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MAPPING INTERVALS TO INTERVALS

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We study the question of mapping intervals to intervals by rational functions which map the real line into the extended real line and the upper half plane into the upper half plane.

Let $\mathscr{R}$ be the set of rational functions which map the upper half plane into the upper half plane and the real line into the extended real line. A function in $\mathscr{R}$ is the upper half plane equivalent of a finite Blaschke product on the unit disk and is sometimes referred to as a rational Cayley inner function. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers with $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and let Λ be the set of points $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ in $\boldsymbol{R}^{2n}$ such that there is a φ in $\mathscr{R}$ with $\varphi([a_i, b_i]) = [A_i, B_i]$ for $i = 1, \cdots, n$. The purpose of this paper is to determine which points P lie in Λ. In other words, we wish to describe those collections of those intervals $[A_1, B_1], \cdots, [A_n, B_n]$ which are images under a rational Cayley inner function of the intervals $[a_1, b_1], \cdots, [a_n, b_n]$.

We note that it is always possible to map points to points, that is, there is a ψ in $\mathscr{R}$ with $\psi(a_i) = A_i$ and $\psi(b_i) = B_i$ for $i = 1, \cdots, n$. However, this function ψ may have a pole in some (a_j, b_j) and so $\psi([a_j, b_j]) \supseteqq \boldsymbol{R}$. The motivation for this research is the following question due to J. Rovnjak (verbal communication): Is it always possible to map intervals to intervals? In terms of the notation above, the question is whether Λ contains every point $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ with $A_i < B_i$ for $i = 1, \cdots, n$. The answer to this question will be shown to be in the negative. For example, we will show that if $n \geqq 2$, if $[A_i, B_i] \subset [a_i, b_i]$ for $i = 1, \cdots, n$, and if $[A_i, B_i] \neq [a_i, b_i]$ for some i, then there is no function φ in $\mathscr{R}$ with $\varphi([a_i, b_i]) = [A_i, B_i]$ for $i = 1, \cdots, n$.

The main result of this paper describes the set Λ, the closure of Λ, and the boundary of Λ in terms of functions in $\mathscr{R}$ with degree less than n and in terms of certain ideal points. This theorem is stated in §1 and three corollaries are established. The theorem is proved in §2 and some further observations are made in §3. We also include in §3 an elementary proof of the assertion mentioned above that it is always possible to map points to points; more general results can be found in [1; Article II].

We would like to point out that the analysis in this paper is similar in certain respects to that of the moment space of a Tchebycheff system as developed in [3]. In particular, the use of the dual cone and the identification of the dual cone with a cone of nonnegative functions are themes in [3]. However, the results here are not consequences of the analysis in [3].

1. The main theorem. Let $U = \boldsymbol{R}\backslash\bigcup_{i=1}^{n}[a_i, b_i]$ and let $\mathscr{M}$ be the set of nonnegative purely atomic measures μ on $U \cup \{\infty\}$ such that the support of μ is a finite set. If (μ, c) is in $\mathscr{M} \times \boldsymbol{R}$, then the function φ given by

$$\varphi(z) = c + \int(1 + tz)(t - z)^{-1}d\mu(t) \tag{1}$$

is in $\mathscr{R}$ and, conversely, any function φ in $\mathscr{R}$ is of form (1) for some pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$; see [2, Chapter II]. In equation (1) and elsewhere in this paper, we adopt the following convention: if f is a continuous function on U with $\lim_{x\to\infty} f(x) = A = \lim_{x\to-\infty} f(x)$, and if μ is in $\mathscr{M}$, then $\int f(t)d\mu(t) = \mu(\infty)A + \int_U f(t)d\mu(t)$. Thus, the expression for φ in (1) can be rewritten as

$$\varphi(z) = c + \mu(\infty)z + \int_U(1 + tz)(t - z)^{-1}d\mu(t) .$$

Define a function Γ from $\mathscr{M} \times \boldsymbol{R}$ into $\boldsymbol{R}^{2n}$ by setting

$$\Gamma(\mu, c) = (\varphi(a_1), \cdots, \varphi(a_n), \varphi(b_1), \cdots, \varphi(b_n))$$

where φ is given by (1). Since the derivative of φ is given by

$$\varphi'(z) = \int(1 + t^2)(t - z)^{-2}d\mu(t) , \tag{2}$$

the function φ is nondecreasing on each interval $[a_i, b_i]$ and therefore $\varphi([a_i, b_i]) = [\varphi(a_i), \varphi(b_i)]$. These remarks show that the set Λ is precisely the range of Γ.

The degree of a rational function φ is the maximum of the degrees of the polynomials p and q where $\varphi = p/q$ and p and q have no common zero. It is elementary to show that the degree of the function φ in (1) is equal to the number of points in the support of μ. We denote this integer by $\#\mu$.

We define an ideal point over the interval $[a_j, b_j]$ to be a point I of the form $I = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ where $A_i = B_i = 0$ for $i \neq j$ and $A_j \leqq 0 \leqq B_j$. For an ideal point I we write $\chi(I) = 1$ if $I \neq 0$ and $\chi(I) = 0$ if $I = 0$.

The following theorem describes the set Λ, the closure of Λ and the boundary of Λ in terms of those points of the form $\Gamma(\mu, c)$ with $\#\mu \leqq n - 1$ and in terms of ideal points.

THEOREM. *Let P be a point in $\boldsymbol{R}^{2n}$.*

(i) *The point P is in the closure of Λ if and only if*

$$P = \Gamma(\mu, c) + \sum_{j=1}^{n} I_j \tag{3}$$

where (μ, c) is in $\mathscr{M} \times \boldsymbol{R}$ and I_j is an ideal point over $[a_j, b_j]$. In this case, the point P can be represented as in (3) *with $\sharp\mu \leqq n-1$ and $\sharp\mu + \sum_{j=1}^{n} \chi(I_j) \leqq n$.*

(ii) *The point P is in the boundary of Λ if and only if it can be represented as in* (3) *with*

$$\sharp\mu + \sum_{j=1}^{n} \chi(I_j) \leqq n - 1 .$$

(iii) *The point P is in both Λ and the boundary of Λ if and only if $P = \Gamma(\mu, c)$ with (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ and $\sharp\mu \leqq n - 1$.*

(iv) *Finally, the representation for P in* (3) *is unique if and only if P is in the boundary of Λ.*

We now establish three corollaries to this theorem.

