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CANONICAL MODELS FOR INVARIANT SUBSPACES

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The invariant subspace structure of an operator algebra $\mathfrak{L}_+$ is completely determined. The non-selfadjoint algebra is constructed from a cyclic transformation acting on a finite set. There is a distinguished (finite) set of invariant subspaces of $\mathfrak{L}_+$ which has been identified elsewhere. These subspaces are used as canonical models; all other invariant subspaces for $\mathfrak{L}_+$ are described in terms of these subspaces. Uniqueness of this representation is also discussed.

1. **Introduction.** This paper is a continuation of [5]. While some acquaintance with [5] would be helpful in reading the present paper, it is not absolutely necessary. Here we determine completely the invariant subspace structure of an operator algebra $\mathfrak{L}_+$ constructed from a cyclic transformation on a finite set. (The definitions of $\mathfrak{L}_+$ and an allied algebra $\mathfrak{R}_+$ are given below.) As shown in [5] there is a distinguished class of subspaces invariant for both $\mathfrak{L}_+$ and $\mathfrak{R}_+$. We shall show that the invariant subspaces for $\mathfrak{L}_+$ can be completely described in terms of these distinguished subspaces in much the same spirit as Buerling's theorem describes the invariant subspaces of the shift.

To define the algebras $\mathfrak{L}_+$ and $\mathfrak{R}_+$, let X be a finite set with elements $x_0, x_1, \cdots, x_{n-1}$ and let τ be the permutation of X defined by $\tau(x_i) = x_{i+1} (i \neq n-1)$ and $\tau(x_{n-1}) = x_0$. Let $\boldsymbol{Z}$ denote the set of integers and let $l^2(\boldsymbol{Z} \times X)$ be the Hilbert space of all (complex-valued) functions f on $\boldsymbol{Z} \times X$ such that $\sum_n \sum_x |f(n, x)|^2 < \infty$. Let f be an element of $l^2(\boldsymbol{Z} \times X)$ and define operators L_δ and R_δ on $l^2(\boldsymbol{Z} \times X)$ as follows:

$$(L_\delta f)(n, x) = f(n-1, \tau^{-1}x);\ (R_\delta f)(n, x) = f(n-1, x)\ .$$

For a complex-valued function φ defined on X (i.e., $\varphi \in l^\infty(X)$), we define operators L_φ and R_φ on $l^2(\boldsymbol{Z} \times X)$:

$$(L_\varphi f)(n, x) = \varphi(x) f(n, x);\ (R_\varphi f)(n, x) = \varphi(\tau^{-n}x) f(n, x)\ .$$

Let $\mathfrak{M}_L = \{L_\varphi | \varphi \in l^\infty(X)\}$ and $\mathfrak{M}_R = \{R_\varphi | \varphi \in l^\infty(X)\}$. The algebra $\mathfrak{L}$ (resp. $\mathfrak{R}$) is defined to be the von Neumann algebra generated by L_δ and $\mathfrak{M}_L$ (resp. R_δ and $\mathfrak{M}_R$). Finally we define the non-selfadjoint algebra $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the weakly closed algebra generated by L_δ and $\mathfrak{M}_L$ (resp. R_δ and $\mathfrak{M}_R$).

The algebras $\mathfrak{L}$ and $\mathfrak{R}$ are crossed products and $\mathfrak{L}_+$ and $\mathfrak{R}_+$ are called *non-selfadjoint crossed products.* We refer the reader to

[5, 6] for discussions of these algebras including some of their elementary properties. It should be noted that one of the principal results of [5] identifies the equivalence classes of $\mathfrak{L}_+$-invariant subspaces that are unitarily equivalent *by a unitary operator in* $\mathfrak{R}$. Each such class can also be described in terms of a multiplicity function. These functions play an important role in this paper (cf. § 3) for, as we shall see, the multiplicity function will allow us to identify a canonical subspace in each equivalence class of $\mathfrak{L}_+$-invariant subspaces.

It is shown in [5] that for certain subsets B of $\boldsymbol{Z} \times X$, subspaces of the form $l^2(B) = \{f \in l^2(\boldsymbol{Z} \times X) \mid f(n, x) = 0 \text{ for } (n, x) \notin B\}$ are invariant under both the algebras $\mathfrak{L}_+$ and $\mathfrak{R}_+$. Such a set B must be invariant under the two maps λ and ρ defined on $\boldsymbol{Z} \times X$ as follows:

$$\lambda(n, x) = (n + 1, \tau x)$$
$$\rho(n, x) = (n + 1, x) .$$

In case τ is a nonperiodic transformation on a measure space X as discussed in [5], all $\mathfrak{L}_+ \vee \mathfrak{R}_+$-invariant subspaces have the form $l^2(B)$. As shown in § 5 of [5], this is not the case when τ is a periodic transformation acting on a finite discrete set, as considered here.

In this paper we shall show that the $\mathfrak{L}_+ \vee \mathfrak{R}_+$-invariant subspaces of the form $l^2(B)$ are sufficiently plentiful to allow us to describe all the pure $\mathfrak{L}_+$-invariant subspaces in terms of these more elementary ones. (A *pure* subspace is one that contains no nonzero subspace reducing $\mathfrak{L}_+$.) The idea is to use subspaces of the form $l^2(B)$ as canonical models for the $\mathfrak{L}_+$-invariant subspaces. This concept is made precise in the following definition. (For a subspace $\mathscr{M}$, $P_{\mathscr{M}}$ denotes the orthogonal projection onto $\mathscr{M}$.)

DEFINITION. A family of full, pure invariant subspaces $\{\mathscr{M}_i\}_{i \in I}$ constitutes a complete set of *canonical models* for the pure $\mathfrak{L}_+$-invariant subspaces in case (a) for no two distinct indices i and j is $P_{\mathscr{M}_i}$ unitarily equivalent to $P_{\mathscr{M}_j}$ by a unitary operator in $\mathfrak{R}$; and (b) for every pure $\mathfrak{L}_+$-invariant subspace $\mathscr{M}$ there is an i in I and a partial isometry R_θ in $\mathfrak{R}$ such that $R_\theta P_{\mathscr{M}_i} R_\theta^* = P_{\mathscr{M}}$.

In particular the last equation implies that $\mathscr{M} = R_\theta \mathscr{M}_i$. The motivation for this concept stems from Beurling's theorem. Recall that this theorem states that if $\mathscr{M}$ is a (nonzero) nonreducing subspace for the bilateral shift on L^2 (of the unit circle), then $\mathscr{M} = \theta H^2$ where θ is a unimodular function on the circle. In this case the singleton $\{H^2\}$ is a complete set of canonical models for the pure invariant subspaces of the shift. A similar result is valid if we consider the non-selfadjoint crossed product determined by a factor (of a special kind) and an automorphism implemented by a unitary operator acting on the factor. In this case also a single subspace, $l^2(\boldsymbol{Z}_+, \mathscr{H})$, forms a complete set of canonical models for the invariant subspaces of the algebra $\mathfrak{L}_+$ as shown in [6]. It is shown in [5], with the particular algebra $\mathfrak{L}_+$ as we have defined it here, that there are pure subspaces $\mathscr{M}$ and $\mathscr{N}$ invariant for $\mathfrak{L}_+$ for which there is no partial isometry R_θ in $\mathfrak{R}$ satisfying $R_\theta P_{\mathscr{M}} R_\theta^* = P_{\mathscr{N}}$; thus a set of canonical models will necessarily consist of more than one subspace in the situation considered here. In this paper it will be shown that a finite collection of subspaces of the form $l^2(B)$ provides a complete set of canonical models for the (pure) $\mathfrak{L}_+$-invariant subspaces. Moreover, it will be seen that in this case the multiplicity function provides the necessary information for explicitly constructing the subspace from the canonical model.

