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HOMOGENEOUS MODELS AND DECIDABILITY

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Fix a countable first order structure $\mathscr{A}$ realizing only recursive types. It is known that if $\mathscr{A}$ is prime or saturated then it is decidable iff the set of types it realizes is recursively enumerable. A natural conjecture was that the techniques of proof for those two cases could be combined to produce the result for those $\mathscr{A}$ that are homogeneous. This paper provides a negative answer to that conjecture.

For a complete decidable theory T, let $\{\theta_i \mid i < w\}$ be some fixed effective enumeration of all the formulas of $L(T)$. Then by an *index* for a recursive n-type $\Gamma(x_1, \cdots, x_n)$ of T we mean a natural number e satisfying:

$$\{e\}(i) = \begin{cases} 0 & \text{if} \quad \theta_i \in \Gamma \\ 1 & \text{otherwise} \end{cases}$$

(for notation, see [5]). Also, if Φ is a set of recursive types of T, then a *witness set* A for Φ is a set of natural numbers satisfying:

(1) $\forall n \in A \; \exists \Gamma \in \Phi$ (n is an index for Γ); and

(2) $\forall \Gamma \in \Phi \; \exists n \in A$ (n is an index for Γ).

If Φ is exactly the set of types $\Gamma(x_1, \cdots, x_n)$ realized in some model $\mathscr{A}$ of T satisfying $(x_i \neq x_j) \in \Gamma(x_1, \cdots, x_n)$, $1 \leqq i < j \leqq n$, $n < \omega$ then we also say that A is a witness set for $\mathscr{A}$. Finally, a model $\mathscr{A}$ of T is *decidable* just in case the theory of $(\mathscr{A}, a_i)_{i<\omega}$ is decidable for some indexing $\{a_i \mid i < \omega\}$ of $|\mathscr{A}|$. An undecidable model is a countable model that is not decidable.

Assume now that $\mathscr{C}$ is a prime model. Harrington [2] proved an equivalent version of (by the definitions, if a set of types has a witness set, then those types are recursive):

(*) $\mathscr{C}$ is decidable iff $\mathscr{C}$ has an r.e. witness set.

From a recursion theoretic point of view, the principal device in the proof is a "wait and see" argument. Millar and Morley independently proved that (*) remains true when $\mathscr{C}$ is assumed to be countable and saturated. The principal recursion theoretic technique employed is a finite injury priority argument. Notice that a prime or saturated model is automatically homogeneous, and that any hemogeneous model is uniquely determined, up to isomorphism, by the set of types it realizes. It was therefore very natural that Morley asked whether (*) remained true under just the assumption that $\mathscr{C}$ was countable and homogeneous. This paper provides a negative answer. Interestingly, the construction exploits an "infinite injury". Sufficient conditions for such $\mathscr{C}$ to be decidable are discussed in, another one of interest is that (*) also holds if $\mathscr{C}$ is countable and recursively saturated:

THEOREM 1. *If $\mathscr{A}$ is a countable, recursively saturated structure realizing only recursive types, then $\mathscr{A}$ is decidable if and only if $\mathscr{A}$ has a Σ_2^0 witness set.*

Proof. In [Millar], the following theorem is proved.

Let T be a complete decidable theory with a countable saturated model. Then T has a decidable, homogeneous model realizing all of the recursive types of T iff the set of recursive types of T has a Σ_2^0 witness set.

In that paper, the only use made of the assumption that T has a countable saturated model is in proving that every recursive, consistent (with T) set of formulas in a finite number of free variables is contained in a recursive type of T. Therefore, if we can show that $\mathrm{Th}(\mathscr{A})$ automatically has that derived property, then the conclusion of the theorem holds for $\mathrm{Th}(\mathscr{A})$. So let $\Phi(x_1, \cdots, x_n)$ be any recursive, consistent (with $\mathrm{Th}(\mathscr{A})$) set of formulas in the displayed free variables. Since $\mathscr{A}$ is recursively saturated, Φ is realized in $\mathscr{A}$ by some $\langle a_1, \cdots, a_n\rangle$. Then by the assumption that $\mathscr{A}$ realizes only recursive types, the type realized by $\langle a_1, \cdots, a_n\rangle$ is recursive and contains Φ. Since obviously every recursive type of $\mathrm{Th}(\mathscr{A})$ is realized in $\mathscr{A}$, we see that $\mathrm{Th}(\mathscr{A})$ has a decidable homogeneous model $\mathscr{B}$ realizing every recursive type. Since recursively saturated structures are ω-homogeneous, $\mathscr{A}$ and $\mathscr{B}$ are countable, homogeneous structures realizing exactly the same types. Therefore by a well known theorem, $\mathscr{A}$ and $\mathscr{B}$ are isomorphic. Thus $\mathscr{A}$ is decidable.

