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IRREDUCIBLE OPERATORS WHOSE SPECTRA ARE SPECTRAL SETS

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In this note those compact subsets of the plane that are the spectra of irreducible subnormal operators are characterized.

In December 1977 John B. Conway presented a colloquium talk at Virginia Tech and asked, "Which compact subsets of the plane are the spectra of irreducible subnormal operators?" If the adjective irreducible is replaced by pure in this question, then Clancey and Putnam [2] have the following answer: A compact set K is the spectrum of a pure subnormal operator if and only if for every open disc Δ that has a nonempty intersection with K, we have $R(K \cap \Delta^-) \neq C(K \cap \Delta^-)$. Similarly, our answer to Conway's question will be a function algebraic characterization. For the basic facts concerning this area we refer to [3] or [10].

If $\mathscr{H}$ is a separable Hilbert space the algebra of continuous operators on $\mathscr{H}$ will be denoted $\mathscr{B}(\mathscr{H})$. An operator $T \in \mathscr{B}(\mathscr{H})$ is irreducible if $\mathscr{H}$ has no nonzero subspace that is invariant under T and its adjoint T^*. For the basic facts concerning subnormal operators we refer to [4]. If $T \in \mathscr{B}(\mathscr{H})$ then $\sigma(T)$ will denote the spectrum of T.

If K is a compact subset of the plane, $\mathscr{R}(K)$ will denote the collection of rational functions with poles off K; the uniform closure of $\mathscr{R}(K)$ in $C(K)$, the algebra of continuous functions on K, is denoted $R(K)$. If φ belongs to the maximal ideal space of $R(K)$, then there exists a point $z \in K$ such that $\varphi(f) = f(z)$ for all $f \in R(K)$. Hence the Gleason parts of $R(K)$ form a partition of K.

If $T \in \mathscr{B}(\mathscr{H})$ and K is a compact set containing $\sigma(T)$, then K is called a spectral set for T if

$$\text{(i)} \qquad \|f(T)\| \leqq \|f\|_K$$

for all $f \in \mathscr{R}(K)$. (Here $\|f\|_K$ denotes the sup norm of f on K). If K is a spectral set for T, it is easy to define $f(T)(\in \mathscr{B}(\mathscr{H}))$ for all $f \in R(K)$. A basic fact about a subnormal operator S is that $\sigma(S)$ is a spectral set and equality holds in (i) with $K = \sigma(S)$ for all $f \in R(\sigma(S))$.

THEOREM 1. *A compact set K is the spectrum of an irreducible operator T whose spectrum is a spectral set if and only if $R(K)$ has one nontrivial Gleason part G and $G^- = K$.*

REMARK. In the sufficiency part of the proof the operator we construct is subnormal. Hence, the theorem answers the question of Conway. For the necessity part of the proof we need two (nontrivial) results and an elementary fact.

Fact. Suppose K_i is a spectral set for $T_i \in \mathscr{B}(\mathscr{H}_i)$ and $K = (\cup K_i)^-$ is compact. Then K is a spectral set for $T = \bigoplus T_i$.

We leave the proof to the reader.

THEOREM 2. (*Mlak, Lautzenheiser, Seever*). *Let K be a spectral set for $T \in \mathscr{B}(\mathscr{H})$ and let $G_1, G_2, \cdots$ be the nontrivial Gleason parts of $R(K)$. Then $T = N \oplus (\bigoplus_i T_i)$, where N is a normal operator with spectrum contained in the boundary of K and G_i^- is a spectral set for T_i. The proof of this and related results can be found in* [5, 7, 8, 9].

The following theorem (combined with the fact and Theorem 2) guarantees that if K is a spectral set for T with at least two nontrivial Gleason parts G_1, G_2 for $R(K)$ such that $\sigma(T) \cap G_i \neq \varnothing$ $i = 1, 2$, then T has a nontrivial reducing subspace.

THEOREM 3. (*Melnikov*). *Let G_1 be a nontrivial Gleason part for $R(K)$. If $a \in G_1^- \backslash G_1$ then a is a peak point. In particular, if G_2 is another Gleason part then $G_2 \cap G_1^- = \varnothing$.*

The proof of this result can be found in [6].

Proof of Theorem 1. The only thing left to establish is if K is a compact set with one nontrivial Gleason part G and $G^- = K$, then K is the spectrum of an irreducible subnormal operator. Choose a sequence $\{z_n\}$ of points belonging to G such that $\{z_n\}^- = K$. Let δ_n denote point mass measure at z_n. Fix $z_0 \in G$.

For each z_n choose a representing measure λ_n for $R(K)$ at z_0 such that δ_n is absolutely continuous with respect to λ_n (such a measure can be found by [10, p. 165]). Then

$$\lambda = \sum_n \frac{\lambda_n}{2^n}$$

is a representing measure for $R(K)$ at z_0 such that the support of λ is K. Let $\mathscr{H} = R^2(\lambda)$, the $L^2(\lambda)$ closure of $R(K)$, and let S be multiplication by z. Clearly $\sigma(S) = K$ so we need to show S is irreducible.

