. Since $d(u, v) \leqq 1$ for every $u, v \in X$, we have $||h(x)||_2^2 \leqq \sum_{m=1}^{\infty} m^{-2} = \Pi^2/6$ for every $x \in S$.

LEMMA 4. *If $x, y \in S$ are such that $h(x) = h(y)\rho t$ for some $t \in R$, then $x = y$ and $t = 0$.*

Proof. Suppose that $h(x) = h(y)\rho t = c^t h(y)$ for some $t \in R$. Without loss of generality we may assume that $t \geqq 0$. Then $f_n(x) = c^t f_n(y) \leqq f_n(y)$ for every n. By Lemma 2, $x = y$. If $x = y$, clearly $t = 0$.

LEMMA 5. *The mapping $H: X \to l_2$ defined by*

$$H(x) = \begin{cases} \bar{0} & \text{if } x = p\ , \\ h(x\Pi\Upsilon(x))\rho(-\Upsilon(x)) & \text{if } x \in X - \{p\}\ , \end{cases}$$

where $\Upsilon: X - \{p\} \to R$ is a continuous mapping defined by $x\Pi\Upsilon(x) \in S$, is a homeomorphism.

Proof. If $x \neq p$, then clearly $H(x) \neq \bar{0} = H(p)$. If $H(x) = H(y)$ with $x \neq p \neq y$, then $h(x\Pi\Upsilon(x))\rho(-\Upsilon(x)) = h(y\Pi\Upsilon(y))\rho(-\Upsilon(y))$ so that

$h(x\Pi\Upsilon(x)) = h(y\Pi\Upsilon(y))\rho(\Upsilon(x) - \Upsilon(y))$. By Lemma 4 we have $h(x\Pi\Upsilon(x)) = h(y\Pi\Upsilon(y))$ and $\Upsilon(x) = \Upsilon(y)$. Since h is one-to-one $x\Pi\Upsilon(x) = y\Pi\Upsilon(y)$. Hence, $x = y$ and H is one-to-one. Since h, Π, Υ, and p are continuous, H is continuous on $X - \{p\}$. We will now show that H is continuous at p. Let $\{z_i\}$ be any sequence in $X - \{p\}$ which converges to p. Since p is stable there is a neighborhood W of ρ such that $W\Pi R^+ \subset L^{-1}([0, \lambda/2])$. Hence, $\Upsilon(x_i) \leqq 0$ for all i sufficiently large. It suffices to consider two cases: $\Upsilon(z_i) \to -\infty$ and $\Upsilon(z_i) \to t \leqq 0$. If $\Upsilon(z_i) \to -\infty$, then $H(z_i) \to \bar{0}$ since $\|h(z_i\Pi\Upsilon(z_i))\| \leqq \Pi^2/6$ for each i and $\|H(z_i)\| = \|h(z_i\Pi\Upsilon(z_i))\rho(-\Upsilon(z_i))\| = c^{-\Upsilon(z_i)}\|h(z_i\Pi\Upsilon(z_i))\| \to 0$. If $\Upsilon(z_i) \to t$ then $0 \neq \lambda = L(z_i\Pi\Upsilon_i(z_i)) \to L(p\Pi t) = L(p) = 0$ which is impossible. Thus, H is continuous. A short calculation shows that $H^{-1}(H(x)) = h^{-1}[H(x)\rho\Upsilon(x)]\Pi(-\Upsilon(x))$ whenever $x \neq p$. Since h^{-1}, H, ρ, Υ, and Π are continuous, H^{-1} is continuous on $H(X) - \{\bar{0}\}$. Let $\{y_i\}$ be any sequence in $X - \{p\}$ such that $H(y_i) \to \bar{0}$. Since $H(y_i) = c^{-\Upsilon(y_i)}h(y_i\Pi\Upsilon(y_i))$ we must have either $\Upsilon(y_i) \to -\infty$ or $h(y_i\Pi\Upsilon(y_i)) \to \bar{0}$. If $h(y_i\Pi\Upsilon(y_i)) \to \bar{0}$, then $d(y_i\Pi\Upsilon(y_i), x_n) \to 0$ for every n, which is impossible. Hence $\Upsilon(y_i) \to -\infty$. Recall that $S = L^{-1}(\lambda) \subset U$, where U is a neighborhood of p such that for any neighborhood $V \subset U$ of p there is a T so that $U\Pi[T, \infty) \subset V$. Then $y_i = (y_i\Pi\Upsilon(y_i))\Pi(-\Upsilon(y_i)) \in U\Pi[-\Upsilon(y_i), \infty)$. From our choice of U and the fact that $\Upsilon(y_i) \to -\infty$, we have $y_i \to p$. The continuity of H^{-1} follows directly. H is a homeomorphism of X onto $H(X)$.

