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ON INTERPOLATION OF $L_p[a, b]$ AND WEIGHTED SOBOLEV SPACES

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The goal of this paper is to characterize the interpolation spaces between $L_p[a, b]$ or $C[a, b]$ and the space of functions for which $W(x)f^{(r)}(x)$ belongs to $L_p[a, b]$ or $C[a, b]$. In order to achieve this, for a class of weights $W(x)$ the Peetre K functional is characterized.

We recall that the Peetre K functional on $f \in B_1 + B_2$ where B_i are Banach spaces, both of which are contained in a linear Hausdorff space, is given by

$$K(\tau, f) \equiv \inf_{f=f_1+f_2} (\|f_1\|_{B_1} + \tau\|f_2\|_{B_2}) . \tag{1.1}$$

The Peetre interpolation spaces $(B_1, B_2)_{\theta,q;K}$ for $0 \leqq \theta \leqq 1$ and $1 \leqq q \leqq \infty$ are given by their norms

$$\|f\|_{\theta;K} \equiv \|f\|_{\theta,\infty;K} = \sup_{\tau>0} \tau^{-\theta}K(\tau, f) \tag{1.2}$$

and

$$\|f\|_{\theta,q;K} = \left\{\int_0^\infty (\tau^{-\theta}K(\tau, f))^q \frac{d\tau}{\tau}\right\}^{1/q} \quad \text{for } 1 \leqq q < \infty . \tag{1.3}$$

It is therefore obvious that to find a characterization of the space $(B_1, B_2)_{\theta,q;K}$ it is enough to characterize the functional $K(\tau, f)$ in terms of $f(x)$. It can be noted that sometimes a natural condition can be given for a function to belong to a specific $(B_1, B_2)_{\theta,q;K}$ without going through the function (see [4]), but it is preferable to attain a description of $K(\tau, f)$, since that will yield results for all $1 \leqq q \leqq \infty$ simultaneously. In this paper $f \in B_1$, and therefore $K(\tau, f) = \inf_g (\|f - g\|_{B_1} + \|g\|_{B_2})$. Moreover, for the sake of convenience, we shall substitute $\tau = t^r$.

The functionals in which we are interested, $K_*(t^r, f)$ and $K(t^r, f)$ are given by:

$$K_*(t^r, f) = \inf_g (\|f - g\|_B + t^r(\|g\|_B + \|W(\cdot)^r g^{(r)}(\cdot)\|_B) \tag{1.4}$$

and

$$K(t^r, f) = \inf_g (\|f - g\|_B + t^r(\|W(\cdot)^r g^{(r)}(\cdot)\|_B) \tag{1.5}$$

where B is $L_p[a, b]$ or $C[a, b]$ and where $g^{(r)}$ exists except perhaps at zeros of $W(x)$, and $g^{(r-1)}$ is locally absolutely continuous for $x \in$

$[a, b]\backslash\{x_0; w(x_0) = 0\}$. Using the K_* and K functionals of (1.4) and (1.5), in (1.2) and (1.3), we have the norm $\|f\|_{\theta,q;K_*}$ and seminorm $\|f\|_{\theta,q;K}$ respectively. For $\theta > 0$, $\|f\|_{\theta,q;K_*}$ is bounded, that is, f belongs to the interpolation space, if and only if $\|f\|_{\theta,q;K}$ is bounded. This follows the simple observations that: (a) $K(\tau, f) \leqq \|f\|$ and $K_*(\tau, f) \leqq \|f\|$; and, since g in both (1.4) and (1.5) can be chosen among $\|g\| \leqq 2\|f\|$ (otherwise $g = 0$ would yield a smaller number), then (b) $K_*(\tau, f) \geqq K(\tau, f) \geqq K_*(\tau, f) - 2\|f\|\tau$. For $\theta > 0$, in (1.2) when the supremum is taken on $\tau > \delta$ and in (1.3) when the integral is $\int_\delta^\infty$, the estimate (a) would imply boundedness. For $\theta > 0$ (b) would imply, for $\tau \leqq \delta$, that the difference between the expressions with K and K_* is bounded.

We shall solve the problem for $W(x)$ having finitely many zeros x_i for which $A_1|x - x_i|^{\alpha_{ij}} \leqq W(x) \leqq A_2|x - x_i|^{\alpha_{ij}}$ for $x < x_i$ or $x > x_i$ when $j = 1$ or 2 respectively. Actually in § 2 we shall show how to reduce the question to that of characterization of $K(t^r, f)$ when the function is defined on $[0, 1]$ and its support is in $[0, 3/4]$ and where the weight function is $W(x) = x^\alpha$. We shall solve this main problem in § 3 for continuous functions and in § 4 for L_p functions. We shall later, in § 5, fully state the general result for the characterization of K. We shall also state the actual interpolation results as a corollary.

For $C[0, 1]$, $W(x) = x^\alpha$ and $\omega_r^*(f, h)$ given by

$$\omega_r^*(f, h) = \operatorname*{Sup}_{\eta \leqq h} \operatorname*{Sup}_{(r/2)\eta < x^{1-\alpha}} |\Delta_{\eta x^\alpha}^r f(x)| \tag{1.6}$$

where $\Delta_t^r f(x) = \Delta_t(\Delta_t^{r-1} f(x))$ and $\Delta_t f(x) = f(x + t/2) - f(x - t/2)$ we will have the relation

$$C_1\omega_r^*(f, t) \leqq K(t^r, f) \leqq C_2\omega_r^*(f, t) \text{ for } 0 < t < \delta\,. \tag{1.7}$$

It is clear that away from the singularity 0 $\omega_r^*(f, t)$ behaves like a modulus of continuity while near 0 much smaller differences are taken, in other words, for $\omega_r^*(f, h)$ to be small the function has to be much less smooth near 0 than away from 0. For example, $f(x) = x^{1/3}$ and $\alpha = 1/2$ will yield $\omega_1^*(f, t) \sim ct^{2/3}$. The result in (1.7), which will be proved in § 3, can be stated also as the following interpolation theorem.

THEOREM. *Let* $f(x) \in C[0, 1]$, Supp $f \subset [0, 3/4]$ *and* A_r *be given by* $A_r = \{f \in C[0, 1]; x^{r\alpha} f^{(r)}(x) \in C[0, 1], f^{(r-1)}$ *is locally absolutely continuous*$\}$ *then* $f \in (C, A_r)_{\theta,K_*}$ *for* $0 \leqq \theta \leqq 1$ *or* $f \in (C, A_r)_{\theta,q,K_*}$ *for* $0 < \theta \leqq 1$ *and* $1 \leqq q < \infty$ *if and only if* $t^{-r\theta}\omega_r^*(f, t)$ *is bounded for* $t<\delta$ *or* $\int_0^\delta (t^{-r\theta}\omega_r^*(f, t))^q dt/t$ *is bounded, respectively where* $\omega_r^*(f, t)$ *is given* *by* (1.6).

For L_p the expression of $\omega_r^*(f, t)$ is somewhat more complicated and the exact characterization of $K(t^r, f)$ will be given in § 4 for the above $W(x)$.

The problem of interpolation between $\|f\|_{B[a,b]}$ and $\|f^{(r)}\|_{B[a,b]}$ where $B = L_p$ (or C) i.e., the case $W(x) = 1$ was solved and treated extensively. (See for instance [3] and [5].)