THEOREM 2. *There is a homogeneous undecidable model with an* r.e. *witness set.*

In fact, we will specify a complete theory T that has a countable homogeneous model $\mathscr{C}$ with an r.e. witness set A, such that no decidable model has A as a witness set. Notice that if $\mathscr{B}$ is a decidable model and, $b_1, b_2 \in |\mathscr{B}|$ realize the types Γ_1, Γ_2 respectively, then we can effectively find a recursive type Σ realized in $\mathscr{B}$ such that $\Gamma_1(x_1) \cup \Gamma_2(x_2) \subset \Sigma(x_1, x_2)$. It is a slight variation on this ability that we "diagonalize across" in constructing T. T will be the complete, model completion of a universal theory T'. The language for T contains unary predicated symbols $\{P_i \mid i < \omega\}$ and binary predicate symbols $\{S_i \mid i < \omega\}$. For recursive $F, G \subset 2^{<\omega}$ and recursive $H \subset 2^{<\omega} \times 2^{<\omega} \times 2^{<\omega}$ to be determined later, the axioms for T' are the universal closures of ($2^{<\omega}$ is the set of finite sequences of 0's and 1's):

(I) $S_i(x, x)$ for all $i < \omega$;

(II) $S_i(x, y) \to S_i(y, x)$ for all $i < \omega$;

(III) $\neg \bigwedge_{i<l(\underline{f})} P_i(x)^{\underline{f}(i)}$ for all $\underline{f} \in 2^{<\omega} - F$ (where $l(\underline{f})'$ is the length of $\underline{f}$);

(IV) $\neg \bigwedge_{i<l(\underline{g})} S_i(x, y)^{\underline{g}(i)}$ for all $\underline{g} \in 2^{<\omega} - G$; and

(V) $\neg[\bigwedge_{i<l(\underline{f}_0)} P_i(x)^{\underline{f}_0(i)} \wedge \bigwedge_{i<l(\underline{f}_1)} P_i(y)^{\underline{f}_1(i)} \wedge \bigwedge_{i<l(\underline{g})} S_i(x, y)^{\underline{g}(i)}]$ for all $\langle \underline{f}_0, \underline{f}_1, \underline{g}\rangle \in H$,

where for any formula ψ, $\psi^0 \equiv \psi$ and $\psi^1 \equiv \neg\psi$. We must now determine F, G, and H.

Let $\{\mu_n : n < \omega\}$ be a fixed effective enumeration of all partial recursive $\mu : \omega \times \omega \times 2^{<\omega} \to 2$. For any partial recursive $v : \omega \times \omega \times 2^{<\omega} \to 2$, we denote by $v^{(s)}$ the function $\{\langle m, n, \underline{f}, k\rangle \mid v(m, n, \underline{f})$ converges within s steps and equals $k\}$. Notice that the domain and range of $v^{(s)}$ are recursive sets, uniformly in s. Also fix a total recursive $M : \omega \to \omega$ with the property that each element of ω has infinitely many pre-images and $M(n) \neq n$ for any $n < \omega$. F_n, G_n, and H_n^m will be defined by an induction, and then we define $F = \bigcup_{n<\omega} F_n$, $G = \bigcup_{n<\omega} G_n$, and $H = \bigcup_{n,m<\omega} H_n^m$. We introduce various bookkeeping devices to facilitate the construction. First there will be markers $\{\square_i^k \mid k < 3; i < \omega\}$ that will occasionally be defined on elements of F and G. For each $n < \omega$, at most one element of ω will be *n-fixed*, whereas various elements might be *n-designated* (although never more than one at any one time). Also, each $n < \omega$ can be in one of three states: active, passive, or retired. We will use "proceed" in the construction to mean "go on to the next step of the induction". At the beginning of the construction, all markers are undefined and all states are active. At step n, F_n, G_n and H_n^m will be completely determined, $m < \omega$.

Step 0: $F_0 = G_0 = \{\langle 0\rangle, \langle 1\rangle\}$, $H_0^m = \varnothing$, $m < \omega$;

Step $n+1$: There are two conditions that automatically put elements into the sets that we are defining:

(1) If $\underline{f} \in F_n^m(G_n^m)$ then $F^\frown\langle 0\rangle \in F_{n+1}^m(G_{n+1}^m)$; and

(2) $\underline{I}^\frown\langle k, 1\rangle \in F_{n+1}$, G_{n+1} for $k = 0$ and 1, where $\underline{I}$ is a sequence of $n-1$ zeros.

After this is done, then some of the markers are immediately defined or redefined according to:

(1) Define $\square_n^k$ on $\underline{I}^\frown\langle 1, k\rangle$ in F_{n+1} for $k = 0, 1$, where $\underline{I}$ is as above;

(2) Define $\square_n^2$ on $\underline{I}^\frown\langle 1\rangle$ in G_{n+1} for the same I; and

(3) For each active $j < n$ and defined marker $\square_j^i$ $i = 0, 1$, and $j \neq M(n)$ where let us say $\square_j^i$ is defined on $\underline{f}_{ij}$:

(a) Just in case there is a j-fixed integer (and by assumption, j is active), redefine $\square_j^0$ on $\underline{f}_{0j}\hat{}\underline{I}'$ in F_{n+1} where $\underline{I}'$ is a sequence of zeroes such that $\underline{f}_{0j}\hat{}I'$ has length $n + 2$; and

(b) If there is presently a j-designated number, then redefine $\square_j^1$ on $\underline{f}_{1j}\hat{}I''$ in F_{n+1} in a similar fashion.

It is easy to check that the required elements are in F_{n+1} in the instructions for (a) and (b) above, we leave this to the reader.