Observe first that if $f \in R(K)$ then

$$\begin{aligned} |f(z_0)| &= \left|\int f d\lambda\right| \\ &= |\langle f, 1\rangle| . \end{aligned}$$

(Here $\langle\ ,\ \rangle$ denotes the inner product in $R^2(\lambda)$.) It is now routine to show that

$$\text{(ii)} \qquad \langle h_1 h_2, 1\rangle = \langle h_1, 1\rangle\langle h_2, 1\rangle$$

for all $h_i \in R^2(\lambda) \cap L^\infty(\lambda)$ for $i = 1, 2$.

Using the techniques of [12], one can show that the commutant of S is the set $\{M_\psi: \psi \in R^2(\lambda) \cap L^\infty(\lambda)\}$, where M_ψ denotes multiplication by ψ. Hence, if P is a projection commuting with S, then $P = M_f$, where f is the characteristic function of a Borel set F. By (ii) we see that $\langle f, 1\rangle$ equals zero or one. We assume that $\langle f, 1\rangle = 1$ (otherwise we work with the projection $1 - P = M_{1-f}$). Hence

$$\begin{aligned} 1 &= \langle f, 1\rangle \\ &= \int_F d\lambda \\ &= \lambda(F) . \end{aligned}$$

Since λ is a probability measure, $f = 1$ almost everywhere; hence $P = 1$.

Unfortunately there are no known topological criteria that characterize the sets which are nontrivial Gleason parts of $R(K)$. (Many necessary topological and measure theoretic facts are known: (1) $\bar{G}$ is connected. (2) G is σ-compact. (3) G has (area) density one at each of its points. (4) G is area connected (consult [6]), etc.) However, the literature contains many interesting examples discussing various properties of these parts. Combining these examples with Theorem 1, one comes up with interesting operator theoretic consequences.

Fix an operator T whose spectrum is a spectral set. If $\sigma(T)$ has two components of its interior, say U_1 and U_2 such that $\sigma(T) = (U_1 \cup U_2)^-$ one may ask if $T = T_1 \oplus T_2$ with $\sigma(T_i) \subset U_i^-$ for $i = 1, 2$. However, since there exists a disconnected nontrivial part [10, Examples 26.24 and 26.25] the answer, in general, is no. (The first person to observe this phenomenon was Lautzenheiser in his thesis [5]. In this interesting work he also discusses how Theorem 2 subsumes many other known results.) One may also wonder if it is possible to write $T = T_1 \oplus T_2$ with $\sigma(T_1) \subset (\operatorname{int} \sigma(T))^-$ and $\sigma(T_2) \subset (\sigma(T) \backslash \operatorname{int} \sigma(T))$. Again the answer is no. Let $D = \{z: |z| < 1\}$, $D_+ = \{z \in D: \operatorname{Re} z > 0\}$ and $D_- = \{z \in D: \operatorname{Re} z < 0\}$. Construct a compact set F from the closed unit disc D^- by removing pairwise disjoint open discs D_i from D_+ such that int $(D_+ \backslash \cup D_i) = \varnothing$ and $\sum r_i < \infty$ (r_i is the radius of D_i). Using Theorem 2 in [11] and Theorem 3 in [1], we can construct F such that zero is not a peak point of $R(F)$. Let $K = G^-$ where G is the nontrivial part of $R(F)$ containing zero. (Note: $G \cap D_+ \neq \varnothing$ and $G \supset D_-$.)

REFERENCES

1. A. Browder, *Point derivations on function algebras*, J. Functional Anal., **1** (1967), 22-27.

2. K. Clancey and C. R. Putnam, *The local spectral behavior of completely subnormal operators*, Trans. Amer. Math. Soc., **163** (1972), 239-244.

3. T. Gamelin, *Uniform Algebras*, Printice-Hall, Englewood Cliffs, NJ., 1969.

4. P. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, NJ., 1967.

5. R. Lautzenheiser, *Spectral sets, reducing subspaces and function algebras*, Thesis, Indiana University, 1973.

6. M. Melnikov, *The Gleason parts of the algebra* $R(X)$, Math. Sb., **101** (143) (1976), 293-300.

7. W. Mlak, *Decompositions and extensions of operator valued representations of function algebras*, Acta Sci. Math. (Szeged), **30** (1969), 181-193.

8. ———, *Partitions of spectral sets*, Ann. Pol. Math., **25** (1972), 273-280.

9. G. L. Seever, *Operator representations of uniform algebras. I.*, preprint.

10. E. L. Stout, *The theory of uniform algebras*, Bogden and Quigley, Tarrytown-on-Hudson NJ., 1971,

11. J. Wermer, *Bounded point derivations on certain Banach algebras*, J. Functional Anal., **1** (1967), 28-36.

12. T. Yoshino, *Subnormal operators with a cyclic vector*, Tohoku Math. J., **21** (1969), 47-55.

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