The problem of interpolation between $L_p(\nu)$ and $L_p(\mu)$ was solved by Stein and Weiss [6] which covers in general the case where no derivatives are involved.

For $C[a, b] = C[0, 1]$ and $W(x) = (x(1-x))^{1/2}$ a characterization of the class $\{f; K(t^{2r}, f)/t^\beta = 0(1),\ t \to 0\}$ was given by the author [4] in order to characterize the class of functions for which Bernstein polynomials of $f(x)$ and their combinations converge to $f(x)$ at a certain rate.

For this particular case the present paper yields a different (but equivalent) result and in addition here the K functional is characterized and not only the class $\{f: K(t^{2r}, f)/t^\beta = 0(1)\}$. It is clear that the difference between K_* and K is bounded by $2\|f\|t^r$ and the cases of interest would occur when $t^r = o(K(t^r, f))$, $t \to 0+$.

2. Some simplifications. We first observe that if $0 < A_1 \leqq W(x) \leqq A_2$

$$K_{W^*}(t^r, f) = \inf_g (\|f - g\|_B + t^r(\|g\|_B + \|W^r g^{(r)}\|)_B) \tag{2.1}$$

where B is $L_p[a, b]$ or $C[a, b]$ and

$$K_{1^*}(t^r, f) = \inf_g (\|f - g\| + t^r(\|g\|_B + \|g^{(r)}\|_B))$$

are equivalent norms independent of t and therefore the situation in which a continuous $W(x)$ has no zero does not interest us in this paper since it has already been solved and discussed elsewhere.

One can mention here that if $W(x)$ is equal to zero on a subinterval of $[a, b]$ the values of f in that subinterval will not affect $K_W(t^r, f)$. In any case the treatment in this paper is for $W(x)$ having only isolated zeros x_i satisfying $A_1|x - x_i|^\alpha \leqq W(x) \leqq A_2|x - x_i|^\alpha$ for x either only on one side of x_i for that or on both sides.

We can define

$$K_i(t^r, f) = \inf_g [\|f - g\|_{B[x_i, x_{i+1}]} + t^r(\|g(x)\|_{B[x_i, x_{i+1}]} + \|W(x)^r g^{(r)}(x)\|_{B[x_i, x_{i+1}]})]$$

where x_i, x_{i+1} are consecutive zeros of $W(x)$ or one of them may be an edge of $[a, b]$ even in case a or b are not zeros of $W(x)$. We observe

$$K_*(t^r, f) = \sum_{i=1}^{n} K_i(t^r, f) .$$

That $K_*(t^r, f) \leqq \sum \cdots$ is clear from the definition of the K functionals being infimums, and the inequality in the other direction follows, since when g, chosen for $[x_i, x_{i+1}]$ it does not affect its choice elsewhere. In fact there is no relation between $K_i(t^r, f)$ and $K_j(t^r, f)(i \neq j)$ and all the information of $f(x)$ can be derived separately.

Moreover, if (a, b) is infinite, that is $a = -\infty$ or $b = \infty$ or both, and x_i are infinitely many zeros of $W(x)$ that do not have an accumulation point, we still have $K_*(t^r, f) = \sum_{i=0}^{\infty} K_i(t^r, f)$.

For a single K_i a linear transformation can bring $[x_i, x_{i+1}]$ to $[0, 1]$.

To simplify even further we would like to separate the problem into two symmetric problems near 0 and near 1.

For that we shall define the C^∞ function $\psi_1(x)$ $0 \leqq \psi_1(x) \leqq 1$, $\psi_1(x) = 1$ on $[0, 1/4]$ and $\psi_1(x) = 0$ on $[3/4, 1]$. Recalling

$$K_*(t^r, f) = \inf_g (||f - g|| + t^r(||g|| + ||W^r g^{(r)}(\cdot)||))$$

we have

$$K_*(t^r, f) \leqq K_*(t^r, f\psi_1) + K_*(t^r, f(1 - \psi_1)) .$$

We shall show

$$(2.2) \quad K_*(t^r, f \cdot \psi_1) \leqq MK_*(t^r, f), \; K_*(t^r, f(1 - \psi_1)) \leqq MK_*(t^r, f) .$$

Therefore characterization of $K_*(t^r, f\psi_1)$ and $K_*(t^r, f(1 - \psi_1))$ separately will suffice. This is the only point where K_* (rather than K) is used since when $f = g$ and $g^{(r)} = 0$ $(g\psi_1)^{(r)}$ is not necessarily equal to zero.

To prove (2.3) we shall need the following lemma.

LEMMA 2.1. *If $f, f^{(r)} \in L_p[a, b]$ $1 \leqq p < \infty$ or $C[a, b]$, ($f^{(r-1)}$ is locally absolutely continuous), then for $0 < k < r$*

$$(2.4) \qquad ||f^{(k)}||_p \leqq M\left(\frac{||f||_p}{(b - a)^k}\right) + (b - a)^{r-k}||f^{(r)}||_p\Big)$$

where M does not depend on p nor on $[a, b]$.

The lemma is well-known (see Adams [2, p. 81]) if M can depend on p and $[a, b]$, which would suffice for this section but not for the following sections. With M not depending on p or $[a, b]$ I was not able to find a reference, so a simple proof is enclosed. For the space $C[a, b]$ the validity of Lemma 2.1 was mentioned to me by S. Riemenschneider who has a different proof (just for $C[a, b]$) using B-splines.

Using Lemma 2.1 we now prove (2.3). There exists g_t satisfying $\|f - g_t\| + t^r(\|g_t\| + \|W^r g_t^{(r)}\|) \leqq 2K_*(t^r, f)$. Therefore

$$\begin{aligned} K_*(t^r, f\psi_1) &\leqq \|f\psi_1 - g_t\psi_1\| + t^r(\|g_t\psi_1\| + \|W^r(g_t\psi_1)^{(r)}\|) \leqq \|f - g_t\| \\ &+ t^r\|W^r g_t^{(r)}\|_{B[0,1/4]} + t^r\|g_t\|_{B[0,1]} + t^r\|W^r(g_t\psi_1)^{(r)}\|_{B[1/4,3/4]} \leqq 2K_*(t^r, f) \\ &+ t^r \max_{1/4 \leqq x \leqq 3/4} W(x)^r \cdot \sum \binom{r}{k} \|g_t^{(k)}\|_{B[1/4,3/4]} \|\psi_1^{(r-k)}\|_\infty \leqq 2K_*(t^r, f) \\ &+ t^r M(\|g_t^{(r)}\|_{B[1/4,3/4]} + \|g_t\|_{B[1/4,3/4]}) \leqq 2K_*(t^r, f) \\ &+ t^r M_1 \|W(x)^r g_t^{(r)}\|_{B[1/4,3/4]} + t^r M\|g_t\|_{B[0,1]} \leqq M_2 K_*(t^r, f) . \end{aligned}$$

In fact we have shown a little more, that is

$$K_*(t^r, f_1) \leqq M_2 \inf_g (\|f - g\|_{B[0,3/4]} + t^r(\|g\|_{B[0,3/4]} + \|W(x)^r g^{(r)}(\cdot)\|_{B[0,3/4]}))$$

and a similar estimate for $K_*(t^r, f(1 - \psi_1))$ and the interval $[1/4, 1]$.

In this section we show the equivalence treating different $K_*(t^r, f)$. In what follows $K(t^r, f)$ will be used rather than K_*, but the difference is at most $O(t^r)$ so that our result will relate to K_* only if $t^r = O(K(t^r, f))$ (in which case $t^r = O(K_*(t^r, f))$ too).