For notational convenience, let m be equal to $M(n)$. There are now several cases ($H_{n+1}^i = \varnothing$ automatically for all $i \neq m$):

(I) If $\square_m^0$ is undefined or m is retired, then put $H_{n+1}^m = \varnothing$ and proceed;

(II) If $\square_m^0$ is defined and m is active, but there is presently no m-fixed integer, then let $\underline{f}$ be the element on which $\square_m^0$ is presently defined, and put $H_n^m = \varnothing$. If there is no $s < n$ such that $\mu_m^{(n)}(s, s, \underline{f}) = 0$, then proceed. Otherwise, m-fix the least such s, and then proceed;

(III) If $\square_m^0$ is defined, m is active, there is an m-fixed integer, but there is no m-designated integer, then let $\underline{f}$, $\underline{g}$ be the elements on which $\square_m^1$, $\square_m^2$ are defined, respectively, let m_0 be the m-fixed integer. If there is no $s < n$ such that $\mu_m^{(n)}(s, s, \underline{f}) = \mu_m^{(n)}(m_0, s, \underline{g}) = 0$, then proceed. If such an s exists that is m_0 or has been previously m-designated, then retire m and proceed. Otherwise m-designate the least such s and then proceed;

(IV) If m is active but none of (I)-(III) apply, then let $\underline{g}$ be the element on which $\square_m^2$ is defined, m_0 the m-fixed integer, and d the m-designated integer.

(A) If $\mu_m^{(n)}(m_0, d, \underline{g}\hat{}\langle k\rangle) \neq 0$ for both $k = 0$ and 1, then put $H_{n+1}^m = \varnothing$ and proceed;

(B) If $\mu_m^{(n)}(m_0, d, \underline{g}\hat{}\langle k\rangle) = 0$ for both $k = 0$ and 1, then retire m, put $H_{n+1}^m = \varnothing$ and proceed;

(C) If neither (A) nor (B), then let $\underline{f}_i$ be the element of F_n (notice "F_n", not "F_{n+1}") that $\square_m^i$ was defined on $i = 0, 1$, and let k be such that $\mu_m^{(n)}(m_0, d, \underline{g}\hat{}\langle k\rangle) = 0$. Now, in this case

(i) $\underline{f}_i\hat{}\langle 1\rangle \in F_{n+1}$ for $i = 0, 1$;

(ii) $\langle \underline{f}_0\hat{}\langle 0\rangle, \underline{f}_1\hat{}\langle 0\rangle, \underline{g}\hat{}\langle k\rangle\rangle \in H_n^m$; and

(iii) m is changed to the passive state;

after this is done, then proceed.

(V) If m is passive, then put $H_{n+1}^m = \varnothing$ and let m_0 and d be as above, and let $\langle \underline{f}_0\hat{}\langle 0\rangle, \underline{f}_1\hat{}\langle 0\rangle, \underline{g}\hat{}\langle k\rangle\rangle$ be the last element put into $\bigcup_{i<n} H_i^m$ (again, that this exists is left to the reader to check). If either of $\mu_m^{(n)}(m_0, m_0, \underline{f}_0\hat{}\langle 1\rangle)$, $\mu_m^{(n)}(d, d, \underline{f}_1\hat{}\langle 0\rangle)$ are undefined, then proceed. Otherwise,

(A) If $\mu_m^{(n)}(m_0, m_0, \underline{f}_0{}^\frown\langle 1\rangle) = 1$ and $\mu_m^{(n)}(d, d, \underline{f}_1{}^\frown\langle 0\rangle) = 0$, then retire m and proceed;

(B) If both values are 1, then:

(i) $\square_m^i$ is redefined on $\underline{f}_i{}^\frown\langle i\rangle$, $i = 0, 1$;

(ii) $\square_m^2$ is redefined on $\underline{g}^\frown\langle k\rangle^\frown\underline{I}'$ in G_n (not G_{n+1}), where $\underline{I}'$ is the appropriate sequence of zeros;

(iii) $\underline{g}^\frown\langle k\rangle^\frown\underline{I}'^\frown\langle 1\rangle \in G_{n+1}$, $\underline{I}'$ as in (ii); and

(iv) m is changed to the active state,

after this is done, then proceed.

(C) If $\mu_m^{(n)}(m_0, m_0, \underline{f}_0{}^\frown\langle 1\rangle) = 0$ and not (A), then:

(i) m is changed to the active state;

(ii) $\square_m^i$ is redefined on $\underline{f}_i{}^\frown\langle 1 - i\rangle$, $i = 0, 1$;

(iii) $\square_m^2$ is redefined on $\underline{g}^\frown\langle 1 - k\rangle^\frown\underline{I}'^\frown\langle 1\rangle \in G_n$ (not G_{n+1}), where $\underline{I}'$ is the appropriate sequence of zeroes;

(iv) $\langle\underline{g}^\frown\langle 1 - k\rangle^\frown\underline{I}'^\frown\langle 1\rangle\rangle \in G_{n+1}$; and

(v) d is no longer m-designated.

This ends the construction.