COROLLARY A. *If $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ with $A_i < B_i$ for $i = 1, \cdots, n$ and if $\bigcap_{i=1}^{n} [A_i, B_i] \neq \varnothing$, then P is in the interior of Λ.*

Proof. Under these assumptions, the point P is of the form $\Gamma(0, c) + \sum_{j=1}^{n} I_j$ where $I_j \neq 0$ for each j and the corollary follows immediately from the theorem.

COROLLARY B. *Let $P = \Gamma(\mu, c) - \sum_{j=1}^{n} I_j$ with (μ, c) in $\mathscr{M} \times \boldsymbol{R}$, $\sharp\mu \leqq n - 1$, and I_j an ideal point over $[a_j, b_j]$. Then P is in the closure of Λ if and only if $I_j = 0$ for each j.*

Proof. If $I_j = 0$ for each j, then P is evidently in Λ and thus in the closure of Λ. On the other hand, if P is in the closure of Λ, then by the theorem, $P = \Gamma(\mu', c') + \sum_{j=1}^{n} I'_j$ with (μ', c') in $\mathscr{M} \times \boldsymbol{R}$, $\sharp\mu' \leqq n - 1$, and I'_j an ideal point over $[a_j, b_j]$. Thus, $\Gamma(\mu, c) = \Gamma(\mu', c') + \sum_{j=1}^{n}(I_j + I'_j)$ and, by the uniqueness assertion of the theorem, $\mu = \mu'$, $c = c'$, and $I_j + I'_j = 0$ for each j. It follows that $I_j = 0$ for each j and this completes the proof of Corollary B.

COROLLARY C. *Let $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ be a point with $[a_j, b_j] \subset [A_j, B_j]$ for each j. Then* (i) *the point P is in the closure of Λ;* (ii) *the point P is in the interior of Λ if and only if $[a_j, b_j] \neq [A_j, B_j]$ for at least $n - 1$ values of j;* (iii) *the point P is in the* *boundary of Λ if and only if $[a_j, b_j] = [A_j, B_j]$ for at least 2 values of j; and* (iv) *the point P is in Λ if and only if $[a_j, b_j] = [A_j, B_j]$ for all j or $[a_j, b_j] \neq [A_j, B_j]$ for at least $n-1$ values of j. On the other hand, suppose that P satisfies $[A_j, B_j] \subset [a_j, b_j]$ for each j. Then P is in the closure of Λ if and only if $[A_j, B_j] = [a_j, b_j]$ for each j.*

Proof. If $[a_j, b_j] \subset [A_j, B_j]$ for each j, then $P = \Gamma(\delta, 0) + \sum_{j=1}^{n} I_j$ where δ is a unit point mass at infinity and I_j is an ideal point over $[a_j, b_j]$. Furthermore, $I_j = 0$ if and only if $[a_j, b_j] = [A_j, B_j]$. Assertions (i), (ii), (iii), and (iv) follow from this and the theorem. If $[A_j, B_j] \subset [a_j, b_j]$ for each j, then $P = \Gamma(\delta, 0) - \sum_{j=1}^{n} I_j$ and the result follows from Corollary B.

2. Proof of the main theorem.

The proof is based on a number of propositions.

PROPOSITION 1. *Every ideal point is in the closure of Λ.*

Proof. Suppose that I is an ideal point over $[a_j, b_j]$, that is, $I = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ with $A_i = 0 = B_i$ for $i \neq j$ and $A_j \leqq 0 \leqq B_j$. Let δ_k be the unit point mass at $a_j - 1/k$, let ρ_k be a unit point mass at $b_j + 1/k$, and set

$$\mu_k = \frac{-A_j}{k(1 + a_j^2)}\delta_k + \frac{B_j}{k(1 + b_j^2)}\rho_k\,, \qquad k = 1, 2, 3, \cdots .$$

Direct computation using the definition of Γ shows that $\Gamma(\mu_k, 0) \to I$ as $k \to \infty$ and this proves the result.

PROPOSITION 2. *If P is in the closure of Λ, then there is a point $\bar{P}$ in the boundary of Λ and an ideal point I over $[a_1, b_1]$ such that $P = \bar{P} + I$.*

Proof. Let $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$, let J be the ideal point $(-1, 0, 0, \cdots, 0)$, and let W be the set of nonnegative real numbers x such that $P - xJ$ is in the closure of Λ. If x is in W and if $0 \leqq y \leqq x$, then $P - yJ = P - xJ + (x - y)J$. Since $(x - y)J$ is in the closure of Λ by Proposition 1 and Λ is a convex cone, it follows that y is in W. Also, any number greater than $B_1 - A_1$ is not in W; and W is evidently closed. Thus, the set W is of the form $[0, \bar{x}]$ for a nonnegative number $\bar{x}$. Let $\bar{P} = P - \bar{x}J$ and put $I = \bar{x}J$. Then $\bar{P}$ is in the boundary of Λ, I is an ideal point over $[a_1, b_1]$, and $P = \bar{P} + I$, as desired.