Proof of Lemma 2.1. We first observe that instead of proving for $0 < k < n$

$$(2.5) \qquad \|f^{(k)}\|_B \leqq M(n, k)\{(b - a)^{-k}\|f\|_B + (b - a)^{n-k}\|f^{(n)}\|_B\} ,$$

it is enough to show

$$(2.6) \qquad \|f^{(k)}\|_B \leqq M(k)\{(b - a)^{-k}\|f\|_B + (b - a)\|f^{(k+1)}\|_B\} ,$$

that is (2.5) with $n = k + 1$ since (2.5) follows (2.6) by induction. For $a \leqq x \leqq (a + b)/2$ and $h = (b - a)/2k$ we use the Taylor formula with integral remainder that for locally integrable $f^{(k+1)}$ with $f^{(k)}$ locally absolutely continuous is given by

$$(2.7) \qquad \begin{aligned} f(x + jh) &= f(x) + \frac{jh}{1!}f'(x) + \cdots + \frac{(jh)^k}{k!}f^{(k)}(x) \\ &+ \frac{1}{k!}\int_0^{jh} (jh - u)^k f^{(k+1)}(x + u)\,\mathrm{du} \end{aligned}$$

to obtain

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f(x+jh)=h^kf^{(k)}(x) \tag{2.8}$$
$$+\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}\int_0^{jh}(jh-u)^kf^{(k+1)}(x+u)\,\mathrm{du}\,.$$

Therefore $f, f^{(k+1)}\in L_p[a, b]$ (or $C[a, b]$) implies $f^{(k)}\in L_p[a, (a+b)/2]$ (or $C[a, (a+b)/2]$) and

$$h^k\|f^{(k)}\|_{L_p[a,a+b/2]}\leqq 2^k\|f\|_{L_p[a,b]}$$
$$+\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}\left\{\int_a^{(a+b)/2}\left|\int_0^{jh}(jh-u)^kf^{(k+1)}(x+u)du\right|^p dx\right\}^{1/p}$$
$$\leqq 2^k\|f\|_{L_p[a,b]}+\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}\frac{(jh)^{k+1}}{k+1}\|f^{(k+1)}\|_{L_p[a,b]}\,.$$

This can be written as

$$\|f^{(k)}\|_{L_p[a,(a+b)/2]}\leqq 2^k(2k)^k\cdot(b-a)^{-k}\|f\|_{L_p[a,b]} \tag{2.9}$$
$$+\frac{1}{(k+1)!}\frac{2^kk^{k+1}}{2k}(b-a)\|f^{(k+1)}\|_{L_p[a,b]}\,.$$

Using $h=-(b-a)/2k$ we obtain a similar estimate for $\|f^{(k)}\|_{L_p[a+b/2,b]}$ or $\|f^{(k)}\|_{C[a+b/2,b]}$, and combining both we obtain (2.6) with the constants in (2.9) for $C[a, b]$ and with twice those constants for L_p. (The exact constants which we arrived at are not important since they are not the best possible.)

3. The $C[0, 1]$ case. In this section functions $f\in C[0, 1]$ for which $\operatorname{Supp} f\in[0, 3/4]$ are investigated but, as discussed in § 2, it is clear that $f\in C[0, 1]$ in general is actually being treated and the condition $\operatorname{Supp} f\subset[0, 3/4]$ is just for convenience.

THEOREM 3.1. *Suppose* $f(x)\in C[0, 1]$, $\operatorname{Supp} f\subset[0, 3/4]$ *and let*

$$K(t^r, f)\equiv\inf_g(\|f-g\|_{C[0,1]}+t^r\|x^{r\alpha}g^{(r)}(\cdot)\|_{C[0,1]}) \tag{3.1}$$

and

$$\omega_r^*(f, h)\equiv\operatorname*{Sup}_{\eta<h}\ \operatorname*{Sup}_{r/2\eta<x^{1-\alpha}}|\Delta^r_{\eta x^\alpha}f(x)|,\ \Delta_\zeta f(x)\equiv f\Big(x+\frac{\zeta}{2}\Big)-f\Big(x-\frac{\zeta}{2}\Big)\,, \tag{3.2}$$

then for $\alpha>0$

$$M_1\omega_r^*(f, t)\leqq K(t^r, f)\leqq M_2\omega_r^*(f, t) \tag{3.3}$$

where M_1 *and* M_2 *depend on* r *and* α *but not on* f *and* t.

Proof. First we will show $M_1\omega_r^*(f, t) \leqq K(t^r, f)$. There exists g_t satisfying $\|f - g_t\| + t^r\|x^{r\alpha}g_t^{(r)}(x)\| \leqq 2K(t^r, f)$. We have

$$\omega_r^*(f, h) \leqq \omega_r^*(f - g_t, h) + \omega_r^*(g_t, h)$$

and clearly $\omega_r^*(f - g_t, h) \leqq 2^r\|f - g_t\| \leqq 2^{r+1}K(t^r, f)$. To estimate $\omega_r^*(g_t, h)$ we note that $r\eta/2 < x^{1-\alpha}$ always and therefore we can estimate $\Delta^r_{\eta x^\alpha} f$ for $r\eta \leqq x^{1-\alpha}$ and for $r\eta/2 < x^{1-\alpha} \leqq r\eta$ separately. We observe also that for $\alpha \geqq 1$ h can be chosen so small that the first case ($r\eta \leqq x^{1-\alpha}$) always applies.

For $x^{1-\alpha} \geqq r\eta$ and $\eta \leqq h = t$ we write

$$|\Delta^r_{\eta x^\alpha} f(x)| = |\eta^r x^{r\alpha} g_t^{(r)}(\xi)| \leqq t^r \left|\frac{x}{\xi}\right|^{r\alpha} |\xi^{r\alpha} g_t^{(r)}(\xi)| \leqq 2^{r\alpha}\cdot 2K(t^r, f)$$

since $x - (r/2)\eta < \xi < x + (r/2)\eta$ and $|x/\xi| < 2$.

Estimating $\omega_r^*(g_t, h)$ for $r\eta/2 < x^{1-\alpha} < r\eta$ (in which case only $\alpha < 1$ has to be considered), we have using Taylor's formula

$$|\Delta^r_{\eta x^\alpha} g_t(x)| \leqq \sum_{l=0}^{r} \binom{r}{l} \frac{1}{(r-1)!} \left| \int_x^{x+(l-r/2)\eta x^\alpha} \left(x + \left(l - \frac{r}{2}\right)\eta x^\alpha - u\right)^{r-1} g_t^{(r)}(u)\, du \right|$$

$$\leqq \|u^{r\alpha} g_t^{(r)}(u)\| \frac{2^r}{(r-1)!} \max_{0 \leqq l \leqq r} \left| \int_x^{x+(l-r/2)\eta x^\alpha} \frac{(x + (l - r/2)\eta x^\alpha - u)^{r-1}}{u^{r\alpha}} du \right| .$$

For $l > r/2$

$$\left| \int_x^{x+(l-r/2)\eta x^\alpha} \frac{(x + (l - r/2)\eta x^\alpha - u)^{r-1}}{u^{r\alpha}} du \right| \leqq \frac{[(l - r/2)\eta x^\alpha]^r}{x^{r\alpha}} = \left(l - \frac{r}{2}\right)^r \eta^r .$$