Taking into account the results of [5], this is what we will have accomplished: we will have classified the invariant subspaces of $\mathfrak{L}_+$ up to a specific kind of equivalence and we will have identified a canonical member of each equivalence class. Moreover we will have done this in a fashion which is identical in spirit with that exhibited in two other contexts in which reasonably successful generalizations of the theory of shifts have been found, namely, the theory of invariant subspaces on multiply connected domains [1] and the theory of compact groups with ordered duals [4].

2. Alternative representations of $\mathfrak{L}$ and $\mathfrak{R}$.

The algebras $\mathfrak{L}$ and $\mathfrak{R}$ have been discussed in [5] under different hypotheses on X and τ. One major difference between the algebras $\mathfrak{L}$ and $\mathfrak{R}$ considered in § 4 of [5] and those in this paper is that in [5] $\mathfrak{L}$ and $\mathfrak{R}$ are factors while here they are not factors. Indeed a computation using the periodicity of τ shows that $L_\delta^n = R_\delta^n$ where, recall, n is the cardinality of the set X. Since $\mathfrak{L}$ and $\mathfrak{R}$ are commutants of one another, it follows that $\mathfrak{L}$ and $\mathfrak{R}$ have nontrivial centers and so are not factors. As a consequence of the representation developed in this section, we show that the center of $\mathfrak{L}$ (and of $\mathfrak{R}$) is the von Neumann algebra generated by L_δ^n.

In this section we shall show how the algebras $\mathfrak{L}$ and $\mathfrak{R}$ may be viewed as the left and right regular representations of the algebra of all $n \times n$-matrices with entries from $L^\infty(\boldsymbol{T})$, the (essentially) bounded functions on the circle. In order to do this, an isomorphism will be constructed between the spaces $l^2(\boldsymbol{Z} \times X)$ and $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. This second space will be viewed as the set of $n \times n$-

matrices with entries from $L^2(\boldsymbol{T})$. It is a Hilbert space with inner product

$$([f_{ij}],\ [g_{ij}]) = \sum_{i,j}\int_{\boldsymbol{T}} f_{ij}(x)g_{ij}(x)dx$$

for elements $[f_{ij}]$ and $[g_{ij}]$ in $L^2(\boldsymbol{T})\otimes \boldsymbol{M}_n$. (The norm on $\boldsymbol{M}_n$ here is the Hilbert-Schmidt norm.) The algebra $\mathfrak{L}$ (resp. $\mathfrak{R}$) will be realized as the algebra $L^\infty(\boldsymbol{T})\otimes \boldsymbol{M}_n$ acting by left (resp. right) multiplication on the space $L^2(\boldsymbol{T})\otimes \boldsymbol{M}_n$.

The isomorphism mentioned above will be constructed by mapping one basis to another. The set $\mathscr{B}_1$ of indicator (or characteristic) functions of singletons $\{1_{\{(n,x)\}}\,|\, n\in \boldsymbol{Z},\ x\in X\}$ is an orthonormal basis for the space $l^2(\boldsymbol{Z}\times X)$. For the other basis, let E_{ij} be the $n\times n$-matrix whose only nonzero entry is a 1 in the (i, j)th position. Let χ_n be the function on the circle defined by $\chi_n(z) = z^n$. The set $\mathscr{B}_2 = \{\chi_k E_{ij}\,|\, i, j = 1, 2, \cdots, n;\ k\in \boldsymbol{Z}\}$ is an orthonormal basis for the Hilbert space $L^2(\boldsymbol{T})\otimes \boldsymbol{M}_n$. (We assume that the measure on the circle is normalized Lebesgue measure.)

There are several ways to define the map W from $\mathscr{B}_1$ to $\mathscr{B}_2$. For an easy way, first let δ be the function defined on $\boldsymbol{Z}\times X$ by $\delta(n, x) = \begin{cases}1 & n = 1,\\ 0 & n\neq 1\end{cases}$ and define

$$W\delta = \begin{bmatrix} 0 & 0 & 0\cdots & \chi_1 \\ 1 & 0 & 0\cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ & & \ddots & \\ 0 & 0 & 0\cdots 1 & 0 \end{bmatrix}.$$

DEFINITION. The map $W\colon \mathscr{B}_1 \to \mathscr{B}_2$ is defined first on basis elements of the form $1_{\{(0,x_i)\}}$: $W1_{\{(0,x_0)\}} = E_{nn}$, $W1_{\{(0,x_i)\}} = E_{ii} (i\neq 0)$. For other elements of $\mathscr{B}_1$ let

$$W1_{\{(k,x_i)\}} = W1_{\{(0,x_i)\}}(W\delta)^k\,.$$

Observe that the operator L_δ (resp. R_δ) acting on the basis $\mathscr{B}_1$ of $l^2(\boldsymbol{Z}\times X)$ is transformed into left (resp. right) multiplication by $W\delta$ on $L^2(\boldsymbol{T})\otimes \boldsymbol{M}_n$.

The action of W on the bases can be extended to a Hilbert space isomorphism from $l^2(\boldsymbol{Z}\times X)$ to $L^2(\boldsymbol{T})\otimes \boldsymbol{M}_n$. The map W has one other important feature: it is multiplicative on the bases on which it is defined. Let $L_0^2 = \{f\in l^2(\boldsymbol{Z}\times X)\,|\, f(n, x) = 0$ for all but finitely many $n\}$. For elements f and g in L_0^2, we define a multiplication as follows:

$$(f*g)(n, x) = \sum_{k=-\infty}^{\infty} f(k, x)g(n-k, \tau^{-k}x) .$$

The *bounded elements* of $l^2(\boldsymbol{Z} \times X)$ are defined to be those functions f such that $g \mapsto f*g$ is a bounded operation on L_0^2. The set of bounded elements is denoted by L^∞. Note that by definition $L^\infty \subseteq l^2(\boldsymbol{Z} \times X)$. (Our notation coincides with that in [6].) An example of a bounded element is the function δ defined above. A computation shows that $g \mapsto \delta * g$ (resp. $g \mapsto g*\delta$) is simply the operator L_δ (resp. R_δ) acting on g. It is shown in [5] that the algebra of left (resp. right) multiplications by elements of L^∞ is precisely the algebra $\mathfrak{L}$ (resp. $\mathfrak{R}$). Accordingly we shall use the notation L_f for left multiplication by an element f in L^∞. Although slightly tedious, the proof of the following theorem is not difficult and is left to the reader.

THEOREM 2.1. *The map W is a Hilbert space isomorphism from $l^2(\boldsymbol{Z} \times X)$ onto $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ such that $WL_fW^{-1} = L_{Wf}$ and $WR_fW^{-1} = R_{Wf}$ for f in L^∞.*

Since W is an isomorphism, it follows immediately that $W\mathfrak{L}W^{-1}$ is a von Neumann subalgebra of the bounded operators on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. This subalgebra is identified in the following corollary. (From now on, most results will be stated for the left algebra $\mathfrak{L}$; corresponding statements for the algebra $\mathfrak{R}$ will be left to the reader.)

COROLLARY 2.2. *The algebra $W\mathfrak{L}W^{-1}$ is isomorphic to the algebra $L^\infty(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ acting by left multiplication on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$.*

Proof. Recall that we treat $L^p(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ as the set of $n \times n$-matrices with entries from $L^p(\boldsymbol{T})$. Consider the subalgebra $\mathfrak{L}_0$ of $\mathfrak{L}$ defined by $\mathfrak{L}_0 = \{L_f \mid f \in L_0^2\}$. Then $W\mathfrak{L}_0W^{-1}$ is the algebra of $n \times n$-matrices with "polynomial" entries acting by left multiplication on the space $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. So for L_f in $\mathfrak{L}_0$, WL_fW^{-1} is left multiplication by a matrix $[p_{ij}]$ where for each i and j, p_{ij} is a polynomial of the form $\sum_{l=-K}^{K} a_l \chi_l (a_l \in \boldsymbol{C})$. The weak closure of $W\mathfrak{L}_0W^{-1}$ is $W\mathfrak{L}W^{-1}$ and thus $W\mathfrak{L}W^{-1}$ is simply the algebra $L^\infty(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ thought of as acting by left multiplication on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$.