In the proof of the following proposition and elsewhere in this paper, the following notation will be useful. Let $F_1, \cdots, F_n$ and $G_1, \cdots, G_n$ be the continuous functions on U defined by $F_i(t) = (1 + a_i t)(t - a_i)^{-1}$ and $G_i(t) = (1 + b_i t)(t - b_i)^{-1}$. Observe that if $\Gamma(\mu, c) = (A_1, \cdots, A_n, B_1, \cdots, B_n)$, then

$$\begin{aligned} A_i &= c + \int F_i(t)d\mu(t) \\ B_i &= c + \int G_i(t)d\mu(t) . \end{aligned} \tag{4}$$

PROPOSITION 3. *If P is in Λ, then there is a point (μ, c) in $\mathscr{M} \times R$ with $\Gamma(\mu, c) = P$ and $\sharp\mu \leqq 2n - 1$.*

Proof. Let (μ_0, c_0) in $\mathscr{M} \times \boldsymbol{R}$ be chosen so that $\Gamma(\mu_0, c_0) = P$ and $\sharp\mu_0 \leqq \sharp\mu$ for any pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ with $\Gamma(\mu, c) = P$. Let $L^1_R(\mu_0)$ denote the real functions in $L^1(\mu_0)$. Define functions $H_1, \cdots, H_{2n-1}$ in $L^1_R(\mu)$ by setting $H_{2i-1}(t) = F_i(t) - G_i(t)$ for $i = 1, \cdots, n$ and $H_{2i}(t) = G_i(t) - F_{i+1}(t)$ for $i = 1, \cdots, n - 1$. We now show that the functions $H_1, \cdots, H_{2n-1}$ span $L^1_R(\mu_0)$. For this, suppose to the contrary that there is a nonzero function f in $L^\infty_R(\mu_0)$ such that $\int f(t)H_i(t)d\mu_0(t) = 0$ for $i = 1, \cdots, 2n - 1$. There is then a λ in $\boldsymbol{R}$ such that the measure $d\nu(t) = (1 - \lambda f(t))d\mu_0(t)$ satisfies $\nu \geqq 0$ and $\sharp\nu < \sharp\mu_0$. Let $c' = c + \lambda\int F_1(t)f(t)d\mu_0(t)$ and let $\Gamma(\nu, c') = P' = (A_1', \cdots, A_n', B_1', \cdots, B_n')$. Then it follows from (4) that

$$\begin{aligned} A_i' - B_i' &= \int F_i(t)\,d\nu(t) - \int G_i(t)d\nu(t) \\ &= \int H_{2i-1}(t)(1 - \lambda f(t))d\mu_0(t) = A_i - B_i \end{aligned}$$

for $i = 1, \cdots, n$ and similarly $B_i' - A_{i+1}' = B_i - A_{i+1}$ for $i = 1, \cdots, n - 1$. Furthermore, by (4)

$$\begin{aligned} A_1' &= c' + \int F_1(t)d\nu(t) \\ &= c + \lambda \int F_1(t)f(t)\,d\mu_0(t) + \int F_1(t)d\nu(t) \\ &= c + \int F_1(t)d\mu_0(t) = A_1 . \end{aligned}$$

Consequently $P = P'$. From the definition of μ_0, it follows that $\sharp\mu_0 \leqq \sharp\nu$, a contradiction. Thus, the functions $H_1, \cdots, H_{2n-1}$ span $L^1_R(\mu_0)$ and therefore $\sharp\mu_0 \leqq 2n - 1$.

PROPOSITION 4. *Let (μ, c) be in $\mathscr{M} \times \boldsymbol{R}$, let $P = (A_1, \cdots, A_n,$*

$B_1, \cdots, B_n) = \Gamma(\mu, c)$, let m be the maximum of $|a_1|$ and $|b_1|$, and let M be the maximum of $|A_1|$ and $|B_1|$. Then

$$\|\mu\| \leqq (1 + m^2)\frac{B_1 - A_1}{b_1 - a_1} \tag{5}$$

and

$$|c| \leqq M + \|\mu\| K \tag{6}$$

where K is a constant which depends upon $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ but is independent of the pair (μ, c).

Proof. By the mean value theorem, there is an x in (a_1, b_1) with

$$\frac{B_1 - A_1}{b_1 - a_1} = \frac{\varphi(b_1) - \varphi(a_1)}{b_1 - a_1} = \varphi'(x) .$$

Hence, by (2),

$$\frac{B_1 - A_1}{b_1 - a_1} = \int (1 + t^2)(t - x)^{-2} d\mu(t) . \tag{7}$$

Elementary calculus shows for any x in $[a_1, b_1]$ and for all t in R

$$(1 + t^2)(t - x)^{-2} \geqq (1 + m^2)^{-1} . \tag{8}$$

Combining (7) with (8) gives (5). Now let $r = (a_1 + b_1)/2$. Since the function φ in (1) is nondecreasing on $[a_1, b_1]$, it follows that $|\varphi(r)| \leqq M$ and therefore

$$\begin{aligned} |c| &= \left|\varphi(r) - \int (1 + tr)/(t - r) d\mu(t)\right| \\ &\leqq M + \|\mu\| K \end{aligned}$$

where $K = \sup\{|(1 + tr)/(t - r)| : t \text{ in } U\}$. This completes the proof.

Let X be the closure of U in the one point compactification of the real line. Thus,

$$X = \{\infty\} \cup (-\infty, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_{n-1}, a_n] \cup [b_n, \infty)$$

and a function f on X is continuous if it is continuous in the usual sense at every finite point and if

$$\lim_{x \to +\infty} f(x) = f(\infty) = \lim_{x \to -\infty} f(x) .$$

In the following, we consider measures μ_k in $\mathscr{M}$ and a measure μ with $\mu_k \to \mu$ weak star. By this we mean that μ is a measure on X and that $\int f(t) d\mu_k(t) \to \int f(t)\, d\mu_k(t)$ for every continuous function f

on X.

PROPOSITION 5. *If* $\{(\mu_k, c_k)\}$ *is a sequence of points in* $\mathscr{M} \times \boldsymbol{R}$ *with* $\# \mu_k \leqq 2n-1$, *if* $\mu_k \to \mu$ *weak star and* $c_k \to c$, *and if* $\Gamma(\mu_k, c_k) \to P$, *then* (μ, c) *is in* $\mathscr{M} \times \boldsymbol{R}$ *and*

$$P = \Gamma(\mu, c) + \sum_{j=1}^{n} I_j$$

where I_j *is an ideal point over* $[a_j, b_j]$.