Define the partial ordering $\leqq$ on $2^{<\omega} \cup 2^\omega$ by $\underline{f} \leqq \underline{g}$ just in case $\underline{f}$ is a proper initial segment of $\underline{g}$, or $\underline{f} = \underline{g}$. Next we list some of the important properties of our end products:

LEMMA 1. (i) *$F = \bigcup_{n<\omega} F_n$, $G = \bigcup_{n<\omega} G_n$, $H^m = \bigcup_{n<\omega} H_n^m$, and $H = \bigcup_{n<\omega} H^m$ are all recursive;*

(ii) *If $\underline{f} \in F(G)$ then $\underline{f}^\frown\underline{I} \in F(G)$ for all finite sequences of zeros I;*

(iii) *Every element $\langle\underline{f}_1, \underline{f}_2, \underline{g}\rangle \in H$ satisfies $l(\underline{f}_1) = l(\underline{f}_2)$ and $\underline{f}_1(r) = \underline{f}_2(r) = 0$, uhere $r = l(\underline{f}_1)$;*

(iv) *If $\langle\underline{f}_1{}^\frown\langle 0\rangle, \underline{f}_2{}^\frown\langle 0\rangle, \underline{g}^\frown\langle k\rangle\rangle \in H$ then $\underline{f}_i{}^\frown\langle 1\rangle \in F$, $i = 1, 2$, $\underline{g}^\frown\langle 1 - k\rangle \in G$ and $\underline{g}$ was marked by a $\square_s^2$ marked for some s;*

(v) *If $\langle\underline{f}_1{}^\frown\langle 0\rangle, \underline{f}_2{}^\frown\langle 0\rangle, \underline{g}\rangle$, $\langle\underline{f}_1', \underline{f}_2', \underline{g}'\rangle \in H^m$ and $l(\underline{g}) < l(\underline{g}')$ then $\underline{f}_i \leqq \underline{f}_i'$ for $i = 1, 2$ and $\underline{f}_1{}^\frown\langle 0\rangle \leqq \underline{f}_1'$ iff $\underline{f}_2{}^\frown\langle 1\rangle \leqq \underline{f}_2'$ iff $\underline{g} \leqq \underline{g}'$; and*

(vi) *There are only countably many $f \in 2^\omega$ such that $\underline{f} \in F(G)$ for all $\underline{f} \leqq f$.*

Proof. The proof amounts to a routine check of the construction, and we leave most of the details to the reader. (ii) is a consequence of (1) at the beginning of step $n + 1$. The second part of (iii) follows essentially from (IV) (C) (ii), since that constitues the only circumstance in which an element is placed in H. (iv) follows from the last remark and (IV) (C) (i) (iii), and (B) (iii). For (v) first notice by the above that when $\langle\underline{f}_1{}^\frown\langle 0\rangle, \underline{f}_2{}^\frown\langle 0\rangle, \underline{g}\rangle$ is placed in H^m during the construction, by (IV) (C) (iii) m is then changed to the passive state. If another element is to appear in H^m after that point, m must return to the active state. This can only occur via (V), (B) or

(C). Thus, that (v) holds follows simply from an examination of the instructions in those two cases and a simple induction. Finally, for (vi) we see that if $g \in 2^\omega$ is such that for infinitely many $\underset{\sim}{g} \leqq g$ there is an f as in (vi) satisfying $\underset{\sim}{g} \leqq f$ and $f \neq g$, then for some m and i, infinitely many $\underset{\sim}{g} \leqq g$ have $\square_m^i$ defined on them at the end of some stage of the construction. (vi) is straightforward now with the observation that if $\underset{\sim}{f}$, $\underset{\sim}{g}$ are two elements on which a particular $\square_m^i$ is defined at the end of different stages, then one is an initial segment of the other. Next, we complete the list of axioms for T. Arbitrarily fix $0 < r < \omega$, $0 \leqq n < \omega$ and a maximal subset $A(x_1, \cdots, x_n)$ of

$$\{P_i(x_j)^t,\ S_i(x_j, x_k)^t \mid t = 0, 1; 1 \leqq j, k \leqq n; i < r\}$$

consistent with the set of those axioms of T' in which only predicate symbols with indices less than r occur. It is easy to check that this can be done effectively, uniformly in r and n. For the same r and any $m > 0$, let $B(x_1, \cdots, x_{n+m})$ be another such set satisfying $A(x_1, \cdots, x_n) \subset B(x_1, \cdots, x_{n+m})$. Then for all such r, n, m, A, and B we include as an axiom for T:

$$\forall x_1 \cdots \forall x_n \exists x_{n+1} \cdots \exists x_{n+m}[\bigwedge A(x_1, \cdots, x_n) \longrightarrow \bigwedge B(x_1, \cdots, x_{n+m})] .$$

The fact that there always is at least one such B for every A follows from the form of the axioms for T' and Lemma 1 (ii)-(iv). We will now outline a proof that T is consistent. By compactness it is enough to show that for arbitrary such $A_0, B_0, \cdots, A_{s-1}, B_{s-1}$ as above,

$$(*) \qquad T' \cup \{\forall x_1 \cdots \forall x_{n_i} \exists x_{n_i+1} \cdots \exists x_{n_i+m_i}(\bigwedge A_i \longrightarrow \bigwedge B_i) \mid i < s\}$$

is consistent. Assume that such a collection is fixed and we define a model $\mathscr{A}$ of (*), with universe a subset $\{a_i \mid i < \omega\}$, by specifying its diagram $\Delta_{\mathscr{A}}$ by an induction.