Since the algebra $\mathfrak{L}$ is a generalization of the algebra $L^\infty(\boldsymbol{T})$ and since $\mathfrak{L}_+$ is designed to generalize $H^\infty(\boldsymbol{T})$, the subspace of $L^\infty(\boldsymbol{T})$ consisting of functions whose Fourier coefficients of negative index vanish, it is tempting to say that $W\mathfrak{L}_+W^{-1}$ is the algebra $H^\infty(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. However, as the next two results show, this is not quite the case.

COROLLARY 2.3. *The algebra $W\mathfrak{L}_+W^{-1}$ consists of the matrices $[a_{ij}]$ in $H^\infty(\boldsymbol{T})\otimes \boldsymbol{M}_n$ having the property that $\hat{a}_{ij}(0)=0$ if a_{ij} lies above the main diagonal. That is,*

$$W\mathfrak{L}_+W^{-1}=\begin{bmatrix} H^\infty & H_0^\infty & \cdots & H_0^\infty \\ H^\infty & H^\infty & \cdots & H_0^\infty \\ \vdots & & \ddots & \vdots \\ H^\infty & H^\infty & \cdots & H^\infty \end{bmatrix}$$

where $H_0^\infty=\{f\in H^\infty \mid \hat{f}(0)=0\}$.

Proof. Since $\mathfrak{L}_+=\{L_f \mid f\in L^\infty,\ f(n,\cdot)=0,\ n<0\}$ and $L^\infty\subseteqq l^2(\boldsymbol{Z}\times X)$, any function f such that L_f is in $\mathfrak{L}_+$ is also in $l^2(\boldsymbol{Z}_+\times X)$. ($\boldsymbol{Z}_+=\{0,1,2,\cdots\}$.) Using the definition of $W\delta$ and the form of $(W\delta)^k$ for $k\geqq 0$, it follows that $W\mathfrak{L}_+W^{-1}\subseteqq H^\infty(\boldsymbol{T})\otimes\boldsymbol{M}_n$.

To see that $W\mathfrak{L}_+W^{-1}$ has the stated matricial form, it suffices to show that if $j\geqq 0$ and $W1_{\{(j,x_l)\}}=\chi_k E_{\alpha\beta}$ with $\alpha<\beta$ then k is strictly greater than zero. But $W1_{\{(j,x_l)\}}=E_{pp}(W\delta)^j$ where $p=l$ if $l\neq 0$ and $p=n$ if $l=0$. For $j\geqq 0$, the nonzero elements on or above the main diagonal of $(W\delta)^j$ are positive powers of χ_1. Since $\chi_1^k=\chi_k$, the result follows.

REMARK 2.4. (1) In view of 2.3, it is clear that the algebras $W\mathfrak{L}_+W^{-1}$ and $H^\infty(\boldsymbol{T})\otimes\boldsymbol{M}_n$ should not be isomorphic. The following proof that they, in fact, are not isomorphic is due to K. R. Fuller. In this remark, H^∞ will always be used to denote $H^\infty(\boldsymbol{T})$. Note first that H_0^∞ is just the principal ideal zH^∞. Let

$$J=\begin{bmatrix} H_0^\infty & H_0^\infty & \cdots & H_0^\infty \\ H^\infty & H_0^\infty & \cdots & H_0^\infty \\ \vdots & & \ddots & \vdots \\ H^\infty & H^\infty & \cdots & H_0^\infty \end{bmatrix}.$$

Then J is a (two-sided) ideal in $W\mathfrak{L}_+W^{-1}$ and $W\mathfrak{L}_+W^{-1}/J$ is the ring

$$\begin{bmatrix} H^\infty/H_0^\infty & H_0^\infty/H_0^\infty & \cdots & H_0^\infty/H_0^\infty \\ H^\infty/H^\infty & H^\infty/H_0^\infty & \cdots & H_0^\infty/H_0^\infty \\ \vdots & & \ddots & \vdots \\ H^\infty/H^\infty & H^\infty/H^\infty & \cdots & H^\infty/H_0^\infty \end{bmatrix}\cong H^\infty/H_0^\infty\oplus\cdots\oplus H^\infty/H_0^\infty$$

(n summands). Thus $W\mathfrak{L}_+W^{-1}$ has a commutative factor ring. Recall that any (two-sided) ideal in $H^\infty\otimes\boldsymbol{M}_n$ has the form $I\otimes\boldsymbol{M}_n$ where I is an ideal in H^∞. It follows that $H^\infty\otimes\boldsymbol{M}_n$ has no commutative factor rings. Hence $H^\infty \otimes M_n$ is not isomorphic to $W\mathfrak{L}_+ W^{-1}$.

(2) It was noted at the beginning of this section that the algebras $\mathfrak{L}$ and $\mathfrak{R}$ are not factors. We now show how to identify $\mathscr{Z}(\mathfrak{L})$, the center of $\mathfrak{L}$, as the von Neumann algebra generated by L_δ^n. To do this we first write L_{M_n} for the von Neumann algebra of left multiplications by $n \times n$-matrices acting on M_n, and similarly write R_{M_n} for the right multiplications. Let $C_{\mathscr{H}}$ denote the algebra of scalar multiples of the identity acting on the space $\mathscr{H}$. Then Corollary 2.3 shows that $\mathfrak{L} \cong L^\infty(T) \otimes L_{M_n}$ and $\mathfrak{L}' = \mathfrak{R} \cong L^\infty(T) \otimes R_{M_n}$. Hence $\mathscr{Z}(\mathfrak{L}) = \mathfrak{L} \cap \mathfrak{L}' \cong (L^\infty(T) \otimes L_{M_n}) \cap (L^\infty(T) \otimes R_{M_n}) = L^\infty(T) \otimes (L_{M_n} \cap R_{M_n}) = L^\infty(T) \otimes C_{M_n} \cong \{L_\delta^n\}''$.

3. Two-sided invariant subspaces and multiplicity functions. Recall that a subspace that is invariant under both the algebras $\mathfrak{L}_+$ and $\mathfrak{R}_+$ is called a *two-sided invariant* or $\mathfrak{L}_+ \vee \mathfrak{R}_+$-*invariant* subspace. Under appropriate assumptions on X and τ, it can be shown that for each two-sided invariant subspace $\mathscr{M}$ there is a subset B of $Z \times X$ invariant for the maps λ and ρ defined in § 1; the subspace $\mathscr{M}$ consists precisely of those functions in the Hilbert space with support contained in B. As noted previously, this result is not valid in case X is a finite discrete set and τ is a permutation. However, it is an easy computation to show that for any set B in $Z \times X$ invariant for λ and ρ, the subspace $l^2(B)$ is a two-sided invariant subspace. The following remark collects several pertinent observations concerning two-sided invariant subspaces of the form $l^2(B)$.

REMARK 3.1. Let $l^2(B)$ be a (nontrivial) two-sided invariant subspace.

(1) If the point (k, x) is in B then, by the ρ-invariance of B, $(k + l, x)$ is in B for any positive integer l.

(2) Let $Z_{(k)} = \{k, k + 1, k + 2, \cdots\}$. There exists an integer N with the property that the set $Z_{(N)} \times X$ contains B. This follows from the fact that $l^2(B)$ is not the entire space $l^2(Z \times X)$ and that B is invariant for both maps λ and ρ. If N is also chosen to be the largest such integer, then B contains the set $Z_{(N+n)} \times X$. In particular $l^2(B)$ is a full, pure subspace for both the algebras $\mathfrak{L}_+$ and $\mathfrak{R}_+$. (Recall, an $\mathfrak{L}_+$-invariant subspace $\mathscr{M}$ is $\mathfrak{L}_+$-*full* in case the smallest $\mathfrak{L}_+$-reducing subspace containing $\mathscr{M}$ is $l^2(Z \times X)$.)