Proof. Let $\Gamma(\mu_k, c_k) = (A_1^k, \cdots, A_n^k, B_1^k, \cdots, B_n^k)$ and let $P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$. First we must prove that μ is in $\mathscr{M}$. Since $\# \mu_k \leqq 2n-1$, it follows that $\# \mu \leqq 2n-1$. It remains to show that $\mu(a_i) = 0 = \mu(b_i)$ for $i = 1, \cdots, n$. We prove that $\mu(a_1) = 0$; a slight alteration of the argument proves that $\mu(a_i) = 0 = \mu(b_i)$ for each i. Let $a = a_1$ and set $F(t) = F_1(t) = (1 + ta)(t-a)^{-1}$. Since $F'(t) = -(1 + a^2)/(t-a)^2$, the function $-F$ is strictly increasing and unbounded on the interval $(-\infty, a)$. Given $\varepsilon > 0$ choose $s \in (-\infty, a)$ so that $-F(t) \geqq 1/\varepsilon$ for t in $[s, a)$. Then

$$\begin{aligned}(1/\varepsilon)\mu_k([s, a]) &\leqq \int_{[s,a)} - F(t)d\mu_k(t) \\ &= -\int F d\mu_k + \int_{X\setminus[s,a)} F d\mu_k \\ &= -A_1^k + c_k + \int_{X\setminus[s,a)} F d\mu_k \,.\end{aligned}$$

Since $\max\{F(t) \colon t \text{ in } X\setminus[s, a)\} = F(b_1)$,

$$(1/\varepsilon)\mu_k([s, a]) \leqq -A_1^k + c_k + F(b_1)\|\mu_k\| \,.$$

Furthermore, since $\mu_k \to \mu$ weak star, the sequence $\{\|\mu_k\|\}$ is bounded. Thus, there is a constant M which is independent of k, s, and ε such that $\mu_k([s, a]) \leqq \varepsilon M$. Let $k \to \infty$; we deduce that $\mu((s, a]) \leqq \varepsilon M$. Then, letting $\varepsilon \to 0$, we conclude that $\mu(a) = 0$.

Let $\Gamma(\mu, c) = (A_1', \cdots, A_n', B_1', \cdots, B_n')$. To show that P has the form stated in the proposition, we must prove that $A_i \leqq A_i'$ and $B_i' \leqq B_i$ for $i = 1, \cdots, n$. Again we prove in detail only that $A_1 \leqq A_1'$. For this, let f be a continuous function on X with $F \leqq f$. Then by (4),

$$\begin{aligned}\int f d\mu = \lim_{k\to\infty} \int f d\mu_k &\geqq \lim_{k\to\infty} \int F d\mu_k \\ &= \lim_{k\to\infty} A_1^k - c_k = A_1 - c \,.\end{aligned}$$

Since f is an arbitrary continuous function with $F \leqq f$, it follows that $A_1' - c = \int F(t)d\mu \geqq A_1 - c$ and thus $A_1' \geqq A_1$. This completes the proof of Proposition 5.

The dual cone for Λ is the set Λ^* of points Q in $\boldsymbol{R}^{2n}$ such that $Q \cdot X \geqq 0$ for each X in Λ. The following result is routine and establishes a useful criterion for a point in the closure of Λ to be in the boundary of Λ.

LEMMA 6. *Let P be a point in the closure of Λ. Then P is in the boundary of Λ if and only if there is a Q in Λ^* with $Q \cdot P = 0$.*

As in [3], the dual cone for Λ corresponds to a cone of non-negative functions. This correspondence is made explicit in the following remarks and in Proposition 7.

For a point $Q = (u_1, \cdots, u_n, v_1, \cdots, v_n)$ in $\boldsymbol{R}^{2n}$, define a rational function $\hat{Q}$ by the equation

$$\hat{Q}(t) = \sum_{i=1}^{n}(u_i F_i(t) + v_i G_i(t)) = \sum_{i=1}^{n}\left(u_i \frac{1 + a_i t}{t - a_i} + v_i \frac{1 + b_i t}{t - b_i}\right) . \tag{9}$$

Observe that $\hat{Q}$ is continuous at ∞, that

$$\hat{Q}(\infty) = \sum_{i=1}^{n} (u_i a_i + v_i b_i) \text{ , and} \tag{10}$$

$$i\hat{Q}(i) = \sum_{i=1}^{n} (u_i + v_i) = -i\hat{Q}(-i) . \tag{11}$$

If $w(t) = \prod_{i=1}^{n}(t - a_i)(t - b_i)$ and $p(t) = w(t)\hat{Q}(t)$, then p is a real polynomial with degree equal to or less than $2n$, $p(i)/w(i)$ is pure imaginary, and

$$\begin{aligned} p(a_i) &= u_i(1 + a_i^2)w'(a_i) \\ p(b_i) &= v_i(1 + b_i^2)w'(b_i) . \end{aligned} \tag{12}$$

Conversely, if p is a real polynomial with degree less than or equal to $2n$, $p(i)/w(i)$ is pure imaginary and if $Q = (u_1, \cdots, u_n, v_1 \cdots, v_n)$ is defined by (9), then $p(t)$ is equal to $w(t)\hat{Q}(t)$. (To see this, note that $p(t) - w(t)\hat{Q}(t)$ vanishes at a_j and b_j for $1 \leqq j \leqq n$ and so is a real constant multiple γ of $w(t)$. However, $p(i)/w(i) - \hat{Q}(i) = \gamma$ and the left hand side of the preceeding equation is pure imaginary. Thus, $\gamma = 0$.) Finally, if (μ, c) is in $\mathscr{M} \times \boldsymbol{R}$, then

$$Q \cdot \Gamma(\mu, c) = ci\hat{Q}(i) + \int \hat{Q}(t)d\mu(t) . \tag{13}$$

Equation (13) is an immediate consequence of (4), (9), and (11).