Step 0:

$$P_i(a_0),\ S_i(a_0, a_0) \in \Delta_{\mathscr{A}} \quad \text{for all} \quad i < \omega .$$

Assume inductively that membership in $\Delta_{\mathscr{A}}$ has been determined for exactly $P_i(a_j)^t$, $S_i(a_j, a_k)^t$ $t = 0, 1$; $j, k < r$; and $i < \omega$, and is such that $\Delta_{\mathscr{A}}$ is consistent with T'.

Step $r + 1$: Suppose $r = i$ [mod s]. Then with respect to some enumeration of ω that we assume has been fixed before the beginning of the construction, let $\langle k_1, \cdots, k_{n_i}\rangle$ be the least element such that

$$\exists x_{n_i+1} \cdots \exists x_{n_i+m}(\bigwedge A(a_{k_1}, \cdots, a_{k_{n_i}}) \longrightarrow \bigwedge B(a_{k_1}, \cdots, a_{k_{n_i}}, x_{n+1}, \cdots, x_{n_i+m_i})]$$

fails with respect to $\Delta_{\mathscr{A}}$, as presently determined. If there is no such element, then go on to the next stage. Otherwise we specify first that $B(a_{k_1}, \cdots, a_{k_{n_i}}, a_r, \cdots, a_{r+m_i-1}) \subset \varDelta_{\mathscr{A}}$. Next, we complete the description with respect to the new constant symbols, by another induction. Assume that membership of $P_i(a_j)^t$, $S_i(a_j, a_k)^t$ has heen determined for all $j, k < r + m_i$ and $i < v$. Fix $j < r + m_i$. Put $\neg P_v(a_j)$ into $\varDelta_{\mathscr{A}}$ unless $P_v(a_j)$ is already in $\varDelta_{\mathscr{A}}$ or $\underline{f}^\frown\langle 1\rangle \notin F$, where $\underline{f} \in 2^v$ is such that $P_i(a_j)^{\underline{f}(i)} \in \varDelta_{\mathscr{A}}$ for $i < v$. In either of the two alternative cases, simply ensure that $P_v(a_j)$ is in $\varDelta_{\mathscr{A}}$. Now fix in addition a $k < r + m_i$. Put $S_v(a_j, a_k)$ into $\varDelta_{\mathscr{A}}$ unless its negation already belongs, in which case do nothing. This is done for each j and k. This ends both inductions. We leave it to the reader to check via Lemma 1 that $\varDelta_{\mathscr{A}}$ can be expanded to the diagram of a model $\mathscr{A}$ of (*), pointing out that the axioms in V are never endangered because of our attempt to put $\neg P_i(a_j)$ into $\varDelta_{\mathscr{A}}$ whenever "possible", the choice of the A_i, B_i's, and Lemma 1(iii)-(iv). The conclusion is that T is consistent.

An obvious modification of the last argument shows that in fact every model of T' can be extended to a model of T. Next we claim that T admits elimination of quantifiers. An equivalent condition is that for all models $\mathscr{A}$, $\mathscr{B}$ of T and substructures $\mathscr{C} \subset \mathscr{A}, \mathscr{B}$, if an existential sentence with parameters from $|\mathscr{C}|$ holds in $\mathscr{A}$, then it holds in $\mathscr{B}$. So arbitrarily fix such $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\exists y_1 \cdots \exists y_m \theta(\underline{c}_1, \cdots, \underline{c}_n, y_1, \cdots, y_m)$ (where $\theta(x_1, \cdots, x_n, y_1, \cdots, y_m)$ is a quantifier free formula of $L(t)$). Therefore there are $\underline{a}_1, \cdots, \underline{a}_n \in |\mathscr{A}|$ such that

$$\langle \mathscr{A}, \underline{\bar{c}}, \underline{\bar{a}} \rangle \models \theta(\bar{c}, \bar{a}) .$$

Fix such $\underline{\bar{a}}$ and let r be the least number greater than zero and all of the indices of predicate symbols that occur in θ. Let $A(c_1, \cdots, c_n)$, $B(c_1, \cdots, c_n, a_1, \cdots, a_m)$ be the diagram of $\mathscr{A}'$, $\mathscr{A}''$ respectively, where $\mathscr{A}'$ is the reduct of the substructure of $\mathscr{A}$ with universe $\{\underline{c}_i \mid 1 \leq i \leq n\}$ to the language with only the predicate symbols whose indices are less than r, and similarly for $\mathscr{A}''$ (omitting equality). Since $\mathscr{A}$ is obviously also a model of T', an axiom for T is:

$$\forall x_1 \cdots \forall x_n \exists x_{n+1} \cdots \exists x_{n+m}[\bigwedge A(x_1, \cdots, x_n) \rightarrow \bigwedge B(x_1, \cdots, x_{n+m})] .$$

Therefore, since

$$(\mathscr{B}, \underline{\bar{c}}) \models \bigwedge A(\bar{c}) .$$

The conclusion follows.