(3) There exists a partition $\{E_k\}_{k=-\infty}^{\infty}$ of X such that $B = \bigcup_{k \in Z} Z_{(k)} \times E_k$. To construct this partition, let $C = B \backslash \rho(B)$. Then $E_k = \{x \in X \mid (k, x) \in C \cap (\{k\} \times X)\}$. Observe that all but finitely many of the sets E_k are empty. Since B is invariant for λ, the partition

$\{E_k\}_k$ has the following "order" property: $\tau(E_k) \subseteq \bigcup_{j \leq k+1} E_j$. This property will be important in our discussion of multiplicity functions in Theorem 3.4.

(4) Recall that if $\mathscr{M}$ is a left-invariant subspace, then $\mathscr{F} = \mathscr{M} \ominus L_\delta \mathscr{M}$ is a wandering subspace for L_δ. From Lemma 3.2 of [5], we know that the projection $P_{\mathscr{F}}$ onto $\mathscr{F}$ lies in $\mathfrak{M}'_L$, the commutant of $\mathfrak{M}_L$. Hence $P_{\mathscr{F}}$ can be written as $P_{\mathscr{F}} = \sum_{x \in X} P(x)$ where each $P(x)$ is a projection in $\mathscr{L}(l^2(\boldsymbol{Z} \times \{x\}))$. The *multiplicity function* of the subspace $\mathscr{M}$ is the function m on X defined by

$$m(x) = \text{rank}\, P(x) = \text{dimension of range of } P(x)\ .$$

PROPOSITION 3.2. *Let $l^2(B)$ be a (nontrivial) two-sided invariant subspace. Then there exists a set $A \subset B$ such that $l^2(A)$ is the wandering subspace for the operator L_δ associated with the subspace $l^2(B)$.*

Proof. The wandering subspace is by definition $\mathscr{F} = l^2(B) \ominus L_\delta l^2(B) = l^2(B) \ominus l^2(\lambda(B)) = l^2(B \backslash \lambda(B))$. Let $A = B \backslash \lambda(B)$.

In the terminology of ergodic theory, the set A is a wandering subset of $\boldsymbol{Z} \times X$ for the transformation λ. We have found it helpful to represent subspaces of the form $l^2(B)$ by means of figures. Since these subspaces consist of functions supported on the set B, all information concerning such a subspace is codified in the set B. To graph such a set, one may simply represent the set $\boldsymbol{Z} \times X$ in the obvious way as the set $\boldsymbol{Z} \times \{0, 1, 2, \cdots, n-1\}$ in the plane. The graphical representation of subsets B in $\boldsymbol{Z} \times X$ can be used to illustrate the items of Remark 3.1 and Proposition 3.2 as well as provide motivation for proofs of several theorems presented here.

PROPOSITION 3.3. *Assume $l^2(B)$ is as in 3.2 and has the multiplicity function m. Then $\sum_{x \in X} m(x) = n$, the cardinality of the set X.*

Proof. Let A be the wandering set produced in 3.2. For an element x in X, define the set A_x to be the intersection of the set A and the "horizontal" slice through the point x: $A_x = A \cap (\boldsymbol{Z} \times \{x\})$.

Let P be the projection onto the wandering subspace $l^2(A)$. We may write $P = \sum_{x \in X} P(x)$ as in 3.1.4. Clearly $P(x)$ is the projection onto $l^2(A_x)$ for each x and so $\text{rank}\, P(x) = \dim l^2(A_x) = \text{card}\, A_x$, the cardinality of the set A_x. Thus it suffices to show that $\sum_{x \in X} \text{card}\, A_x = n$.

Using Remark 3.1.2, there exists a smallest integer k_0 such that $\{k_0\} \times X$ is contained in B. For any x there exists a positive integer

$j = j(x)$ such that $\lambda^{-j}(k_0, x)$ is in B but $\lambda^{-(j+1)}(k_0, x)$ is not in B. By left invariance and the definition of the wandering set A, the point $\lambda^{-j}(k_0, x) = (k_0 - j, \tau^{-j}x)$ lies in A. Thus associated with the n distinct points $(k_0, x), (k_0, x_1), \cdots, (k_0, x_{n-1})$, we have their pre-images under λ in A and these pre-images must be distinct. This shows that $\sum_{x \in X} \operatorname{card} A_x \geqq n$. But since L_δ is a shift of multiplicity n, it follows (cf. [7]) that the dimension of the wandering subspace is no larger than n and hence $\sum_{x \in X} \operatorname{card} A_x = n$.

It is shown next that if m is any nonnegative integer-valued function on X whose values sum to n, then m is a multiplicity function. Moreover we construct an explicit subspace for such a function.

THEOREM 3.4. *Let m be a function on X having values in the nonnegative integers. If m has the property that $\sum_{x \in X} m(x) = n$, then there exists a two-sided invariant subspace with multiplicity function m.*

Proof. The proof is somewhat lengthy and is broken into three steps. The first step consists of constructing a partition $\{E_k\}_{k=-\infty}^{\infty}$ of X, in which all but finitely many of the sets E_k will be empty. We define the set B to be $\bigcup_{k \in Z} \boldsymbol{Z}_{(k)} \times E_k$. The second step will be to show that the set B is invariant for the maps λ and ρ. This invariance can be translated into a property of the partition which is then verified. The final step consists of showing that the subspace $l^2(B)$ has the desired multiplicity function.

Step 1. Let $\{x_{i_0}, x_{i_1}, \cdots, x_{i_L}\}$ be the support of m so that $m(x) \neq 0$ if and only if $x = x_{i_k}$ for some k, $0 \leqq k \leqq L$. We will assume $i_0 < i_1 < \cdots < i_L$. Define the supplementary function $s(\cdot)$ on the support by

$$s(x_{i_k}) = \begin{cases} 0 & k = 0 \\ i_k - i_0 - \sum_{j=1}^{k} m(x_{i_j}) & k \neq 0 \,. \end{cases}$$

Let $s = \min\{s(x_{i_k}) \mid k = 0, 1, \cdots, L\}$. We can extend the function $s(\cdot)$ to all of X as follows. For x outside the support of the multiplicity function, there exists a smallest positive integer k such that $x = \tau^k y$ with $y \in \operatorname{supp} m$. For such x define $s(x) = s(y) + k$.

Define the sets $E_k = \{x \in X \mid s(x) = s + k\}$, $k = 0, 1, \cdots, \eta$. Clearly $\{E_k\}_{k=0}^{\eta}$ is a partition of X where η is the smallest positive integer such that $\bigcup_{k=0}^{\eta} E_k = X$. Let $B = \bigcup_{k=0}^{\eta} \boldsymbol{Z}_{(k)} \times E_k$. Then $l^2(B) = \sum_{k=0}^{\eta} l^2(\boldsymbol{Z}_{(k)} \times E_k)$ is a subspace contained in $l^2(\boldsymbol{Z}_+ \times X)$.

Step 2. Clearly $l^2(B)$ is invariant for the right algebra $\mathfrak{R}_+$. To show the subspace is invariant for $\mathfrak{L}_+$, it suffices to show that B is invariant for the map λ. To accomplish this last objective, it suffices to show that the partition has the following "order" property:

$$\tau(E_k) \subseteqq \bigcup_{j \leqq k+1} E_j, \qquad k = 0, 1, 2 \cdots .$$

Finally, it follows from the definition of the partition that this property will be demonstrated once we show $s(\tau(x)) \leqq s(x) + 1$. The proof of this inequality depends on whether or not x and $\tau(x)$ lie in the support of the multiplicity function. In case $\tau(x)$ is not in the support of m, the desired inequality follows immediately from the definition of the supplementary function $s(\cdot)$.