PROPOSITION 7. *Let Q be a point in $\boldsymbol{R}^{2n}$. Then Q is in Λ^* if and only if $\hat{Q}(t) \geqq 0$ for t in U and $\hat{Q}(i) = 0$. In this case, the point $Q = (u_1, \cdots, u_n, v_1, \cdots, v_n)$ has the property that $u_i \leqq 0 \leqq v_i$ for $i = 1, \cdots, n$.*

Proof. Suppose that Q is in Λ^*. By (13), $ci\hat{Q}(i) = Q \cdot \Gamma(0, c) \geqq 0$ for each c in $\boldsymbol{R}$ and therefore $\hat{Q}(\mathrm{i}) = 0$. For t in U, let δ_t be the unit point mass at t. Then by (13), $\hat{Q}(t) = Q \cdot \Gamma(\delta_t, 0) \geqq 0$ and thus one direction of the proposition is proved. For the reverse assertion, assume that $\hat{Q}(i) = 0$ and $\hat{Q}(t) \geqq 0$ for t in U. It follows by continuity at ∞ that $\hat{Q}(\infty) \geqq 0$. Hence, the point Q is in Λ^* by (13). To show that $u_j \leqq 0 \leqq v_j$, let E_j be the vector in $\boldsymbol{R}^{2n}$ with all zero entries except for a -1 in the jth slot and let F_j be vector in $\boldsymbol{R}^{2n}$ with all zero entries except for a 1 in the $(n + j)$th slot. By Proposition 1, the vectors E_j and F_j are in the closure of Λ. Hence, $0 \leqq Q \cdot E_j$ and $0 \leqq Q \cdot F_j$. These inequalities imply that $u_j \leqq 0 \leqq v_j$ and this completes the proof of the proposition.

PROPOSITION 8. *If $P = \Gamma(\mu, c) + \sum_{j=1}^n I_j$ where I_j is an ideal point over $[a_j, b_j]$ and if P is in the boundary of Λ, then $\sharp\mu + \sum_{j=1}^n \chi(I_j) \leqq n - 1$.*

Proof. By Lemma 6, there is a point $Q = (u_1, \cdots, u_n, v_1, \cdots, v_n)$ in Λ^* such that $Q \cdot P = 0$. Let p be the polynomial $p(t) = w(t)\hat{Q}(t)$ where $w(t) = \prod_{i=1}^n (t - a_i)(t - b_i)$. The proposition is proved by counting certain zeros of p. Equation (13) shows that $Q \cdot P = \int \hat{Q}(t) d\mu(t) + \sum_{j=1}^n Q \cdot I_j$. Since each term on the right is nonnegative and the sum is zero, each term must be equal to zero. Thus, (i) $\mu(\infty)\hat{Q}(\infty) = 0$, (ii) $\int_U \hat{Q}(t) d\mu(t) = 0$, and (iii) $Q \cdot I_j = 0$ for each $j = 1, \cdots, n$. Suppose that $I_j \neq 0$. Since $u_j \leqq 0 \leqq v_j$ by Proposition 7, it follows from (iii) that either $u_j = 0$ or $v_j = 0$. Hence, by (12), either $p(a_j) = 0$ or $p(b_j) = 0$. Since $\hat{Q}(t) \geqq 0$ for t in U and $w(t) > 0$ for t in U, it follows that $p(t) \geqq 0$ for t in U. Thus, the polynomial p must have at least two zeros, counting multiplicities, on $[a_j, b_j]$. If m is the number of nonzero I_j's, that is, if $m = \sum_{j=1}^n \chi(I_j)$, then p has at least $2m$ zeros counting multiplicities on $\bigcup_{i=1}^n [a_i, b_i]$.

Suppose for the moment that $\mu(\infty) = 0$ and set $k = \sharp\mu$. Then (ii) implies that p has at least k zeros on U. Since $p(t) \geqq 0$ for t in U, each of these zeros is of even order. Thus, the polynomial p has at least $2k$ zeros counting multiplicities on U. It follows from Proposition 7 that $p(\mathrm{i}) = 0 = p(-i)$ and therefore the polynomial p has at least $2m + 2k + 2$ zeros. However, the degree of p is equal to or less than $2n$. Hence $2m + 2k + 2 \leqq 2n$ and thus $m + k + 1 \leqq n$.

This proves the proposition for the case $\mu(\infty) = 0$.

Assume now that $\mu(\infty) > 0$. In this case, (ii) implies that p has at least $k - 1$ zeros on U and, as before, each of these zeros has even order. Thus, the number of zeros on U counting multiplicities is at least $2k - 2$. As before, $p(i) = 0 = p(-i)$ and thus p has at least $2m + 2k$ zeros. Since $\mu(\infty) > 0$, (i) implies that $\hat{Q}(\infty) = 0$. We conclude that the degree of p is less than $2n$. However, since $p(t) \geqq 0$ for large $|t|$, the degree of p is even. Thus, the degree of p is equal to or less than $2n - 2$. It follows that $2m + 2k \leqq 2n - 2$ or $m + k + 1 \leqq n$, as before.

PROPOSITION 9. *If $P = \Gamma(\mu, c) + \sum_{j=1}^{n} I_j$ where I_j is an ideal point over $[a_j, b_j]$ and $\#\mu + \sum_{j=1}^{n} \chi(I_j) \leqq n - 1$, then there is point Q in Λ^* such that* (i) $Q \cdot P = 0$; (ii) *for $x \in U$, $\mu(x) > 0$ if and only if $\hat{Q}(x) = 0$;* (iii) *if $I_j = 0$, then $u_j < 0 < v_j$; and* (iv) *if $I_j \neq 0$, then $u_j = 0 = v_j$.*

Proof. Let $x_1, \cdots, x_k$ be the points of support of μ in U, let $\alpha_1, \cdots, \alpha_m$ be the subscripts of the nonzero I_j's, and define a polynomial p by the equation

$$p(t) = (1 + t^2) \prod_{i=1}^{k} (t - x_i)^2 \prod_{i=1}^{m} (t - a_{\alpha_i})(t - b_{\alpha_i}) .$$

Define $Q = (u_1, \cdots, u_n, v_1, \cdots, v_n)$ by (12). Since $p(i) = 0$, the remarks preceeding Proposition 7 imply that $p(t) = w(t)\hat{Q}(t)$. It follows that Q is in Λ^* by Proposition 7 and properties (i)-(iv) are straightforward consequences of the construction of Q.