Now let ψ be an arbitrary sentence in $L(T)$ consistent with T. Let $\mathscr{A}$ be a model of $\{\psi\} \cup T$ and $\mathscr{B}$ an arbitrary model of T. By elimination of quantifiers there is a quantifier free $\theta(y)$ equivalent under T to $(y = y \wedge \psi)$. If we simply repeat the last argument for $\mathscr{A}$, $\mathscr{B}$, θ and $\mathscr{C} = \varnothing$, we see that T is complete. Since T is axiomitizable it is also decidable. We will now specify the desired r.e. witness set.

By the elimination of quantifiers, to uniquely determine an n-type $\Gamma(x_1, \cdots, x_n)$ of T, it is sufficient to determine the maximal subset of

$$\{P_i(x_j)^t, S_i(x_j, x_k)^t, (x_j = x_k)^t \mid t = 0, 1; 1 \leqq j, k \leqq n; i < \omega\}$$

contained in $\Gamma(x_1, \cdots, x_n)$. Therefore, by the axioms in I and II and the decidability of T, an index for a recursive $\Gamma(x_1, \cdots, x_n)$ can effectively be obtained from indices for the (recursive) functions

$$\{f_i, g_{jk} \in 2^\omega, h \in 2^{n \times n} \mid 1 \leqq j < k \leqq n; 1 \leqq i \leqq n\} ,$$

where

$$P_s(x_i)^{f_i(s)}, S_s(x_j, x_k)^{g_{jk}(s)}, (x_j = x_k)^{h(j-1,k-1)} \in \Gamma(x_1, \cdots, x_n)$$

for the obvious values. Let $\mu\colon \omega \to B$ be a recursive, onto function, where

$$B = \{\langle \underset{\sim}{f}_1, \cdots, \underset{\sim}{f}_n, \underset{\sim}{g}_{12}, \cdots, \underset{\sim}{g}_{1n}, \underset{\sim}{g}_{23}, \underset{\sim}{g}_{23}, \cdots, \underset{\sim}{g}_{(n-1)n}\rangle \mid \underset{\sim}{f}, \underset{\sim}{g}'s \in 2^{<\omega} ,$$
$$\text{all lengths equal; } 0 < n < \omega\} .$$

We will now effectively specify, uniformly in s, a type Γ_s corresponding to $\mu(s)$.

Fix an $s < \omega$ and let

$$\mu(s) = \langle \underset{\sim}{f}_1, \cdots, \underset{\sim}{f}_n, \underset{\sim}{g}_{12}, \cdots, \underset{\sim}{g}_{(n-1)n}\rangle ,$$

where the lengths are say m (all of which of course can be uniformly effectively determined). Check first that:

(i) $\underset{\sim}{f}_i \in F$ for $1 \leqq i \leqq n$;

(ii) $\underset{\sim}{g}_{jk} \in G$ for $1 \leqq j < k \leqq n$; and

(iii) for each $j, k, 1 \leqq j < k \leqq n$ there is no $\langle \underset{\sim}{f}_1', \underset{\sim}{f}_2', \underset{\sim}{g}'\rangle \in H$, $l(\underset{\sim}{f}_1') \leqq m$ such that $\underset{\sim}{f}_1' \leqq \underset{\sim}{f}_j$, $\underset{\sim}{f}_2' \leqq \underset{\sim}{f}_k$ and $\underset{\sim}{g}' \leqq \underset{\sim}{g}_{jk}$.

If any of these fail, then the type $\Gamma(x_1, \cdots, x_n)$ will be the one containing

$$\{P_i(x_j), S_i(x_j, x_k), (x_j \neq x_k) \mid i < \omega; 1 \leqq j, k \leqq n\} ,$$

Otherwise we define the required $f_i, g_{jk} \in 2^\omega$, $1 \leqq i \leqq n$, $1 \leqq j < k \leqq n$ by an induction.

Step 0: $f_i(r) = \underset{\sim}{f}_i(r)$ and $g_{jk}(r) = \underset{\sim}{g}_{jk}(r)$ for all $r < m$, $1 \leqq i \leqq n$, and $1 \leqq j < k \leqq n$. So assume their values have been determined for all arguments less than p, such that (i)-(iii) hold for the extensions.

Step p': First of all $g_{jk}(p) = 0$ for all $1 \leqq j < k \leqq n$. Let f_i' be f_i as determined so far. Now check (iii) again with respect to $\langle f_i'^\frown\langle 0\rangle, f_k'^\frown\langle 0\rangle, g_{jk}'\rangle$, for $1 \leqq j < k \leqq n$. For any j, k for which (iii) fails, define $f_j(p) = f_k(p) = 1$. After this is done define $f_i(p) = 0$ for any $f_i(p)$ not yet defined. That the induction hypotheses are maintained can be checked via Lemma 1 (ii)-(iv). This ends the induction. Γ_s is now defined to be the unique type determined by its containment of

$$\{P_t(x_i)^{f_i(t)},\ S_t(x_j, x_k)^{g_{jk}(t)},\ x_j \neq x_k \mid t < \omega; 1 \leqq i \leqq n; 1 \leqq j < k \leqq n\} .$$

LEMMA 2. *$\{\Gamma_s \mid s < \omega\}$ has an* r.e. *witness set.*

Proof. Straightforward. Let $\{\hat{\Gamma}_s \mid s < \omega\}$ be an r.e. witness set for $\{\Gamma_s \mid s < \omega\}$, where $\hat{\Gamma}_s$ is an index for Γ_s.