For $l = r/2$ the above is zero. For $l < r/2$, we have, since $x + (l - r/2)\eta x^\alpha > 0$,

$$\left| \int_x^{x+(l-r/2)\eta x^\alpha} \frac{(x + (l - r/2)\eta x^\alpha - u)^{r-1}}{u^{r\alpha}} \right| du \leqq \int_0^x u^{r-r\alpha-1} du = \frac{1}{r(1-\alpha)} x^{r(1-\alpha)} < \frac{1}{r(1-\alpha)} r^r \eta^r .$$

Therefore using $\eta \leqq t$

$$|\Delta^r_{\eta x^\alpha} g_t(x)| \leqq K(t^r, f)\frac{\eta^r}{t^r} \cdot \frac{2^r}{(r-1)!} \max\left(\left(r - \frac{r}{2}\right)^r, \frac{1}{r(1-\alpha)} r^r\right) \leqq MK(t^r, f) .$$

To prove now $K(t^r, f) \leqq M_2\omega_2^*(f, t)$ we construct $g_t(x)$ such that

$$\|f - g_t\|_{C[0,1]} + t^r\|x^{\alpha r} g_t^{(r)}\| \leqq M_2\omega_r^*(f, t) .$$

To accomplish the construction of g_t we have to define the functions $\psi_l(x) \equiv \psi(4^l x)$ where $\psi(x) \in C^\infty$, $0 \leqq \psi(x) \leqq 1$, $\psi(x)$ is decreasing, $\psi(x) = 1$ $x \leqq 1$ and $\psi(x) = 0$ $x \geqq 3$.

We also construct

$$(3.4)\quad f_h(x)=\left(\frac{r}{h}\right)^r \int_0^{h/r} \cdots \int_0^{h/r} \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} f(x+k(u_1 \cdots + u_r))du_1 \cdots du_r$$

and

$$(3.5)\quad \begin{aligned} f_h^*(x) &= \left(\frac{r}{h}\right)^r \left[1 - \left(\frac{1}{2}\right)^r\right]^{-1} \\ &\times \int_{h/2r}^{h/r} \cdots \int_{h/2r}^{h/r} \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} f(x+k(u_1 + \cdots + u_r))du_1 \cdots du_r \,. \end{aligned}$$

For $\alpha < 1$ and t satisfying $4^{-(l+1)(1-\alpha)} < t \leqq 4^{-l(1-\alpha)}$ we write

$$(3.6)\quad g_t(x) = \sum_{k=1}^{l} f_{t\cdot 4^{-k\alpha}} \psi_{k-1}(x)(1 - \psi_k(x)) + f^*_{t\cdot 4^{-l\alpha}M} \psi_l(x)$$

where M will be chosen later and for $\alpha \geqq 1$ we write

$$(3.7)\quad g_t(x) = \sum_{k=1}^{\infty} f_{t,4^{-k\alpha}} \psi_{k-1}(x)(1 - \psi_k(x)) \,.$$

We now have to show

$$(3.8)\quad \|f - g_t\|_{C[0,1]} \leqq K_1 \omega_r^*(f, t)$$

and

$$(3.9)\quad t^r \|x^{r\alpha} g_t^{(r)}\|_{C[0,1]} \leqq K_2 \omega_r^*(f, t) \,.$$

We recall that

$$f(x) = \sum_{k=1}^{l} f(x)\psi_{k-1}(x)(1 - \psi_k(x)) + f(x)\psi_l(x)$$

or

$$f(x) = \sum_{k=1}^{\infty} f(x)\psi_{k-1}(x)(1 - \psi_k(x)) \,.$$

(Both expressions are correct independently of α but will be used respectively for $\alpha < 1$ and $\alpha \geqq 1$.)

Since in (3.6) and (3.7) at most two terms of the sum differ from zero for any x we will prove (3.8) when we show for $4^{-k} < x < 3\cdot 4^{-k+1}$

$$(3.10)\quad |f(x) - f_{t4^{-k\alpha}}(x)| \leqq \omega_r^*(f, t)$$

for all k when $\alpha \geqq 1$ and for $k \leqq l$, l given by $4^{-(l+1)(1-\alpha)} < t \leqq 4^{-l(1-\alpha)}$ only for $\alpha < 1$; but in the latter case for $x < 3\cdot 4^{-l}$ we have to show also

$$(3.11)\qquad |f(x) - f^*_{t4^{-l\alpha}}| \leqq \omega^*_r(f, t) .$$

To prove (3.10) we have

$$\begin{aligned}|f(x) - f_{t4^{-k\alpha}}(x)| &\leqq \operatorname*{Sup}_{\substack{\eta\leqq t\\ 4^{-k}<x<3\cdot 4^{-k+1}}} \left|\Delta^r_{\eta 4^{-k\alpha}} f\left(x + \frac{r}{2}\eta 4^{-k\alpha}\right)\right| \\ &\leqq \operatorname*{Sup}_{\substack{x-(r/2)\eta x^\alpha > 4^{-k}\\ \eta\leqq t}} |\Delta^r_{\eta x^\alpha} f(x)| \leqq \omega^*_r(f, t) .\end{aligned}$$

We derive (3.11) as follows

$$\begin{aligned}|f - f^*_{t4^{-l\alpha}}| &\leqq \operatorname*{Sup}_{t/2\leqq\eta\leqq t} \left|\Delta^r_{\eta 4^{-l\alpha}M} f\left(x + \eta\cdot\frac{r}{2}4^{-l\alpha}\right)\right| \\ &\leqq \operatorname*{Sup}_{\substack{t/2\leqq\eta\leqq t\\ \zeta>(r/2)\eta 4^{-l\alpha}M}} |\Delta^r_{\eta 4^{-l\alpha}M} f(\zeta)| = \operatorname*{Sup}_{\substack{\eta\leqq t\\ \zeta\geqq(r/2)\eta\zeta^\alpha}} |\Delta^r_{\eta\zeta^\alpha} f(\zeta)| \leqq \omega^*_r(f, t)\end{aligned}$$

for $M = \min(1, (r/8)^{\alpha/1-\alpha})$, since for such M, $\eta(r/2)4^{-l\alpha}M \geqq (r/2)4^{-l\alpha}(t/2)M \geqq (r/4)4^{-l\alpha}4^{-l(1-\alpha)}4^{-(1-\alpha)} \geqq 4^{-l}r/8\cdot M \geqq 4^{-l}(r/8)^{1/1-\alpha}$, (or $\geqq 4^{-l}$ if $M = 1$).