We turn now to the proof of the theorem. By Proposition 1 and the fact that Λ is a convex cone, every point of form (3) is in the closure of Λ. To prove the reverse assertion, suppose that P is in the closure of Λ. By Proposition 2, there is a point $\bar{P}$ in the boundary of Λ and an ideal point I over $[a_1, b_1]$ with $P = \bar{P} + I$. Since $\bar{P}$ is in the boundary of Λ, there is a sequence of points (μ_k, c_k) in $\mathscr{M} \times \boldsymbol{R}$ such that $\Gamma(\mu_k, c_k) \rightarrow \bar{P}$. Furthermore, by Propositions 3 and 4, we may choose (μ_k, c_k) with $\#\mu_k \leqq 2n - 1$, and we know that there are constants K and L such that $||\mu_k|| \leqq K$ and $|c_k| \leqq L$ for each k. By weak star compactness of the set of measures $\{\nu$ on X: $0 \leqq \nu, ||\nu|| \leqq K\}$ and by the compactness of the interval $[-L, L]$, we may assume there is a measure μ on X and a point c in $\boldsymbol{R}$ such that $\mu_k \rightarrow \mu$ weak star and $c_k \rightarrow c$. By Proposition 5, the pair (μ, c) is in $\mathscr{M} \times \boldsymbol{R}$ and $\bar{P} = \Gamma(\mu, c) + \sum_{j=1}^{n} I_j$ where I_j is an ideal point over $[a_j, b_j]$. Since $\bar{P}$ is in the boundary of Λ, it follows from Proposition 8 that $\#\mu + \sum_{j=1}^{n} \chi(I_j) \leqq n - 1$. Thus, $P = \Gamma(\mu, c) + (I + I_1) + \sum_{j=2}^{n} I_j$ with $\#\mu \leqq n - 1$ and $\#\mu + \chi(I + I_1) + \sum_{j=2}^{n} \chi(I_j) \leqq n$. This proves that P is of form (3) and in fact that P can be represented in form (3) with the additional properties asserted in the theorem.

If the point P of form (3) is in the boundary of Λ, then Proposition 8 implies that $\# \mu + \sum \chi(I_j) \leqq n - 1$. The reverse assertion is immediate from Lemma 6 and Proposition 9.

We prove next the uniqueness assertion of the theorem. Suppose that P is in the boundary of Λ. Then $P = \Gamma(\mu, c) + \sum_{j=1}^{n} I_j$ with I_j an ideal point over $[a_j, b_j]$ and $\# \mu + \sum \chi(I_j) \leqq n - 1$. Let Q in $\boldsymbol{R}^{2n}$ be the point constructed in Proposition 9. Assume now that $P = \Gamma(\mu', c') + \sum_{j=1}^{n} I'_j$ where (μ', c') is in $\mathscr{M} \times \boldsymbol{R}$ and I'_j is an ideal point over $[a_j, b_j]$. Since Q is in Λ^*, $Q \cdot \Gamma(\mu', c') \geqq 0$ and $Q \cdot I'_j \geqq 0$ for each j. However, by (i) of Proposition 9, $0 = Q \cdot P = Q \cdot \Gamma(\mu', c') + \sum_{j=1}^{n} Q \cdot I'_j$ and therefore $Q \cdot \Gamma(\mu', c') = 0$ and $Q \cdot I'_j = 0$ for each j. From the equation $Q \cdot \Gamma(\mu', c') = 0$, from (13), and from (ii) of Proposition 9, we deduce that the finite points in the support of μ' lie in the support of μ. Hence, $\# \mu' \leqq \# \mu + 1$. Let φ be defined by (1) with respect to the pair (μ, c) and let ψ be defined by (1) with respect to the pair (μ', c'). Then the rational function $\varphi - \psi$ has degree equal to or less than $2k + 1$ where $k = \# \mu$. If $I_j = 0$, then by (iii) of Proposition 9 and the fact that $Q \cdot I'_j = 0$, we conclude that $I'_j = 0$. Hence, if $I_j = 0$, then $\varphi(a_j) = \psi(a_j)$ and $\varphi(b_j) = \psi(b_j)$. Since $I_j = 0$ for at least $k + 1$ values of j, the function $\varphi - \psi$ has at least $2(k + 1)$ zeros. Thus, $\varphi = \psi$ and it follows that $\mu = \mu'$ and $c = c'$. From this, we learn immediately that $I_j = I'_j$ for each j.

Now suppose that P is in the interior of Λ. Then $P = \Gamma(\mu, c)$ for some pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$. Let x be a point in $U \cup \{\infty\}$ which is not in the support of μ and let δ be the unit point mass at x. Since P is in the interior of Λ, there is a $y > 0$ such that $P + y(P - \Gamma(\delta, 0))$ is in the interior of Λ. Thus, there is a (μ_1, c_1) in $\mathscr{M} \times \boldsymbol{R}$ with $P + y(P - \Gamma(\delta, 0)) = \Gamma(\mu_1, c_1)$. It follows that $P = \Gamma(\nu, c_1)$ where $\nu = (1 + y)^{-1}\mu_1 + y(1 + y)^{-1}\delta$. Since x is in the support of ν, the measures μ and ν are different. This proves the nonuniqueness of the representation in (3) for points P in the interior of Λ.

It remains to prove the assertion about points in both Λ and the boundary of Λ. We have already shown that if $P = \Gamma(\mu, c)$ with $\# \mu \leqq n - 1$, then P is in the boundary of Λ and it is evidently in Λ. Suppose, on the other hand, that the point P is in Λ and in the boundary of Λ. Since P is in Λ, $P = \Gamma(\mu, c)$ for some (μ, c) in $\mathscr{M} \times \boldsymbol{R}$. Since P is in the boundary of Λ, $P = \Gamma(\mu', c') + \sum_{j=1}^{n} I_j$ with $\# \mu' + \sum_{j=1}^{n} \chi(I_j) \leqq n - 1$. But the uniqueness assertion implies that $\mu = \mu'$, $c = c'$, and $I_j = 0$ for each j. Hence, $\# \mu \leqq n - 1$ and this completes the proof of the theorem.