LEMMA 3. *For all $s < \omega$ there is an m such that no negated atomic formula containing a predicate symbol with index greater than m belongs to Γ_s.*

Proof. For the S_t predicate symbols this is immediate, since the instructions specify that $g_{ij}(t) = 0$ for all t greater than the length of the corresponding $\underline{g}_{ij}$. Notice next that since the $f_j(p)$ and $f_k(p)$ are defined to be 1 whenever (iii) fails with respect to $\langle \underline{f}_j'{}^\frown 0, \underline{f}_k'{}^\frown 0, \underline{g}_{jk}' \rangle$, it follows by Lemma 1 (v) that (iii) can fail at most once for each such pair j, k. The lemma now follows by the instructions in the induction for what to do when (iii) does not fail. Call such types as in the lemma *eventually zero* (e.z.) types.

LEMMA 4. *$\{\hat{\Gamma}_s \mid s < \omega\}$ is a witness set for a countable model of T.*

Proof. By a standard argument, $T \cup \{\Gamma_s(a_1^s, \cdots, a_{n_s}^s) \mid s < \omega\} \equiv T^*$ is a consistent theory, where the a_j^i's are distinct new constant symbols. We first claim that this theory has a countable model that omits every type of T that is not e.z. By elimination of quantifiers for T and Lemma 1 (vi), T has only a countable number of types altogether. Therefore by the Omitting Types Theorem [1] and elimination of quantifiers it is sufficient to show that for every 2-type $\Gamma(x_1, x_2)$ of T that is not e.z., and every formula $\theta(b_1, \cdots, b_n, x_1, x_2)$ (where the b_i's are a_k^j's), if $\exists x_1 \exists x_2 \theta(b_1, \cdots, b_n, x_1, x_2)$ is consistent with T^*, then so is $\exists x_1 \exists x_2 [\theta(b_1, \cdots, b_n, x_1, x_2) \wedge \neg \gamma(x_1, x_2)]$ for some $\gamma(x_1, x_2) \in \Gamma(x_1, x_2)$. So fix such a $\Gamma(x_1, x_2)$ and θ, and we will find a $\gamma(x_1, x_2)$. Let ψ be the quantifier free formula such that

$$T \vdash [\theta(b_1, \cdots, b_n, x_1, x_2) \leftrightarrow \psi(b_1, \cdots, b_n, x_1, x_2)] .$$

Let m be larger than any index of any predicate symbol occurring in ψ. By the assumption of consistency there are $\underline{f}_i$ and $\underline{g}_{jk}$ of length m such that the following is consistent:

$$(\#) \qquad T^* \cup \{\psi(b_1, \cdots, b_{n+2})\} \cup \{P_t(b_i)^{\underline{f}_i(t)}, S_t(b_j, b_k)^{\underline{g}_{jk}(t)} \mid t < m; \\ 1 \leqq i \leqq n+2; 1 \leqq j < k \leqq n+2\},$$

where b_{n+1}, b_{n+2} are *new* constant symbols. Now for the noneffective step. Essentially by Lemma 1 (iv)-(v), there is an $m' > m$ and $\underline{f}_i'$, $\underline{g}_{jk}'$ such that for all $i, j, k, 1 \leqq i \leqq n+2$, $1 \leqq j < k \leqq n+2$:

(a) $\underline{f}_i \leqq \underline{f}_i'$ and $\underline{g}_{jk} \leqq \underline{g}_{jk}'$;

(b) $\underline{f}_i' \in F$ and $\underline{g}_{jk}' \in G$;

(c) if $S_0(b_j, b_k)^{\underline{g}_{jk}(0)} \notin T^*$, then no marker $\square_s^2$ is ever defined on a $\underline{g}$, $\underline{g}_{jk} \nleqq \underline{g} \leqq \underline{g}_{jk}'$ or $\underline{g}_{jk}' \leqq \underline{g}$;

(d) if $S_0(b_j, b_k)^{\underline{g}_{jk}(0)} \notin T^*$ and a $\square_s^2$ marker is defined on $\underline{g}_{jk}$ at stage m', then it is defined there at all later stages;

(e) if $S_0(b_j, b_k)^{\underline{g}_{jk}(0)} \in T^*$, then $S_t(b_j, b_k) \in T^*$ for all $t \geqq m'$;

(f) $P_t(b_i) \in T^*$ for all $i, \Gamma, 1 \leqq i \leqq n$, $t \geqq m'$;

(g) (#) remains consistent when $m, \underline{f}_i, \underline{g}_{jk}$ are replaced by m', $\underline{f}_i'$, $\underline{g}_{jk}'$ respectivly.

Notice for (e) and (f) that all of the Γ_s are e.z. types. Now let f_i, $g_{jk} \in 2^\omega$ be functions satisfying:

(i) $\underline{f}_i' \leqq f_i$, $\underline{g}_{jk}' \leqq g_{jk}$; and

(ii) $f_i(t) = g_{jk}(t) = 0; 1 \leqq i \leqq n+2, 1 \leqq j < k \leqq n+2\ m' \leqq t < \omega$.