Assume next that both x and $\tau(x)$ are in the support of the multiplicity function. Because τ is cyclic, we shall assume initially that $x = x_{i_k}$ where $k < L$. Then $\tau(x) = x_{i_{k+1}} = x_{i_k+1}$. But then

$$\begin{aligned} s(\tau(x)) &= i_{k+1} - i_0 - \sum_{j=1}^{k+1} m(x_{i_j}) \\ &= i_k + 1 - i_0 - \sum_{j=1}^{k} m(x_{i_j}) - m(x_{i_{k+1}}) . \end{aligned}$$

If $k = 0$, it follows that

$$s(\tau(x)) = i_0 - i_0 + 1 - m(x_{i_1}) \leqq 0 = s(x_{i_0}) \leqq s(x) + 1 .$$

If $k \neq 0$, then we have

$$s(\tau(x)) = s(x) + 1 - m(x_{i_{k+1}}) \leqq s(x) + 1 .$$

Now suppose that $x = x_{i_L}$ and both x and $\tau(x)$ are in the support of m. Then x_{i_L} must be x_{n-1} and $\tau(x) = x_0$ so both x_{n-1} and x_0 are in the support of m. We know $s(\tau(x)) = s(x_0) = 0$, so we need to show $s(x_{n-1}) + 1 \geqq 0$. This is done as follows:

$$s(x_{n-1}) = (n-1) - 0 - \sum_{j=1}^{L} m(x_{i_j}) = n - 1 - (n - m(x_0)) \geqq 0 .$$

Thus $s(x_{n-1}) + 1 \geqq 0$ and hence $s(x_0) \leqq s(x_{n-1}) + 1$.

Lastly, assume that $x \notin \operatorname{supp} m$ and $\tau(x) \in \operatorname{supp} m$. In this case $s(x) = s(y) + k$ for some y as in the definition of the function $s(\cdot)$. A computation similar to the preceding one shows that $s(\tau(x)) \leqq s(x) + 1$.

Step 3. The proof will be completed by showing that the multiplicity function m_1 for the two-sided invariant subspace $l^2(B)$ is equal to the original function m. It follows immediately from the definition of the partition that the supports of the functions m and m_1 are identical. To show $m = m_1$, it suffices to identify the λ-wandering set A and show that $m(x) = \operatorname{card} A_x$.

We know that the wandering set is $B \backslash \lambda(B)$. Let

$$A = \bigcup_l \{(l, x),\ (l+1, x),\ \cdots,\ (l + m(x) - 1, x) \mid x \in E_l \cap \operatorname{supp} m\} .$$

We shall show $A = B \backslash \lambda(B)$. Clearly $A \subseteqq B$. Suppose $A \cap \lambda(B) \neq \varnothing$. Specifically, we may assume (by invertibility) the point $(k + n_2, x_{i_q})$ in the intersection $A \cap \lambda(B)$ satisfies $\lambda^\alpha(l + n_1, x_{i_p}) = (k + n_2, x_{i_q})$ where $x_{i_p} \in E_l$, $x_{i_q} \in E_k$, $0 \leqq n_1 < m(x_{i_p})$, $0 \leqq n_2 < m(x_{i_q})$, and α is a positive integer less than n. Observe that

$$\tau^\alpha(x_{i_p}) = x_{i_q} \text{ (so that } \alpha = (i_q - i_p)(\operatorname{mod} n))$$

and

$$(\dagger) \qquad l + n_1 + \alpha = k + n_2 .$$

Due to the cyclic nature of τ, we are forced into considering separate cases. In each case, the point $(k + n_2, x_{i_q})$ cannotbe in $A \cap \lambda(B)$ and hence this intersection will be empty.

For the first case, suppose $i_p < i_q$ and $i_p \neq i_0$. We have

$$s(x_{i_p}) = i_p - i_0 - \sum_{j=1}^{p} m(x_{i_j})$$

and

$$s(x_{i_q}) = i_q - i_0 - \sum_{j=1}^{q} m(x_{i_j}) .$$

Thus

$$\begin{aligned} k - l &= s + k - (s + l) \\ &= s(x_{i_q}) - s(x_{i_p}) \\ &= i_q - i_p - \sum_{j=p+1}^{q} m(x_{i_j}) \\ &= \alpha - \sum_{j=p+1}^{q} m(x_{i_j}) . \end{aligned}$$

Hence $k = \alpha + l - \sum_{j=p+1}^{q} m(x_{i_j}) \neq \alpha + l + n_1 - n_2$ for any n_1, n_2 since $n_1 \in \{0, 1, 2, \cdots, m(x_{i_p}) - 1\}$ and $n_2 \in \{0, 1, 2, \cdots, m(x_{i_q}) - 1\}$. This contradicts ($\dagger$). So no element in $A \cap \lambda(B)$ can satisfy the hypotheses of this case. The remaining cases are all based on the demonstration of this first case and are left to the reader.

We now have $A \subseteqq B \backslash \lambda(B)$. To show $A = B \backslash \lambda(B)$, simply observe that, by construction, the cardinality of the set A is the same as that of the set X. Since the wandering set $B\backslash\lambda(B)$ has cardinality n, we must have $A = B\backslash\lambda(B)$.

Lastly observe that, by construction, $m(x) = \operatorname{card} A_x$ for each element x in X.

REMARK 3.5. (1) The subspace $l^2(B)$ constructed in the proof of the theorem has the property that $l^2(B) \subseteq l^2(\boldsymbol{Z}_+ \times X)$ but $L_\delta^{-1} l^2(B) \nsubseteq l^2(\boldsymbol{Z}_+ \times X)$. Such a subspace is said to be *left-justified*. This is equivalent of course to the property that $B \cap (\{0\} \times X) \neq \varnothing$ but $B \cap (\{-1\} \times X) = \varnothing$.

(2) Clearly a multiplicity function for a two-sided invariant subspace can be identified with an ordered n-tuple of nonnegative integers $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ with the property that $\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} = n$. The number of distinct n-tuples with this property (and hence the number of distinct left-justified two-sided invariant subspaces $l^2(B)$) is $\binom{2n-1}{n} = (2n-1)!/(n-1)!n!$, [8, p. 139]. In §4 we shall show how to use this finite collection of subspaces as canonical models for all left-pure invariant subspaces for the algebra $\mathfrak{L}_+$.

The final theorem of this section illustrates how to obtain a multiplicity function directly from a two-sided invariant subspace of the form $l^2(B)$. Recall that the multiplicity function of such a subspace was defined using a decomposition of the projection onto the wandering subspace.

THEOREM 3.6. *Let $l^2(B)$ be a (nontrivial) two-sided invariant subspace with associated ordered partition $\{E_k\}_{k=-\infty}^{\infty}$ (cf. 3.1.3). Then the multiplicity function for the subspace $l^2(B)$ can be formed as follows: for x in E_l, $m(x) = \min_k \{n_k + k - l \mid n_k$ is the first positive integer with the property that $x \in \tau^{n_k}(E_k)$; $k \in \boldsymbol{Z}\}$.*

Proof. Recall that $B = \bigcup_{k \in Z} \boldsymbol{Z}_{(k)} \times E_k$ and there is an integer N such that $E_k = \varnothing$, $k < N$ (cf. Remark 3.1). Note next that m as defined above is a nonnegative-valued function since the partition $\{E_k\}$ has the property that $\tau(E_k) \subseteq \bigcup_{l \leq k+1} E_l$. We will assume throughout the proof that the element x is in E_l. Let $A = B\backslash\lambda(B)$ be the wandering set for the subspace and recall that we need only show that $m(x)$ is the cardinality of the set A_x, the x-section of the set A.

We first show that the equation $m(x) = 0$ is equivalent to the equation $A_x = \varnothing$. Suppose that $m(x) = 0$. Then there exists an integer k such that $n_k + k = l$ (so that in particular $k < l$) and hence x is in $\tau^{n_k}(E_k)$. But then $\tau^{-n_k}x$ is an element of the set E_k so that the point $(k, \tau^{-n_k}x)$ is in B. We now have $\lambda^{n_k}(k, \tau^{-n_k}x) = (l, x)$ so that the point (l, x) is not in A_x. Hence A_x must be empty since if A_x is to be nonempty, it must contain at least the point (l, x) (along with possibly other elements of the form $(l + j, x)$ for $j > 0$).