3. An example and some final comments.

EXAMPLE. We work out in detail the case $n = 2$. To state the result recall that for distinct extended complex numbers z_2, z_3, z_4 and for an arbitrary extended complex number z_1, the cross ratio $\langle z_1, z_2, z_3, z_4\rangle$ is the extended complex number $S(z_1)$ where S is the linear fractional transformation determined by the conditions $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$. In particular, if z_1, z_2, z_3, and z_4 are finite, then $\langle z_1, z_2, z_3, z_4\rangle = (z_1 - z_3)(z_2 - z_4)(z_1 - z_4)^{-1}(z_2 - z_3)^{-1}$. It follows that given distinct extended complex numbers z_2, z_3, z_4 and distinct extended complex numbers w_2, w_3, w_4 and given the extended complex number z_1, then the extended complex number w_1 defined by the equation $\langle w_1, w_2, w_3, w_4\rangle = \langle z_1, z_2, z_3, z_4\rangle$ is the value at z_1 of the linear fractional transformation T satisfying $T(z_2) = w_2$, $T(z_3) = w_3$, and $T(z_4) = w_4$.

PROPOSITION 10. *Let* $n = 2$. *For* $j = 1, 2, \cdots, 5$ *let* Λ_j *be the set of points* (A_1, A_2, B_1, B_2) *satisfying condition* j *defined in the following way*:

(1) $A_1 = A_2 = B_1 = B_2$;

(2) $(A_1 < B_1$ *and* $A_1 \leqq A_2 = B_2 \leqq B_1)$ *or* $(A_2 < B_2$ *and* $A_2 \leqq A_1 = B_1 \leqq B_2)$;

(3) $A_1 < B_1$, $A_2 < B_2$ *and* $[A_1, B_1] \cap [A_2, B_2] \neq \varnothing$;

(4) $A_1 < B_1$, $A_2 < B_2$, $[A_1, B_1] \cap [A_2, B_2] = \varnothing$ *and* $A_1 = k$ *where* $\langle k, A_2, B_1, B_2\rangle = \langle a_1, a_2, b_1, b_2\rangle$;

(5) $A_1 < B_1$, $A_2 < B_2$, $[A_1, B_1] \cap [A_2, B_2] = \varnothing$ *and* $A_1 < k$. *Then the closure of* Λ *is* $\Lambda_1 \cup \cdots \cup \Lambda_5$; *the interior of* Λ *is* $\Lambda_3 \cup \Lambda_5$; *the boundary of* Λ *is* $\Lambda_1 \cup \Lambda_2 \cup \Lambda_4$; *and* $\Lambda = \Lambda_1 \cup \Lambda_3 \cup \Lambda_4 \cup \Lambda_5$.

Proof. Let (μ, c) be a pair in $\mathscr{M} \times \boldsymbol{R}$ and let I_1, I_2 be ideal points over $[a_1, b_1]$, $[a_2, b_2]$, respectively. Define conditions (1′) through (5′) on the tuple (μ, c, I_1, I_2) as follows:

(1′) $\#\mu = 0$ and $\chi(I_1) = \chi(I_2) = 0$; (2′) $\#\mu = 0$ and $\chi(I_1) + \chi(I_2) = 1$;

(3′) $\#\mu = 0$ and $\chi(I_1) + \chi(I_2) = 2$; (4′) $\#\mu = 1$ and $\chi(I_1) = \chi(I_2) = 0$;

(5′) $\#\mu = 1$ and $\chi(I_1) + \chi(I_2) \geqq 1$. Let Λ_i' be the set of points P in $\boldsymbol{R}^4$ such that there is a tuple (μ, c, I_1, I_2) satisfying (i′) with $P = \Gamma(\mu, c) + I_1 + I_2$. The main theorem shows that the closure of Λ is $\Lambda_1' \cup \cdots \cup \Lambda_5'$; the interior of Λ is $\Lambda_3' \cup \Lambda_5'$; the boundary of Λ is $\Lambda_1' \cup \Lambda_2' \cup \Lambda_4'$; and $\Lambda = \Lambda_1' \cup \Lambda_3' \cup \Lambda_4' \cup \Lambda_5'$. Thus, it is sufficient to prove that $\Lambda_i = \Lambda_i'$ for $i = 1, \cdots, 5$. The equalities $\Lambda_i = \Lambda_i'$ for $i = 1, 2, 3, 4$ and the inclusion $\Lambda_5 \subset \Lambda_5'$ are straightforward. The only inclusion which requires any comment is $\Lambda_5' \subset \Lambda_5$. Let T be the (unique) linear fractional transformation with $T(a_2) = A_2$, $T(b_1) = B_1$, $T(b_2) = B_2$. Then T lies in $\mathscr{R}$ since T maps the real line into the extended real line and $T(a_2) < T(b_2)$ so that T is increasing on $\boldsymbol{R}$. Note that $k = T(a_1)$ and the pole of T is not in $[a_1, b_1] \cup [a_2, b_2]$. Thus, there is a pair (ν, d) in $\mathscr{M} \times \boldsymbol{R}$ with $\#\nu = 1$ and

$$\Gamma(\nu, d) = (T(a_1), T(a_2), T(b_1), T(b_2)) .$$

Suppose, to reach a contradiction, that $k \leqq A_1$. Then

$$\Gamma(\nu, d) = P + \tilde{I}_1$$

where $\tilde{I}_1$ is an ideal point over $[a_1, b_1]$. Thus,

$$\Gamma(\nu, d) = \Gamma(\mu, c) + I_1 + I_2 + \tilde{I}_1 .$$

But the theorem implies $\Gamma(\nu, d)$ lies in the boundary of Λ since $\sharp\nu = 1$ and thus by the theorem has a unique representation. This contradiction shows that $k > A_1$, as desired.

We now prove the assertion in the introduction that it is always possible to map points to points.