It follows that

$$(+) \quad \{P_t(b_i)^{f_i(t)}, S_t(b_j, b_k)^{g_{jk}(t)} \mid 1 \leqq i \leqq n+2; 1 \leqq j < k \leqq n+2 t < \omega\}$$

is consistent with T^*. We leave the details for the reader to check, noting that the axioms of T' in V are not endangered because of (c) and (d) and Lemma 1 (iv). (In case (d) obtains, an easy check of the construction shows that no element is listed in H_t^s for any $t \geqq m'$.)

Now let h be such that

$$(+') \qquad \{(b_i = b_j)^{h(i,j)} \mid 1 \leqq i \leqq n+2\}$$

is consistent with (#), and let $\Sigma(x_1, \cdots, x_{n+2})$ be the unique $(n+2)$-type of T determined by the union of (#) and (+'), after the obvious substitution of variables for constants. By construction Σ is e.z. Therefore so is the 2-type $\Phi(x_{n+1}, x_{n+2})$ contained in $\Sigma(x_1, \cdots, x_{n+2})$. Since $\Phi(x_1, x_2)$ is e.z., there must be a formula in $\Gamma(x_1, x_2)$ that is not in $\Phi(x_1, x_2)$. This formula is the desired $\gamma(x_1, x_2)$. Thus the lemma is proved, since it is straightforward to show that every e.z. type of T has an index in $\{\hat{\Gamma}_s \mid s < \omega\}$.

By Lemma 4, let $\mathscr{A}$ be a countable model of T with witness set $\{\hat{\Gamma}_s \mid s < \omega\}$. Notice that the above omitting types argument can easily be amalgamated with the usual [1] proof that any countable model has a countable homogeneous elementary extension to show that $\mathscr{A}$ has a countable homogeneous elementary extension $\mathscr{C}$ which omits all types that are not e.z. Thus $\mathscr{C}$ is our desired countable homogeneous model with r.e. witness set $\{\hat{\Gamma}_s \mid s < \omega\}$. Therefore the theorem is finished with

LEMMA 5. *No decidable model of T realizes exactly the set of types $\{\Gamma_s \mid s < \omega\}$.*

Proof. In order to obtain a contradiction, assume that $\mathscr{B}$ is a decidable model of T realizing exactly the types $\{\Gamma_s \mid s < \omega\}$. Since $\mathscr{B}$ is decidable, the function $\mu\colon \omega \times \omega \times 2^{<\omega} \to 2$ such that

(1) $\mu(n, n, \underline{f}) = 0$ iff $(\mathscr{B}, \underline{b}_n) \models \bigwedge_{i<l(\underline{f})} P_i(b_n)^{\underline{f}(i)}$; and

(2) $\mu(n, m, \underline{g}) = 0$ iff $(\mathscr{B}, \underline{b}_n, \underline{b}_m) \models \bigwedge_{i<l(\underline{g})} S_i(b_n, b_m)^{\underline{g}(i)}$ for $n \neq m$

is recursive for some indexing $\{\underline{b}_i \mid i < \omega\}$ of $|\mathscr{B}|$. Fix an $m < \omega$ such that μ_m is such a μ. It is easy to see that because $\square_m^0$ is first defined on an element in F and $\mathscr{B}$ is a model of T, eventually in the construction of F an m-fixed number m_0 appears. Fix an r_0 such that

$$(1\sharp) \qquad (\mathscr{B}, \underline{b}_{m_0}) \models P_i(b_{m_0}) \quad \text{for all} \quad i \geqq r_0 .$$

Such an r_0 exists, since $\mathscr{B}$ realizes only e.z. types. By the same argument there is an m-designated number d at some stage $r_0' \geqq r_0$ and an $s_0 \geqq r_0'$ such that

$$(2\sharp) \qquad (\mathscr{B}, \underline{b}_d) \models P_i(b_d) \quad \text{for all} \quad i \geqq s_0 .$$

Now, the only circumstance under which a number can cease being m-designated is the one covered in (V)(C). Since (1♯) holds, it follows that d is the m-designated number for all stages greater than r_0'. But now, essentially by (IV)(C) and (1♯), (2♯) there is an $\langle \underline{f}_0, \underline{f}_1, \underline{g} \rangle \in H$ such that

$$(\mathscr{B}, \underline{b}_{m_0}, \underline{b}_d) \models \bigwedge_{i<l(\underline{f}_0)} P_i(b_{m_0})^{\underline{f}_0(i)} \wedge \bigwedge_{i<l(\underline{f}_1)} P_i(b_d)^{\underline{f}_1(i)} \wedge \bigwedge_{i<l(\underline{g})} S_i(b_{m_0}, b_d)^{\underline{g}(i)} .$$

But then $\mathscr{B}$ is not a model T', which is the desired contradiction.

It should be noted that $\{\Gamma_s \mid s < \omega\}$ also has a recursive witness set, this follows easily from the fact that each given partial recursive function has an infinite number of effectively recognizable indices. Therefore a strengthening along this line fails to produce a sufficient condition for ensuring decidability.

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