On the other hand if A_x is empty then (l, x) is in B but not in A. Thus (l, x) must be in $\lambda(B)$ and hence $\lambda^{-1}(l, x) = (l - 1, \tau^{-1}x)$ is in B. By left-invariance and the definition of the partition, $(l - 2, \tau^{-1}x) \notin B$. Thus $n_{l-1} = 1$ so that the smallest positive integer α such that x is in $\tau^{\alpha}(E_{l-1})$ is $\alpha = 1$. Thus $0 \in \{n_k + k - l \mid n_k$ is the first positive integer with the property that $x \in \tau^{n_k}(E_k)\}$ and hence $m(x) = 0$.

Assume now that $m(x)$ is different from zero. Let j be an integer such that $0 \leqq j < m(x)$ and set $n_0 = l + j$. Suppose the point (n_0, x) does not belong to A_x. This means that $(n_0, x) = \lambda(n_0 - 1, \tau^{-1}x)$ with $(n_0 - 1, \tau^{-1}x)$ in B. Then $\tau^{-1}x$ lies in E_k for some $k \leqq n_0 - 1$. Hence $m(x) \leqq n_k + k - l \leqq 1 + (n_0 - 1) - l = j < m(x)$. This contradiction shows that (n_0, x) lies in A and hence $m(x) \leqq \operatorname{card} A_x$.

Finally the point $(l + m(x), x)$ is in B. Let n_k and k be the integers as in the statement of the theorem so that x is in $\tau^{n_k}(E_k)$ and $m(x) = n_k + k - l$. Then $(l + m(x), x) = (n_k + k, x) = \lambda^{n_k}(k, \tau^{-n_k}x)$ and the point $(k, \tau^{-n_k}x)$ is in B since $\tau^{-n_k}x$ is in E_k. Thus $(l + m(x), x)$ is in B but not in A so that $(l + m(x), x)$ is not in A_x. This completes the proof that $m(x) = \operatorname{card} A_x$.

4. Canonical models. In this section we show how to use the finite number of left-justified $\mathfrak{L}_+ \vee \mathfrak{R}_+$-invariant subspaces of the form $l^2(B)$ as canonical models for the pure $\mathfrak{L}_+$-invariant subspaces.

THEOREM 4.1. *Let $\mathscr{M}$ be a nontrivial left-pure invariant subspace of $l^2(\boldsymbol{Z} \times X)$. Then there exists a two-sided invariant subspace of the form $l^2(B)$ and a partial isometry R_v in the algebra $\mathfrak{R}$ such that $P_{\mathscr{M}} = R_v P_{l^2(B)} R_v^*$ and hence $\mathscr{M} = R_v l^2(B)$.*

Proof. Let $\mathscr{F}$ be the left-wandering subspace associated with $\mathscr{M}$ $(\mathscr{F} = \mathscr{M} \ominus L_\delta \mathscr{M})$. The dimension of $\mathscr{F}$ is less than or equal to n. Let $m(\cdot)$ be the multiplicity function for $\mathscr{M}$. If $\sum_{x \in X} m(x) = n$, then there exists a left-justified two-sided invariant subspace $l^2(B)$ with multiplicity function m. The theorem now follows from Theorem 3.4 in [5] and, in fact, R_v is a unitary operator in this case.

If $\sum_{x \in X} m(x) < n$, then there exists an element x_k in X such that $m(x_k) = 0$. Define a new function m_1 by

$$m_1(x) = \begin{cases} m(x) & x \neq x_k\,, \\ n - \sum_{x \in X} m(x) & x = x_k\,. \end{cases}$$

Then m_1 is a multiplicity function for a left-justified two-sided invariant subspace $l^2(B)$. Let $\mathscr{F}_1 = l^2(B) \ominus L_\delta l^2(B)$, the wandering subspace associated with $l^2(B)$, and let $P_{\mathscr{F}_1} = \sum_x \oplus P_1(x)$ (cf. Remark 3.1). By construction rank $P_1(x) = m_1(x)$. Now let $Q = \sum_{x \neq x_k} \oplus P_1(x)$. We may also decompose the projection onto the wandering subspace for $\mathscr{M}$: $P_{\mathscr{F}} = \sum_{x \in X} \oplus P(x)$. It follows that the projection Q is equivalent to the projection $P_{\mathscr{F}}$ in $\mathfrak{M}_L'$ since the corresponding dimensions are identical (see [2] for a discussion of equivalence of projections). Since $Q < P_{\mathscr{F}_1}$, it follows that $P_{\mathscr{F}} \prec P_{\mathscr{F}_1}$ in $\mathfrak{M}_L'$. So by Theorem 3.4 of [5], there exists a partial isometry R_v in $\mathfrak{R}$ such that $P_{\mathscr{M}} = R_v P_{l^2(B)} R_v^*$ and this completes the proof.

As a corollary, we are able to show that any two-sided invariant subspace that is not left-reducing is (left) pure and full. Lemma 4.2 will be useful for the proof. Let L^p and H^p denote the usual Lebesgue and Hardy spaces on the circle with normalized Lebesgue measure. Let H_0^p denote the space of H^p-functions vanishing at the origin. Recall that Szego's theorem asserts that if ω is a positive integrable function on the circle, T, then the $\inf \int_T |1-f|^2 \omega dm$, which is taken over all polynomials f vanishing at zero, is precisely $\exp\left(\int \log \omega dm\right)$ where the expression is zero by fiat if $\log \omega$ is not integrable. It follows from this fact that if f is an L^2-function then the distance in L^2 from f to the closed subspace $\overline{(fH_0^\infty)}$ is

$$\exp\left(\int \log |f|^2 dm\right).$$

LEMMA 4.2. *Let E be a measurable subset of the circle such that both E and its complement $T \backslash E$ have positive measure. Then the closure of $1_E H^2$ in L^2, $\overline{1_E H^2}$, equals $1_E L^2$.*

Proof. From Szego's theorem we know that the distance from the indicator function 1_E to the space $1_E H_0^2$ is $\exp\left(\int \log 1_E dm\right)$, which equals zero since 1_E vanishes on a set of positive measure. Thus $1_E \in \overline{1_E H_0^2}$. From this it is easy to see that the subspace $\overline{1_E H^2}$ is a reducing subspace for the operator of multiplication by z (i.e., the shift operator). Thus by [7] there exists a measurable subset F such that $\overline{1_E H^2} = 1_F L^2$. It is easy to show that the sets E and F differ by at most a null set so that $\overline{1_E H^2} = 1_E L^2$.

COROLLARY 4.3. *Let $\mathscr{M}$ be a two-sided invariant subspace that is not left-reducing. Then $\mathscr{M}$ is (left) pure and full.*

Proof. We shall assume first that $\mathscr{M}$ is (left) pure and show that $\mathscr{M}$ is full. Subsequent to this we shall show how this assumption can be replaced by the assumption that $\mathscr{M}$ is not left-reducing.

Since $\mathscr{M}$ is a pure subspace we can apply Theorem 4.1 and write $\mathscr{M} = R_v l^2(B)$ where R_v is a partial isometry in the algebra $\mathfrak{R}$ and $l^2(B)$ is a two-sided invariant subspace.