We shall prove (3.9) now. First let us observe

$$(3.14)\qquad |f^{(r)}_h(x)| = \left|\left(\frac{r}{h}\right)^r \sum_{j=1}^r \binom{r}{j}(-1)^{k+1}\Delta^r_{jh/r} f(x + jh/2)\right|$$

which can be proved following Achieser [1, p. 174] where the case in which f_h is translated to be centered at zero and $r = 2$ is treated. Therefore, for $4^{-k} \leqq x \leqq 3\cdot 4^{-k+1}$ (and $k < l$ for $\alpha < 1$)

$$\begin{aligned}|t^r x^{r\alpha} f^{(r)}_{4^{-k\alpha}}(x)| &\leqq t^r 3^{r\alpha}|4^{-kr\alpha} f^{(r)}_{t4^{-k\alpha}}(x)| \leqq 3^{r\alpha} r^r \sum_{j=1}^r \binom{r}{j} |\Delta^r_{t4^{-k\alpha}(j/r)} f(x + jt4^{-k\alpha/2}) \\ &\leqq 3^{r\alpha} r^r\cdot 2^r \max_j |\Delta^r_{t4^{-k\alpha}(j/r)} f\left(x + jt4^{-k\alpha}\left(\frac{1}{2}\right)\right)\Big| \leqq M \operatorname*{Sup}_{\substack{\eta\leqq t\\ x-(r/2)\eta x^\alpha>4^{-k}}} |\Delta^r_{\eta x^\alpha} f(x)| \\ &\leqq M\omega^*_2(f, t) .\end{aligned}$$

For $f^*_h(x)$ we have

$$(3.15)\qquad \begin{aligned}|f^*_k(x)| = \left(\frac{r}{h}\right)^r \left(1 - \left(\frac{1}{2}\right)^r\right)^{-1} \Bigg| \sum_{j=1}^r \binom{r}{j}(-1)^{k+1} \\ \times \left\{\Delta^r_{jh/r} f\left(x + jh\frac{1}{2}\right) - \Delta^r_{jh/2r} f(x + jh/4)\right\}\Bigg| .\end{aligned}$$

For $h = t\cdot 4^{-l\alpha}M$, $t \geqq 4^{-(l+1)\alpha}$ and $x < 3\cdot 4^{-l}$ we derive $t^r\|x^{r\alpha} f^{*(r)}_{t,4^{-l\alpha}M}(x)\| \leqq M_1\omega^*_r(f, t)$ similar to our earlier calculation. To complete the proof one has to check $g^{(r)}_t(x)$ at points x for which $g_t(x)$ is equal to the sum of two terms or in other words, $\{x: 4^{-k}<x<3\cdot 4^{-k+1}\}\cap\{x: 4^{-k+1}< x < 3\cdot 4^{-k+2}\} = \{x: 4^{-k+1} < x < 3\cdot 4^{-k+1}\}$ on which $g_t(x)=\psi_{k-1}(x)f_{t\cdot 4^{-k\alpha}}(x)+(1-\psi_{k-1}(x)f_{t\cdot 4^{-k\alpha+\alpha}}(x) = f_{t\cdot 4^{-k\alpha+\alpha}}(x)+\psi_{k-1}(x)[f_{t\cdot^{-k\alpha}}(x)-f_{t\cdot 4^{-k\alpha+\alpha}}(x)]$. Since $|\psi_{k-1}^{(j)}(x)| \leqq M4^{kj}$ we have to estimate only $f_{t4^{-k\alpha}}^{(r-j)}(x) - f_{t4^{-k\alpha+\alpha}}^{(r-j)}(x)$ and we will use on this function Lemma 2.1 where $b - a = 2\cdot 4^{-k+1}$. Using (3.14) (for $r = n$ in the lemma) and using (3.10) for k and $k - 1$, we obtain in $4^{-k+1} < x < 3\cdot 4^{-k+1}$

$$t^r x^{r\alpha}\psi_{k-1}^{(j)}(x)|f_{t\cdot 4^{-k\alpha}}^{(r-j)}(x) - f_{t\cdot 4^{-k\alpha+\alpha}}^{(r-j)}(x)| \leqq M_* t^r x^{r\alpha} 4^{kj}(4^k)^{r-j}\omega_r^*(f, t) + M_* 4^{kj}\cdot 4^{-kj}\omega_r^*(f, t) .$$

Recalling $t^r x^{r\alpha}4^{kr} \leqq 12^{r\alpha}t^r 4^{kr(1-\alpha)}$ which is bounded for $\alpha \geqq 1$ or otherwise $k < l$ and $t \leqq 4^{-l(1-\alpha)}$ which still implies that $t^r x^{r\alpha}4^{kr}$ is bounded, we have $t^r x^{r\alpha}|\psi_{k-1}^{(j)}(x)(f_{t4^{-k\alpha}}^{(r-j)}(x) - f_{t4^{-k\alpha+\alpha}}^{(r-j)}(x))| \leqq M\omega_r^*(f, t)$. Similarly we can treat $g_t(x)$ in $4^{-l} < x < 3\cdot 4^{-l}(\alpha < 1)$, (using (3.15) instead of (3.14)).

4. The L_p case. The expression for ω_r^* for the L_p case is more complicated. Possible different expressions for ω_r^* will be discussed in § 5 but a complete result will be obtained here with ω_r^* given by

$$\omega_r^*(f, t) = \operatorname*{Sup}_{\eta\leqq t}\left\{\sum_{k=1}^{k_0(t)}\int_{4^{-k}}^{4^{-k+1}}|\Delta_{\eta 4^{-k\alpha}}^r f(x)|^p dx\right\}^{1/p} + \operatorname*{Sup}_{\eta\leqq t^{1/1-\alpha}}\left\{\int_0 |\Delta_\eta^r f(x)|^p dx\right\}^{1/p}\delta(\alpha) \tag{4.1}$$

where $\Delta_\mu f$ in this section is a forward difference given by $\Delta f(x) = f(x + \mu) - f(x)$, $\delta(\alpha) = 1$ for $\alpha < 1$ $\delta(\alpha) = 0$ for $\alpha \geqq 1$ and $k_0(t)$ given by $k_0(t) = \operatorname{Max}\{k: 4^{-4} + \operatorname{tr} 4^{-k\alpha} \leqq 4^{-k+1}\}$. One can observe that for $\operatorname{tr} < 1/4$ and $\alpha \geqq 1$ there is no bound on k and we replace $k_0(t)$ by ∞. In accordance with the discussion in § 2 we have Supp $f \subset [0, 3/4]$ with no loss of generality.

The functional $\omega_r^*(f, t)$ represents the L_p smoothness of f in exactly the same way as the r modulus of continuity does when away from the singular point, in this case 0. Near the singular point the function need not be as smooth. The expression (4.1) is a quantitative measure of smoothness needed near 0 (the singular point) as well as elsewhere that expresses the above qualitative description. For $K(t^r, f)$ given by

$$K(t^r, f) = \inf_g (\|f - g\|_p + t^r\|x^{r\alpha}g^{(r)}\|_p) \tag{4.2}$$

we can derive the following theorem.

THEOREM 4.1. *For* $f(x)\in L_p$ *and* Supp $f\subset[0, 3/4]$ *we have*

$$(4.3) \qquad A\omega_r^*(f, t) \leqq K(t^r, f) \leqq B\omega_r^*(f, t)$$

where $K(t^r, f)$ *and* $\omega_r^*(f, t)$ *are given by* (4.2) *and* (4.1) *respectively.*

Proof. We first show $\omega_r^*(f, t) \leqq A^{-1}K(t^r, f)$ for some $A > 0$. By definition of $K(t^r, f)$ there exists g_t such that $\|f-g_t\| \leqq 2K(t^r, f)$ and $t^r\|x^{r\alpha}g_t^{(r)}(x)\| \leqq 2K(t^r, f)$. Obviously $\omega_r^*(f, t) \leqq \omega_r^*(f-g_t t) + \omega_r^*(g_t, t)$.