THEOREM 5. *Let Π be a dynamical system on a separable metric space X which has a globally assymptotically stable critical point p. Let $c \in (0, 1)$ and ρ be the dynamical system on l_2 defined by $x\rho t = c^t x$. Then Π can be embedded into ρ if and only if ρ is uniformly asymptotically stable.*

Proof. Suppose that Π can be embedded into ρ. Evidently the origin is uniformly asymptotically stable with respect to ρ. Since Π is embedded into ρ, it is easy to show that p is uniformly asymptotically stable. Now suppose that p is uniformly asymptotically stable. In light of Lemma 4 it remains to show that $H(x\Pi t) = H(x)\rho t$. It is easy to show that $\Upsilon(x\Pi t) = \Upsilon(x) - t$. Hence,

$$\begin{aligned} H(x\Pi t) &= h((x\Pi t)\Pi\Upsilon(x\Pi t))\rho(-\Upsilon(x\Pi t)) \\ &= h(x\Pi\Upsilon(x))\rho(-\Upsilon(x) + t) \\ &= (h(x\Pi\Upsilon(x))\rho(-\Upsilon(x)))\rho t \\ &= H(x)\rho t \end{aligned}$$

for every $x \neq p$ and $t \in R$. Clearly $H(p\Pi t) = H(p) = \bar{0} = \bar{0}\rho t$ for every $t \in R$.

COROLLARY 6. ([1]) *Let Π be a dynamical system on a locally compact space X. If Π has a globally asymptotically stable critical point, then Π can be embedded into ρ.*

LEMMA 7. *Let A be a compact subset of l_2 with $\bar{0} \notin A$. Then $(A\rho R) \cup \{\bar{0}\}$ is locally compact in the relative topology.*

Proof. Since A is a compact with $\bar{0} \notin A$, for any N, $\varepsilon > 0$ there are $t_1, t_2 \in R$ such that $\|A\rho t\| > N$ for $t < t_1$ and $\|A\rho t\| < \varepsilon$ for $t_2 < t$. It easily follows that $A\rho R$ is locally compact since $A\rho B$ is compact whenever B is a compact subset of R. Next we will show that $(A\rho R^+) \cup \{\bar{0}\}$ is a compact neighborhood of $\bar{0}$ in $A\rho R$. Clearly $A\rho R^+$ is a neighborhood of $\bar{0}$ in $(A\rho R) \cup \{\bar{0}\}$. Let $\{x_i\}$ and $\{t_i\}$ be any sequences in A and R^+ respectively. Without loss of generality we may assume that there is an $x \in A$ such that $x_i \to x$. If $\{t_i\}$ has an accumulation point t, then $x\rho t$ is an accumulation point of $\{x_i \rho t_i\}$. If $t_i \to \infty$, then $x_i \rho t_i \in A\rho t_i \to \bar{0}$. It follows that any sequence in $(A\rho R^+) \cup \{\bar{0}\}$ has an accumulation. Hence, $(A\rho R^+) \cup \{\bar{0}\}$ is compact. The desired result follows immediately.

THEOREM 8. *Let Π be a dynamical system on a separable metric space X which has a globally asymptotically stable critical point p. Let $c \in (0, 1)$ and ρ be the dynamical system on l_2 defined by $x\rho t = c^t x$. If Π can be embedded into ρ, then there exists a locally compact subset Y of l_2 such that Π can be embedded into ρ restricted to Y.*

Proof. Let the notation be as before. Evidently $h(S)$ is a subset of the Hilbert cube, $T = \{x \in l_2 \colon x = (x_1, x_2, \cdots, x_n, \cdots)$ with $|x_n| \leqq n^{-1}$ for each $n\}$, which is a compact subset of l_2. Since $S = L^{-1}(\lambda)$, the section S is a closed subset of X with $p \notin S$. Hence $\bar{0} \notin h(S)$. Since $\overline{h(S)}$ is a closed subset of T, it is compact. Set $Y = \overline{(h(S)\rho R)} \cup \{\bar{0}\}$. By Lemma 7, Y is a locally compact subset of l_2. Clearly $H(X) \subset Y$. The desired result follows directly.

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