Let P be the final projection of R_v so that $P = R_v R_v^*$ is an element of $\mathfrak{R}$. In this proof we shall use the notation L^2 to refer to $l^2(\boldsymbol{Z} \times X)$. Thus $PL^2 = R_v L^2 = R_v \bigvee_{n\leqq 0} L_\delta^n l^2(B) = \bigvee_{n\leqq 0} L_\delta^n R_v l^2(B) = \bigvee_{n\leqq 0} L_\delta^n \mathscr{M}$. Observe also that P commutes with $\mathfrak{R}$. To see this let R_φ be an element of $\mathfrak{M}_R$. Then $R_\varphi PL^2 = R_\varphi \bigvee_{n\leqq 0} L_\delta^n \mathscr{M} = \bigvee_{n\leqq 0} L_\delta^n R_\varphi \mathscr{M} \subseteqq \bigvee_{n\leqq 0} L_\delta^n \mathscr{M} = PL^2$. So PL^2 is invariant for $\mathfrak{M}_R$ and hence the projection P is in the commutant of $\mathfrak{M}_R$. To show P commutes with R_δ observe that $R_\delta P R_\delta^* L^2 = R_\delta PL^2 \subseteqq PL^2$ so that $R_\delta P R_\delta^* \leqq P$. But since the von Neumann algebra $\mathfrak{R}$ is finite and the projections $R_\delta P R_\delta^*$ and P are equivalent in $\mathfrak{R}$, we must have $R_\delta P R_\delta^* = P$. Thus P is in $\mathfrak{R}'$, the commutant of $\mathfrak{R}$. But $\mathfrak{R}' = \mathfrak{L}$ and so P lies in $\mathscr{Z}(\mathfrak{R})$, the center of $\mathfrak{R}$ (which is also $\mathscr{Z}(\mathfrak{L})$, the center of $\mathfrak{L}$).

By Corollary 2.2 we can represent P as an operator of the form $1_E \otimes I$ acting on the Hilbert space $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. If E is almost all of the circle, we may take P to be the identity and it follows that $\mathscr{M}$ is full. If $\mu(\boldsymbol{T} \backslash E) > 0$ so that the indicator function 1_E vanishes on a set of positive measure, we show that $\mathscr{M}$ is not pure, contrary to our assumption.

First note that P is also the initial projection of R_v since $\mathfrak{R}$ is a finite von Neumann algebra and P is a central projection. It is easy to see that we can choose an integer $N > 0$ such that $L_\delta^N W^{-1}(H^2 \otimes \boldsymbol{M}_n) \subseteqq l^2(B)$ where W is the isomorphism of § 2. Thus we have

$$
\begin{aligned}
PL^2 &\supseteqq \mathscr{M} \\
&= R_v P l^2(B) \quad (P \text{ is the initial projection}) \\
&\supseteqq R_v P L_\delta^N W^{-1}(H^2 \otimes \boldsymbol{M}_n) \\
&= R_v L_\delta^N P W^{-1}(H^2 \otimes \boldsymbol{M}_n) \\
&= R_v L_\delta^N W^{-1}(WPW^{-1})(H^2 \otimes \boldsymbol{M}_n) \\
&= R_v L_\delta^N W^{-1}\overline{(1_E \otimes I)(H^2 \otimes \boldsymbol{M}_n)} \\
&= R_v L_\delta^N W^{-1}(\overline{1_E H^2} \otimes \boldsymbol{M}_n) \\
&= R_v L_\delta^N W^{-1}(1_E L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n) \quad (4.2) \\
&= R_v L_\delta^N W^{-1}(1_E \otimes I)(L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n)
\end{aligned}
$$

$$= R_v L_\delta^N P L^2$$
$$= P R_v L_\delta^N L^2$$
$$= PL^2 \quad (P \text{ is the final projection}).$$

But this contradicts the purity of $\mathfrak{M}$ since PL^2 is a reducing subspace. This completes the proof under the assumption that $\mathscr{M}$ is a pure subspace.

Now assume that $\mathscr{M}$ is a two-sided invariant subspace that is not left-reducing. To complete the proof it suffices to show that $\mathscr{M}$ is pure. We shall show that if $\mathscr{M}$ is not pure then, in fact, $\mathscr{M}$ must reduce $\mathfrak{L}_+$. We may break $\mathscr{M}$ into its reducing and pure pieces: $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ where $\mathscr{M}_1 = \bigcap_{n \geq 0} L_\delta^n \mathscr{M}$ and $\mathscr{M}_2 = \sum_{n \geq 0} L_\delta^n \mathscr{F}$ ([5], Prop. 3.3; or [7]). If $\mathscr{M}$ is not a pure subspace then $\mathscr{M}_1 \neq \{0\}$. The subspace $\mathscr{M}_1$ is left-reducing; in addition, $\mathscr{M}_1$ is right-reducing. To see this we need only show $\mathscr{M}_1$ reduces R_δ. Since $\mathscr{M}_1$ reduces $\mathfrak{L}$, we have $\mathscr{M}_1 = R_p L^2$ where R_p is a projection in $\mathfrak{R}$, the commutant of $\mathfrak{L}$. (Recall $L^2 = l^2(\boldsymbol{Z} \times X)$.) But since $\mathscr{M}_1$ is invariant for R_δ, $R_\delta R_p R_\delta^* L^2 = R_\delta R_p L^2 \subseteqq R_p L^2$. Thus $R_\delta R_p R_\delta^* \leqq R_p$ and so $R_\delta R_p R_\delta^* = R_p$ as in the first part of the proof. It follows that R_p commutes with $\mathfrak{R}$ and so $R_p \in \mathfrak{L} \cap \mathfrak{R} = \mathfrak{L} \cap \mathfrak{L}' = \mathscr{Z}(\mathfrak{L})$, the center of $\mathfrak{L}$. The projection R_p may be represented as $1_E \otimes I$ acting on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$. Since $\mathscr{M}_2$, the pure part of $\mathscr{M}$, is orthogonal to $\mathscr{M}_1$, $W\mathscr{M}_2$ must lie in the range of $1_{E'} \otimes I$ $(E' = \boldsymbol{T} \backslash E)$. Hence $W\mathscr{M}_2 \subseteqq L^2(E') \otimes \boldsymbol{M}_n$. Representing $W\mathscr{M}_2$ as a matrix of subspaces, $W\mathscr{M}_2 = [\mathscr{M}_{ij}]$, we have $\mathscr{M}_{ij} \subseteqq L^2(E')$ for each i, j.

Since $\mathscr{M}_2$ is invariant under L_δ, it is easy to see that each subspace $\mathscr{M}_{ij}$ is invariant under multiplication by χ_1, the bilateral shift on $L^2(\boldsymbol{T})$. Using the fact that $\mathscr{M}_2$ is pure, a calculation shows that the subspaces $\mathscr{M}_{ij}$ do not reduce the shift, unless $\mathscr{M}_{ij} = \{0\}$. Hence for each i, j, either $\mathscr{M}_{ij} = \varphi_{ij} H^2(\boldsymbol{T})(|\varphi_{ij}| = 1$ a.e.) or $\mathscr{M}_{ij} = \{0\}$. But since $\mathscr{M}_{ij} \subseteqq L^2(E')$, we conclude that $\mathscr{M}_{ij} = \{0\}$ for all i, j. Hence $\mathscr{M}_2 = \{0\}$ and so $\mathscr{M}$ is a (left) reducing subspace.

REMARK 4.4. The preceding proof shows that a two-sided invariant subspace is either reducing or pure. Thus in the decomposition $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ as above, at most one of the spaces $\mathscr{M}_1$, $\mathscr{M}_2$ is a nonzero subspace.

5. Uniqueness. In this last section, we comment on the degree of nonuniqueness in the construction of canonical models. Theorem 4.1 not only states that if $\mathscr{M}$ is a left-pure invariant subspace then $\mathscr{M} = R_v l^2(B)$, but its proof and the proof of Theorem 3.4 actually construct the two-sided invariant subspace $l^2(B)$. Since $\mathscr{M}$ also equals $R_v R_\delta^{-k} l^2(\rho^k(B)) = R_w l^2(C)$, the subspace $l^2(B)$ is not unique.

However we can choose a specific subspace as follows.