To estimate $\omega_r^*(f - g_t, t)$ we write $f - g_t = F_t$ and

$$\omega_r^*(F_t, t) \leqq r \operatorname*{Sup}_{\eta\leqq t} \operatorname*{Sup}_{j} \binom{r}{j} \times \left\{\sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} |F_t(x + \eta j 4^{-k\alpha})|^p dx\right\}^{1/p} + 2^r\|F_t\| .$$

Since $4^{-k} + \operatorname{tr} 4^{-k\alpha} < 4^{-k+1}$ (also $4^{-k+1} + \operatorname{tr} 4^{-k\alpha} < 4^{-k+2}$), each point $\zeta = x + \eta j 4^{-k\alpha}$ $x \in [4^{-k}, 4^{-k+1}]$ appears for fixed η and j at most twice and therefore $\omega_r^*(F_t, t) \leqq r \sup_j \binom{r}{j} 4K(t^r, f) + 2^r 2K(t^r, f)$.

Somewhat more complicated is the estimate of $\omega_r^*(g_t, t)$. Using Taylor's formula (and forward differences), we have

$$\begin{aligned} I_1 &\equiv \operatorname*{Sup}_{\eta\leqq t}\left(\sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} |\Delta_{\eta 4^{-k\alpha}}^r g_t(x)|^p dx\right)^{1/p} \\ &\leqq \operatorname*{Sup}_{\eta\leqq t}\left(\sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} \left| \sum_{j=1}^{r} \binom{r}{j} \frac{1}{(r-1)!} \right.\right. \\ &\qquad \times \left.\left. \int_x^{x+j\eta 4^{-k\alpha}} (x + j\eta 4^{-k\alpha} - u)^{r-1} g_t^{(r)}(u) du \right|^p dx \right)^{1/p} \\ &\leqq M_1(r) \operatorname*{Sup}_{\eta\leqq t} \operatorname*{Sup}_{j\leqq r} \\ &\qquad \times \left(\sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} \left| \int_x^{x+j\eta 4^{-k\alpha}} [(x+j\eta 4^{-k\alpha} - u)^{r-1}/u^{r\alpha}] u^{r\alpha} g_t^{(r)}(u) du \right|^p dx\right)^{1/p} . \end{aligned}$$

Observing that

$$\frac{(x + j\eta u^{-k\alpha} - u)^{r-1}}{u^{r\alpha}} \leqq \frac{(j\eta 4^{-k\alpha})^{r-1}}{(4^{-k\alpha})^r} \leqq j\eta^{r-1} \frac{4^{k\alpha}}{4^\alpha}$$

and writing $M[u^{r\alpha}g_t^{(r)}](x) = \operatorname{Sup}_h 1/h \int_x^{x+h} |u^{r\alpha}g_t^{(r)}(u)| du$, the Hardy-Littlewood maximal function of $u^{r\alpha}g_t^{(r)}(u)$, we have for $1 < p < \infty$

$$\begin{aligned} I_1 &\leqq M_1(r) \operatorname*{Sup}_{\eta\leqq t} \operatorname*{Sup}_{j\leqq r} \eta^r \left(\sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} |M[u^{r\alpha}g_t^{(r)}](x)|^p dx\right)^{1/p} \\ &\leqq M_1(r) t^r 2K(t^r, f) . \end{aligned}$$

For $p = 1$ we estimate I_1 by Fubini's theorem (using $k_0(t)$)

$$I_1 \leqq M_r(r)t^{r-1} \operatorname*{Sup}_{j \leqq r} j \sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} 4^{k\alpha} \int_x^{x+j\eta 4^{-k\alpha}} |u^{r\alpha} g_t^{(r)}(u)| du dx$$
$$< M_1(r)t^r r^2 \sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+2}} |u^{r\alpha} g_t^{(r)}(u)| du \leqq t^r M_2(r) K(t^r, f) .$$

For $\alpha < 1$ we have to estimate one more term i.e.,

$$I_2 = \operatorname*{Sup}_{\eta \leqq t^{1/1-\alpha}} \left\{ \int_0 |\Delta_\eta^r g_t|^p dx \right\}^{1/p} .$$

Following the above and using Taylor's formula around $x + (r/2)\eta$,

$$I_2 \leqq M \operatorname*{Sup}_{\substack{\eta \leqq t^{1/1-\alpha} \\ j \leqq r}} \left[\left\{ \int_0^\eta + \int_\eta \right\} \left| \int_{x+(r/2)\eta}^{x+r\eta-j\eta} (x + (r - j)\eta - u)^{r-1} g_1^{(r)}(u) | du \right|^p dx \right]^{1/p}$$
$$\equiv J_1 + J_2 .$$

For $x > \eta$ or $j < r$

$$\left| \frac{(x + r\eta - j\eta - u)^{r-1}}{u^{r\alpha}} \right| \leqq \frac{(|j - r/2|\eta)^{r-1}}{(\eta)^{r\alpha}} \leqq c\eta^{r-r\alpha-1}$$

and the estimate of J_2 proceeds as that of I_1 since $\eta^{r(1-\alpha)} \leqq t^r$. For $x < \eta$ and $j = r (u > x)$

$$\frac{(x + r\eta - r\eta - u)^{r-1}}{u^{r\alpha}} \leqq u^{r-r\alpha-1}$$

and

$$\int_{x+(r/2)\eta}^{x} u^{r-r\alpha-1} du \sim \eta^{r(1-\alpha)} .$$

Therefore we have

$$J_1 \leqq C \left\{ \int_0^\eta \eta^{r(1-\alpha)p/q} \int_x^{x+(r/2)\eta} |u^{r\alpha} g_t^{(r)}(u)|^p u^{r(1-\alpha)-1} du dx \right\}^{1/p}$$
$$\leqq C \left\{ \eta^{r(1-\alpha)p/q} \eta^{r(1-\alpha)} \cdot \int_0^{\eta+(r/2)\eta} |u^{r\alpha} g_t^{(r)}(u)| du \right\}^{1/p}$$
$$\leqq C\eta^{r(1-\alpha)} ||u^{r\alpha} g_t^{(r)}(u)|| \leqq Ct^r ||u^{r\alpha} g_t^{(r)}(u)|| \leqq 2CK(t^r, f) .$$

To prove $K(t^r, f) \leqq B\omega_r^*(f, t)$ we define g_t which will satisfy $||f - g_t||_p \leqq B_1\omega_r^*(f, t)$ and $t^r ||x^{r\alpha} g_t^{(r)}||_p \leqq B_2\omega_r^*(f, t)$. Define f_h, f_h^* and g_t the same as in § 3 by (3.4), (3.5), (3.6) and (3.7) with possibly different M in (3.6).

To show $||f - g_t|| \leqq B\omega_r^*(f, t)$ we write

$$||f - g_t||^p \leqq C \left\{ \sum_{k=1}^{l} \int |f(x) - f_{t \cdot 4^{-k\alpha}}(x)|^p |\psi_{k-1}(x)(1 - \psi_k(x))|^p dx \right.$$
$$\left. + \int |f(x) - f_{t \cdot 4^{-l\alpha_M}}|^p |\psi_l(x)|^p dx \right\}$$

which follows since the sum is finite for every x.