PROPOSITION 11. *Let $c_1, \cdots, c_m$ be distinct points in $\boldsymbol{R}$ and let $C_1, \cdots, C_m$ be any points in $\boldsymbol{R}$. Then there is a ψ in $\mathscr{R}$ with $\psi(c_j) = C_j$ for $j = 1, \cdots, m$.*

Proof. Let $F_j(t) = (1 + tc_j)(t - c_j)^{-1}$ for $j = 1, \cdots, m$ and let Φ be the convex cone of all positive measures on $\boldsymbol{R}\backslash\{c_1, \cdots, c_m\}$ with a finite number of points of support. Map Φ into $\boldsymbol{R}^m$ by

$$\mu \longrightarrow \left\{\int F_j d\mu\right\}_{k=1}^{m} .$$

The image of Φ is a convex cone in $\boldsymbol{R}^m$. If it is not all of $\boldsymbol{R}^m$, then there are scalars $r_1, \cdots, r_m$ not all zero with

$$0 \leqq \Sigma_1^m r_j \int F_j d\mu , \quad \mu \in \Phi$$

so that $\Sigma_1^m r_j F_j$ is nonnegative on $\boldsymbol{R}\backslash\{c_1, \cdots, c_m\}$. However, $\lim_{t \uparrow c_j} F_j = -\infty$ while $\lim_{t \downarrow c_j} F_j = +\infty$. These imply that $r_j = 0$ for all j, and hence the image of Φ is all of $\boldsymbol{R}^m$.

Finally, we consider the following question: What is the smallest integer $q(n)$ such that every point P is Λ is of the form $P = \Gamma(\mu, c)$ for some pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ with $\sharp\mu \leqq q(n)$? It is evident that $q(1) = 1$, a careful study of Proposition 10 shows that $q(2) = 2$, and Proposition 3 asserts that $q(n) \leqq 2n - 1$. The following proposition shows that $q(n) \geqq 2n - 2$.

PROPOSITION 12. *There is point P in Λ such that $\sharp\mu \geqq 2n - 2$ for every pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ with $\Gamma(\mu, c) = P$.*

To prove this proposition, we first establish the following lemma.

LEMMA. *Assume that* (μ_k, c_k) *are in* $\mathscr{M} \times \boldsymbol{R}$, *that* $\mu_k \to 0$ *weak star, that* $c_k \to c$, *and that* $\Gamma(\mu_k, c_k) \to P = (A_1, \cdots, A_n, B_1, \cdots, B_n)$. *If there is an* $\alpha \in \{1, \cdots, n\}$ *and an* $s < a_\alpha$ *such that* $\mu_k((s, a_\alpha)) = 0$ *for each* k, *then* $A_\alpha = c$. *If there is a* $\beta \in \{1, \cdots, n\}$ *and an* $s > b_\beta$ *such that* $\mu_k((b_\beta, s)) = 0$ *for each* k, *then* $B_\beta = c$.

Proof. To simplify the notation, we prove in detail only the case where there is an $s < a_1$ with $\mu_k((s, a_1)) = 0$ for each k. Let $a = a_1$, let $F(t) = F_1(t) = (1 + ta)(t - a)^{-1}$, and let $\Gamma(\mu_k, c_k) = (A_1^k, \cdots, A_n^k, B_1^k, \cdots, B_n^k)$. Then by (4),

$$A_1^k = c_k + \int_{X \setminus (s,a]} F(t) d\mu_k(t) .$$

Letting $k \to \infty$ and recalling that $\mu_k \to 0$ weakstar, we deduce that $A_1 = c$, which proves the lemma.

Turning to the proposition, we may assume that $n \geqq 2$. Let $I = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ with $A_1 = \cdots = A_{n-1} = -1$, $A_n = 0$, $B_1 = \cdots = B_{n-1} = 1$, and $B_n = 0$ and let $\Lambda(2n - 3)$ be the set of points P in Λ such that $P = \Gamma(\mu, c)$ for some pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ with $\#\mu \leqq 2n - 3$. We prove the proposition by showing that I is not in the closure of $\Lambda(2n - 3)$. Since I is in the closure of Λ by the main theorem, it then follows that $\Lambda(2n - 3)$ does not equal Λ.

To show that I is not in the closure of $\Lambda(2n - 3)$, assume that it is and let (μ_k, c_k) be points in $\mathscr{M} \times \boldsymbol{R}$ with $\#\mu_k \leqq 2n - 3$ and $\Gamma(\mu_k, c_k) \to I$. By Propositions 4 and 5, we may assume that there is a pair (μ, c) in $\mathscr{M} \times \boldsymbol{R}$ with $\mu_k \to \mu$ weak star, $c_k \to c$, and $I = \Gamma(\mu, c) + \sum_{j=1}^n I_j$ where I_j is an ideal point over $[a_j, b_j]$. However, since $A_n = 0 = B_n$, it follows from (2) that $\mu = 0$ and with $\mu = 0$ it follows that $c = 0$. Since the conclusion to the lemma fails for $\alpha = 1, \cdots, n - 1$ and $\beta = 1, \cdots, n - 1$, the hypothesis also fails. In fact the hypothesis must fail for any subsequence of the sequence $\{(\mu_k, c_k)\}$. It follows that for any $\varepsilon > 0$, there is a K such that $\mu_k((a_k - \varepsilon, a)) > 0$ and $\mu_k((b_k, b_k + \varepsilon)) > 0$ for all $i = 1, \cdots, n - 1$ and all $k \geqq K$. In particular, one can choose ε small enough so that the intervals $(a_1 - \varepsilon, a_1), \cdots, (a_{n-1} - \varepsilon, a_n), (b_1, b_1 + \varepsilon), \cdots, (b_{n-1}, b_{n-1} + \varepsilon)$ are pairwise disjoint. For this choice of ε and for $k \geqq K$, it follows that $\#\mu_k \geqq 2n - 2$, a contradiction. Hence, the point I is not in the closure of $\Lambda(2n - 3)$ and this completes the proof of the proposition.

We have not been able to decide whether $q(n) = 2n - 2$ or $2n - 1$ for $n > 2$.

REFERENCES

1. N. Achiezer and M. G. Krein, *Some Questions in the Theory of Moments*, Amer. Math. Soc. Translation Series, Vol. 2, Providence, R. I., 1962.

2. W. F. Donoghue, Jr., *Monotone matrix functions and analytic continuation*, Springer, New York, 1974.
3. S. Karlin and W. J. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics,* John Wiley and Sons, New York, 1966.

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