For any multiplicity function such that $\sum m(x) = n$, construct the left-justified two-sided invariant subspace $l^2(B)$ as in the proof of 3.4. In case $\sum m(x) < n$, extend the function to another multiplicity function m_1 by defining $m_1(x_k) = n - \sum_{x \neq x_k} m(x)$, where x_k is the element of X outside the support of m with the smallest index k. Now construct the subspace $l^2(B)$ for this multiplicity function as in 3.4. Using this procedure, any given multiplicity function will yield a unique left-justified two-sided invariant subspace of the form $l^2(B)$. It remains only to consider the partial isometry in the expression $\mathscr{M} = R_v l^2(B)$. The following proposition is a reformulation of a corollary found in [3, p. 64].

PROPOSITION 5.1. *Let U be a bilateral shift on a Hilbert space $\mathscr{H}$, let $\mathscr{H}_+$ be a full, pure invariant subspace for U, and let $\mathscr{M}$ be a pure invariant subspace for U. If V_1 and V_2 are partial isometries on $\mathscr{H}$ which commute with U and satisfy $V_i P_{\mathscr{H}_+} V_i^* = P_{\mathscr{M}}$, then there is a partial isometry W on $\mathscr{H}$ such that:* (1) *the initial space of W is the initial space of V_2;* (2) *the final space of W is the initial space of V_1;* (3) *W commutes with U;* (4) *W is reduced by $\mathscr{H}_+$; and* (5) *$V_2 = V_1 W$.*

Proof. Let $\mathscr{F} = \mathscr{H}_+ \ominus U\mathscr{H}_+$ and $\mathscr{G} = \mathscr{M} \ominus U\mathscr{M}$. Then $V_i P_{\mathscr{F}} V_i^* = V_i(P_{\mathscr{H}_+} - UP_{\mathscr{H}_+}U^*)V_i^* = V_i P_{\mathscr{H}_+} V_i^* - UV_i P_{\mathscr{H}} V_i^* U^* = P_{\mathscr{M}} - UP_{\mathscr{M}}U^* = P_{\mathscr{G}}$. Thus when restricted to $\mathscr{F}$, each V_i is a partial isometry mapping onto $\mathscr{G}$. For $i = 1, 2$, let $\mathscr{F}_i$ be the initial space of $V_i|\mathscr{F}_i$. Then $\mathscr{F}_i$ is a subspace of $\mathscr{F}$ and V_i maps $\mathscr{F}_i$ isometrically onto $\mathscr{G}$. Because V_i commutes with U and $\mathscr{F}_i$ is contained in the complete wandering subspace $\mathscr{F}$, it is easy to check that the initial space of V_i is $\sum_{n=-\infty}^{\infty} \oplus U^n \mathscr{F}_i$. Since $V_i \mathscr{F}_i = \mathscr{G}$, we can find a partial isometry W_0 mapping $\mathscr{F}$ to $\mathscr{F}$ such that the initial space of W_0 is $\mathscr{F}_2$, the final space of W_0 is $\mathscr{F}_1$, and $(V_1|\mathscr{F}_1)W_0 = (V_2|\mathscr{F}_2)$. Define W on all of $\mathscr{H} = \sum_{n=-\infty}^{\infty} U^n \mathscr{F}$ by the following formula

$$W\left(\sum U^n e_n\right) = \sum U^n W e_n ,$$

where $\{e_n\}_{n=-\infty}^{\infty}$ is a sequence in $\mathscr{F}$ satisfying $\sum \|e_n\|^2 < \infty$. (For details on this definition see the proof of Theorem 3.4 in [5]. It is helpful to note that if the spaces $U^n \mathscr{F}$ are identified and operators on $\mathscr{H}$ are written as operator matrices, W is diag$(\cdots, W_0, W_0, W_0, \cdots)$.) It is immediate that W satisfies the conclusions of the proposition.

Our uniqueness theorem is a simple translation of Proposition

5.1. We include a precise statement for completeness.

COROLLARY 5.2. *Let $\mathscr{M}$ be a left-pure invariant subspace and let $l^2(B)$ be a two-sided invariant subspace. If R_{v_1} and R_{v_2} are partial isometries in $\mathfrak{R}$ which satisfy $R_{v_i}P_{l^2(B)}R^*_{v_i} = P_{\mathscr{M}}$, then there is a partial isometry R_w in $\mathfrak{R}$ whose initial space is the initial space of R_{v_2}, whose final space is the initial space of R_{v_1}, which is reduced by $l^2(B)$, and which satisfies $R_{v_2} = R_{v_1}R_w$.*

Proof. Let $\tilde{W}$ be the partial isometry obtained from Proposition 5.1 for the bilateral shift L_δ. We need only show that $\tilde{W}$ is in $\mathfrak{R}$. But since the initial space of R_{v_1} is the final space of $\tilde{W}$, we have $R^*_{v_1}R_{v_1} = R^*_{v_1}R_{v_1}\tilde{W} = \tilde{W}$ and hence $\tilde{W}$ is in $\mathfrak{R}$.

The preceding corollary does not answer completely all questions of uniqueness. In particular, it does not describe the partial isometries in $\mathfrak{R}$ that are reduced by a canonical model. A satisfactory description can be given in case the subspace of Corollary 5.2 is both full and pure. In this case $\mathscr{M} = R_u l^2(B)$ where R_u is a unitary operator in $\mathfrak{R}$. The task is to describe those unitary operators in $\mathfrak{R}$ that are reduced by a particular canonical model. Expressing R_u in its matricial form as discussed in § 2, we can show that R_u has a specific form, which depends on the canonical model reducing it. The following two examples illustrate this.

EXAMPLE 5.3. Consider the subspace $l^2(\boldsymbol{Z}_+ \times X)$ and let P be the projection onto the subspace. It is easy to show that $\mathfrak{R} \cap \{P\}' = \mathfrak{R}_+ \cap \mathfrak{R}^*_+ = \mathfrak{M}_R$. The image of $\mathfrak{M}_R$ in $L^\infty(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ is the algebra of (right multiplications by) diagonal matrices with constant entries along the diagonal. Thus the unitary operators in $\mathfrak{R}$ that reduce $l^2(\boldsymbol{Z}_+ \times X)$ can be represented as right multiplications on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ by matrices of the form

$$\begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}$$

with $|\alpha_i| = 1$.

EXAMPLE 5.4. A more typical example can be obtained by considering the case that X has cardinality 3. Let $B = \{(0, x_2), (1, x_0), (1, x_2)\} \cup (\boldsymbol{Z}_{(2)} \times X)$ and consider the subspace $l^2(B)$. The image of this (two-sided invariant) subspace in $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$ (under the isomorphism discussed in § 2) is

$$\begin{bmatrix} H_0^2 & H_0^2 & H_1^2 \\ H^2 & H^2 & H_0^2 \\ H^2 & H^2 & H_0^2 \end{bmatrix}.$$

(Recall $H_0^2 = \chi_1 H^2$ and $H_1^2 = \chi_2 H^2$.) A computation shows that if R_u is a unitary operator in $\mathfrak{R}$ reduced by $l^2(B)$, then R_u must be right multiplication by a matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\chi_1 \\ a_{21} & a_{22} & a_{23}\chi_1 \\ a_{31}\bar{\chi}_1 & a_{32}\bar{\chi}_1 & a_{33} \end{bmatrix}$$

with the constants a_{ij} chosen so that the matrix is unitary (recall $\chi_k(z) = z^k$).

This second example exhibits the general characteristics of the form of a unitary operator reduced by a canonical model. In general if we represent R_u as right multiplication by the matrix $[\varphi_{ij}]$ on $L^2(\boldsymbol{T}) \otimes \boldsymbol{M}_n$, one can show that each φ_{ij} has the form $c_{ij}\chi_k$ where c_{ij} is a complex constant and k is -1, 0, or 1. In particular φ_{ii} can be shown always to be a constant; if $i > j$, $\varphi_{ij} = c_{ij}\chi_k$ and $k = 0$ or 1; and if $i < j$ then $\varphi_{ij} = c_{ij}\chi_k$ and $k = 0$ or -1. Moreover one can construct a simple algorithm, based on the form of the canonical model for deciding the value of k. Although we feel that these results are of value, their statements and proofs are notationally cumbersome and so are omitted.

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