Since $f_{t\cdot 4^{-k\alpha}}(x)$ can be written as

$$f_{t,4^{-k\alpha}}(x) = \left(\frac{r}{t}\right)^r \int_0^{t/r} \cdots \int_0^{t/r} \sum_{k=1}^{r} (-1)^{k+1} \times \binom{r}{k} f(x+k(u_1+\cdots+u_r)4^{-k\alpha}) du_1 \cdots du_r$$

and since $0 \leqq \psi_k \leqq 1$ and $\psi_{k-1}(1-\psi_k) \neq 0$ in $[4^{-k}, 3\cdot 4^{-k+1}]$, the kth term

$$\int_{4^{-k}}^{3\cdot 4^{-k+1}} |f - f_{t\cdot 4^{-k\alpha}}|^p dx \leqq \left(\frac{r}{t}\right)^r \int_0^{t/r} \cdots \int_0^{t/r} \times \int_{4^{-k}}^{4^{-k+1}} |\Delta^r_{(u_1+\cdots+u_r)4^{-k\alpha}} f|^p dx du_1 \cdots du_r + \left(\frac{r}{t}\right)^r \int_0^{t/r} \cdots \int_0^{t/r} \int_{4^{-k+1}}^{4^{-k+2}} |\Delta^r_{(u_1\cdots u_r)4^{-(k-1)\alpha}/4^\alpha} f(x)|^p dx du_1 \cdots du_r\,.$$

We observe now that with $\eta = u_1 + \cdots + u_r$ or $\eta = 4^{-\alpha}(u_1 + \cdots + u_r)$ and since the integral is the same for all terms, we have on $L_p[4^{-l+1}, 1]$

$$\|f - g_t\| \leqq C\left(\frac{r}{t}\right)^r \int_0^{t/r} \cdots \int_0^{t/r} [\omega_r^*(f, t) + \omega_r^*(f, t/4^\alpha)] du_1 \cdots du_r \leqq C_1 \omega_r^*(f, t)\,.$$

Similarly we can treat the remaining integral remembering that $4^{-(l+1)(1-\alpha)} < t \leqq 4^{-l(1-\alpha)}$ and $t\cdot 4^{-l\alpha} \leqq 4^{-l}$ and $4^{-l}M < t^{1/1-\alpha}$ for appropriate M. To estimate $\|x^{r\alpha} g_t^{(r)}\|$ we shall observe first that (3.14) and (3.15) are still valid for $f \in L_p$ except that the result is valid almost everywhere rather than everywhere.

Rewritten to take into account forward difference, we have for (3.14) and (3.15)

$$f^{(r)}_{t\cdot 4^{-k\alpha}}(x) = \left(\frac{r}{t}\right)^r 4^{k\alpha} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \Delta^r_{j(t/r)4^{-k\alpha}} f(x) \quad \text{a.e.} \tag{4.4}$$

and

$$f^{*(r)}_{t\cdot 4^{-l\alpha}M}(x) = \left(\frac{r}{t}\right)^r \left(1-\left(\frac{1}{2}\right)^r\right)^{-1} \sum_{j=1}^{r} \binom{r}{j} (-1)^{j+1} \times \{\Delta^r_{j(t/r)4^{-l\alpha}M} f(x) - \Delta^r_{j(t/2r)4^{-l\alpha}M} f(x)\} \quad \text{a.e.} \tag{4.5}$$

Using (4.4) and (4.5), we have

$$
\begin{aligned}
(4.6)\qquad & t^{rp}\int_{4^{-k}}^{3\cdot 4^{-k+1}} |x^{r\alpha} f^{(r)}_{t\cdot 4^{-k\alpha}}(x)|^p dx \\
&\leqq M(r) \max_{1\leqq j\leqq r} \int_{4^{-k}}^{3\cdot 4^{-k+1}} |\Delta^r_{j(t/r)4^{-k\alpha}} f(x)|^p dx \\
&\leqq M(r)\left\{ \max_{1\leqq j\leqq r} \int_{4^{-k}}^{4^{-k+1}} |\Delta^r_{j(t/r)4^{-k\alpha}} f(x)|^p dx \right. \\
&\left. + \max_{1\leqq j\leqq r} \int_{4^{-k+1}}^{4^{-k+2}} |\Delta^r_{j(t/r)\cdot 4^{-\alpha}\cdot 4^{-\alpha(k-1)}} f(x)|^p dx \right\} .
\end{aligned}
$$

We notice that it is a maximum or a finite number of terms and $j(t/r)$ and $j(t/r)4^{-\alpha}$ are smaller than t and moreover it is a maximum on the same terms for all k. Similarly one can estimate

$$t^{rp}\int_0^{4^{-l+1}} |x^{r\alpha} f^{(r)}_{t\cdot 4^{-l\alpha_M}}|^p dx .$$

To conclude the proof let us follow Lemma 2.1 in much the same way as was done in the proof of Theorem 3.1.

To calculate the L_p norm of $g_t^{(r)}(x)$ we recall that in

$$
\begin{aligned}
\{x;\, 4^{-k+1} < x < 3\cdot 4^{-k+1}\}\quad g_t(x) &= f_{t\cdot 4^{-k\alpha+\alpha}}(x) \\
&+ \psi_{k-1}(x)[f_{t\cdot 4^{-k\alpha}}(x) - f_{t\cdot 4^{-k\alpha+\alpha}}(x)] ,
\end{aligned}
$$

and since $|\psi^{(j)}_{k-1}| \leqq M4^{kj}$, we have to estimate in $L_p[4^{-k+1}, 3\cdot 4^{-k+1}]$ $f^{(r-j)}_{t\cdot 4^{-k\alpha}}(x) - f^{(r-j)}_{t\cdot 4^{-k\alpha+\alpha}}(x)$ and for this we use (4.4) and earlier estimates in this section together with Lemma 2.1 where $b - a = 2\cdot 4^{-k+1}$.

It can be seen that the estimate for L_p norm in $[4^{-k+1}, 3\cdot 4^{-k+1}]$ is given by a maximum of a finite number of terms that depend on j and r but not on k. Using this and the fact that in the sums (3.6) or (3.7) we have for any x only two nonzero terms, we can conclude the proof i.e., $t^r\|x^{r\alpha} g_t^{(r)}\| \leqq B\omega_r^*(f, t)$.

If r is even, we can write $\omega_{2r}(f, p, t)$

$$
\begin{aligned}
(4.7)\qquad \omega_{2r}(f, p, t) &= \operatorname*{Sup}_{\eta\leqq t}\left\{ \sum_{k=1}^{k_0(t)} \int_{4^{-k}}^{4^{-k+1}} |\Delta^{2r}_{\eta 4^{-k\alpha}} f(x)|^p dx \right\}^{1/p} \\
&+ \operatorname*{Sup}_{\eta\leqq t^{1/1-\alpha}} \left\{ \int_{r\eta}^{1-r\eta} |\Delta^{2r}_{\eta} f(x)|^p \right\}^{1/p} ,
\end{aligned}
$$

where the differences are symmetric ($\Delta_\eta f(x) = f(x+\eta/2) - f(x-\eta/2)$) and $k_0(t) = \operatorname{Max}(k\colon 4^{-k} - \mathrm{tr}\, 4^{-k\alpha} > 4^{-k-1})$. In this case one can prove similarly:

THEOREM 4.2. *For* $f(x) \in L_p$ $\operatorname{Supp} f \subset [0, 3/4]$, *we have for* $t < t_0$

$$(4.8)\qquad A\omega_{2r}(f, p, t) \leqq K(t^{2r}, f) \leqq B\omega_{2r}(f, p, t) .$$

Actually Theorem 4.2 does not yield a new result, just a similar characterization which is proved following the same method, but I believe that (4.7) and $\omega_{2r}(f, p, t)$ will be convenient using symmetric rather than forward differences.

5. Conclusions. In this section we will use the two main results for §§ 3 and 4 as well as considerations of § 2 to obtain a global description of the K functional (which is a sum of translates of the local case) and also the interpolation theorem involved.

DEFINITION 5.1. A weight function $W(x)$ on $[a, b]$ is of class A if it is a continuous nonnegative function with finitely many zeros at $a \leqq x_1 < x_2 < \cdots < x_n \leqq b$ such that $0 < A_{ij}|x - x_i|^{\alpha_{ij}} \leqq W(x) \leqq B_{ij}|x - x_i|^{\alpha_{ij}}$ in $0 < (x - x_i)(-1)^j < \delta$ where $\alpha_{ij} > 0$ $i = 1, \cdots, n$ and $j = 0, 1$ and where, in case $x_1 = a$ or $x_n = b$, the above condition for $i = 1, j = 1$ or $i = n, j = 0$ is void. (a and b might be $-\infty$ or ∞ respectively.)

For $W(x)$ of class A we may define the modified modulus of continuity as follows:

For $f \in C$ and $t \leqq t_0$

$$(5.1)\qquad \omega_r^*(f, t; W, C) = \sum_{i,j} \operatorname*{Sup}_{\eta < t} \operatorname*{Sup}_{\substack{(r/2)\eta < x^{1-\alpha_{ij}} \\ x < d/2}} |\Delta^r_{\eta x^{\alpha_{ij}}} f(x_i + (-1)^j x)| + \operatorname*{Sup}_{\eta < t} \left\{ |\Delta^r_\eta f(x)|; x \pm r\frac{\eta}{2} \in [a, b] \text{ and } |x - x_i| > \frac{d}{4} \right\}.$$

For $f \in L_p$ and $t \leqq t_0$ we have

$$(5.2)\qquad \omega_r^*(f, t, w; L_p) = \sum_{ij} \omega^*_{r,i,j}(f, t) + \operatorname*{Sup}_{\eta < t} \left\{ \int_{|x - x_i| > d/16} |\Delta^r_\eta f|^p \, dx \right\}^{1/p}$$

where $\omega^*_{r,i,j}$ are the expressions given by (4.1) with α_{ij} replacing α, $f(x_i + (-1)^j x)$ replacing $f(x)$ and k starting from k_1 rather than 1, (chosen so that $4^{-k_1+1} \leqq d/2$, and therefore the distance between x_i and $x_i + (-1)^j x$ is less than $d/2$). Both expressions are measurements of smoothness showing that near a zero of $W(x)$ less smoothness is needed and that the amount of relaxation in smoothness depends on the rate at which $W(x)$ tends to zero near x_i.

Now using the introduction, § 2 and the main result in §§ 3 and 4 we can conclude the following interpolation results:

THEOREM 5.1. *For* $W(x)$ *of class* A, $f \in C[a, b]$ *or* $f \in L_p[a, b]$, *and the expressions* $K(t^r, f)$, $\omega_r^*(f, t; w; C)$ *and* $\omega_r^*(f, t; w; L_p)$ *given by* (1.5), (5.1) *and* (5.2) respectively, *we have for* $t \leqq t_0$(t_0 *small enough*)

(5.3) $M_1\omega_r^*(f, t; w, B) \leqq K(t^r, f) \leqq M_2\omega_r^*(f, t, w, B)$, $0<M_1<M_2<\infty$

where B is either $C[a, b]$ or $L_p[a, b]$.

THEOREM 5.2. *Under the conditions of Theorem 5.1 and when the interpolation space $(B, B(r, w))_{\theta,q;K_*}$ is given by the norm in (1.2) and (1.3) using the functional $K_*(f, t)$ defined in (1.4) for $B = C$ or $B = L_p$, we have $f \in (B, B(r, w))_{\theta,q;K_*}$ if and only if*

(5.4) $$\operatorname*{Sup}_{0<t\leqq t_0} t^{-r\theta}\omega_r^*(f, t, w, B)\leqq M(f) \text{ for } q=\infty \text{ and } B=C \text{ or } B=L_p$$

respectively and

(5.5) $$\int_0^{t_0}(t^{-r\theta}\omega_r^*(f, t, w, B))^q\frac{dt}{t}\leqq M(f) \text{ for } 1\leqq q<\infty \text{ and } B=C \text{ or } B=L_p$$

respectively.

6. Remarks and generalizations.

1. In an earlier paper [4] the author proved for Bernstein polynomials, $B_n(f, x)$ for $\beta < 2$ $\|B_n(f) - f\|_{C[0,1]} = 0(1/n^{\beta/2})$ if and only if $|[x(1 - x)]^{\beta/2}\Delta_h^2 f| \leqq Mh^\beta$, as a result of the equivalence of $K(t^2, f)/t^\beta \leqq M$ and $\operatorname{Sup}_{h<x<1-h}|[x(1 - x)]^{\beta/2}\Delta_h^2 f| \leqq Mh^\beta$ where $K(t^2, f) = \inf_g (\|f - g\|_C + t^2\|x(1 - x)g''(x)\|_C)$. This paper yields the new characterization of $\|B_n f - f\|_{C[0,1]} = 0(n^{-\beta/2})$, that is $\|B_n f - f\| = 0(n^{-\beta/2})$ if and only if $\|\Delta_{hx^{1/2}}^2 f\|_{C(h^2, 1-h^2)} \leqq Mh^\beta$ where α of our Theorem 3.1 is 1/2. Similarly with respect to other results of [4] one can deduce additional results from Theorem 3.1. (Results on conditions for rate of convergence of combinations of Bernstein polynomials.)

2. For the case $C[0, 1]$ given in § 3 the condition $K(t^r, f)/t^\beta \leqq M$ (which is an important case) is equivalent to

$$\operatorname*{Sup}_{(r/2)h<x<1-(r/2)h} |x^{r\alpha\beta}\Delta_h^r f| \leqq Mh^\beta .$$

We did not go that route in order to characterize the K functional completely and not just the case $K(t^r, f)/t^\beta \leqq M$.

3. An alternative for $\omega_{2r}^*(f, t)$ could be

(6.1) $$\omega_{2r}^{**}(f, t) = \operatorname*{Sup}_{\eta\leqq t}\left(\int_{(r\eta)^{1/1-\alpha}}^{1-C}|\Delta_{\eta x^\alpha}^{2r} f(x)|^p dx\right)^{1/p} + \operatorname*{Sup}_{\eta\leqq t^{1/1-\alpha}}\left(\int_{r\eta}|\Delta_\eta^{2r} f(x)|^p dx\right)^{1/p} \quad \text{for } \alpha < 1$$

and

$$\omega_{2r}^{**}(f, t) = \operatorname*{Sup}_{\eta\leqq t}\left(\int_0^{1-C}|\Delta_{\eta x^\alpha}^{2r} f(x)|^p dx\right)^{1/p} \text{ for } \alpha \geqq 1 .$$

While in proving $\omega_{2r}^{**}(f, t) \leqq AK(t^{2r}, f)$ there was no problem, the author was not able to show $K(t^{2r}, f) \leqq A_1\omega_{2r}^{**}(f, t)$.

4. Various α were treated and while the case $\alpha = 1/2$ has already yielded a result about the rate of approximation of Bernstein polynomials, the rate of approximation of the Post-Widder inversion formula for Laplace transforms or the Gamma operators relate to $\alpha = 1$ and together with a much wider class of operators will be treated elsewhere.

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