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THE ASYMMETRIC PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS

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Let $L_i = \sum_{j=1}^{3} a_{ij}x_j$, $i = 1, 2, 3$, be three linear forms in the variables x_1, x_2, x_3 with real coefficients a_{ij}. A theorem of Davenport asserts that, if $|\det(a_{ij})| = 7$, then there exist integers u_1, u_2, u_3, not all zero, such that

$$\left|\prod_{i=1}^{3} L_i(u_1, u_2, u_3)\right| \leqq 1 .$$

Under the same hypothesis, W. H. Adams has asked whether, given a positive real number u, there exist integers u_1, u_2, u_3, not all zero, such that

$$-u^{-1} \leqq L_1(u_1, u_2, u_3)L_2(u_1, u_2, u_3)\,|L_3(u_1, u_2, u_3)| \leqq u .$$

Our objective is to prove this conjecture.

Davenport gave several proofs of his theorem [3], and other proofs have been given by Chalk and Rogers [2] and Mordell [8]. Isolation results, notably those of Davenport [6] and Swinnerton-Dyer [10], show that Adams conjecture is true for real u in some open interval containing 1.

The set of points (L_1, L_2, L_3) in R_3, formed as the variables range over all integral values, is a lattice Λ of determinant $d(\Lambda) = |\det(a_{ij})|$. In terms of Λ, our result is as follows.

THEOREM. *If $d(\Lambda) = 7$, then there exists a point (x_1, x_2, x_3) of Λ, other than the origin, such that*

$$-u^{-1} \leqq x_1x_2|x_3| \leqq u ,$$

with the equality sign being necessary only if $u = 1$.

The method of proof is the projective one due to Davenport [3]. We begin with three lemmas.

LEMMA 1. *If x, y, z, t are real nuumbers with $1 < t^2 \leqq 1.9$, such that the inequality*

$$(1) \qquad -t^2 < (n + x)(n + y)|n + z| < 1$$

is not solvable in integers n, then

$$(2) \qquad \varphi = (x - y)^2 + (y - z)^2 + (z - x)^2 > 14t .$$

We note that this is a generalization of a lemma due to

Davenport [3].

Proof. We may assume that none of x, y, z is an integer, for otherwise inequality (1) is solvable for an integer n. We distinguish cases according to the comparative sizes of $[x]$, $[y]$, $[z]$.

Case 1. *Two of* $[x]$, $[y]$, $[z]$ *are equal.*

As x, y, z may be replaced by $x+n$, $y+n$, $z+n$ respectively, for any integer n, without altering either the hypothesis or the conclusion of the lemma, we may assume that two of $[x]$, $[y]$, $[z]$ are zero. Inequality (1) implies that

$$|(n+x)(n+y)(n+z)| < 1 \tag{3}$$

has no solution in integers n.

If $[x] = [y] = 0$, then $xy(1-x)(1-y) \leqq 1/16$. If, further, $|xyz(x-1)(y-1)(z-1)| < 1$, then (3) is solvable for one of the values $n = 0, -1$. Hence, we must have $|z(z-1)| \geqq 16$, whence $z(z-1) \geqq 16$, so that either $z < -3.5$ or $z > 4.5$. As $0 < x, y < 1$, it follows that $|x-z| > 3.5$ and $|y-x| > 3.5$ and therefore also $\varphi > 24.5$. Thus, if $\varphi \leqq 14t$, then $t > 1.75$ and $t^2 > 1.9$, contrary to hypothesis. Hence $\varphi > 14t$.

As (3) is symmetric in x, y, z the other two possibilities follow by the same argument.

Case 2. *Two of* $[x]$, $[y]$, $[z]$ *differ by* 1 *and no two are equal.*

Suppose first $[x]$, $[y]$ differ by 1. As we may replace x, y, z by $x+n$, $y+n$, $z+n$ respectively, for any integer n, without altering either the hypothesis or the conclusion of the lemma, we may assume that $[x]+[y] = -1$. Again, we may replace x, y, z by $-x$, $-y$, $-z$ respectively, without alternating the lemma, so we may assume that $z > 0$. Finally, by the symmetry of x and y in the lemma, we may assume that $-1 < x < 0 < y < 1$.

If $z < 1$ then $-1 < xyz < 0$, contrary to inequality (1). Therefore $z > 1$. Putting $f(n) = (x+n)(y+n)(z+n)$, we have $f(1) \geqq 1$, $f(0) \leqq -t^2$ and $f(-1) \geqq 1$, so that $f(1) = 1 + e_1$, $f(0) = -t^2 - e_2$, $f(-1) = 1 + e_3$, where e_1, e_2, e_3 are nonnegative real numbers. Introducing the new variables $\xi = xyz$, $\eta = xy + yz + zx$ and $\zeta = x + y + z$, these equations become

$$\begin{aligned} \xi + \eta + \zeta &= e_1 \\ \xi &= -t^2 - e_2 \\ \xi - \eta + \zeta &= 2 + e_3 , \end{aligned}$$

from which it follows that

$$\zeta = 1 + t^2 + \frac{1}{2}e_1 - e_2 + \frac{1}{2}e_3$$

$$\eta = -1 + \frac{1}{2}e_1 - \frac{1}{2}e_3 .$$

Hence

$$\begin{aligned}\frac{1}{2}\varphi = \zeta^2 - 3\eta &= \left(1 + t^2 + \frac{1}{2}e_1 + e_2 + \frac{1}{2}e_3\right)^2 + 3\left(1 - \frac{1}{2}e_1 + \frac{1}{2}e_3\right) \\ &\geqq (1 + t^2)^2 + 3 \\ &> 7t ,\end{aligned}$$

since the last inequality may be written in the form

$$(t - 1)(t^3 + t^2 + 3t - 4) > 0 ,$$

which is true as $t > 1$. Thus $\varphi > 14t$ as required.

We may therefore assume that $[x]$, $[y]$ do not differ by 1. By the symmetry of x and y we may suppose that $[y]$, $[z]$ differ by 1. As before, we may assume that $-1 < z < 0 < y < 1$. Since we are assuming that the previous cases do not arise, it follows that either $x > 2$ or $x < -1$.

Suppose first that $x > 2$. Then $f(1) = 1 + e_1$, $f(0) = -1 - e_2$ and $f(-1) = t^2 + e_3$ where e_1, e_2, e_3 are nonnegative real numbers. As before, solving these three equations for ζ, η gives

$$\begin{aligned}2\varphi = (2\zeta)^2 - 6(2\eta) &= (3 + t^2 + e_1 + 2e_2 + e_3)^2 + 6(1 + t^2 - e_1 + e_3) \\ &\geqq (3 + t^2)^2 + 6(1 + t^2) \\ &> 28t ,\end{aligned}$$

since the last inequality may be written in the form

$$(t - 1)(t^3 + t^2 + 13t - 15) > 0 .$$

Hence $\varphi > 14t$, as required.

Now suppose that $x < -1$. Then $f(1) = -t^2 - e_1$, $f(0) = t^2 + e_2$, $f(-1) = -1 - e_3$ where e_1, e_2, e_3 are nonnegative real numbers. Proceeding as before, we obtain

$$\begin{aligned}2\varphi &= (1 + 3t^2 + e_1 + 2e_2 + e_3)^2 + 6(1 + t^2 + e_1 - e_3) \\ &\geqq (1 + 3t^2)^2 + 6(1 + t^2) \\ &> 28t ,\end{aligned}$$

since the last inequality may be written as

$$(t - 1)(9t^3 + 9t^2 + 21t - 7) > 0 .$$

This completes Case 2.

The preceding two cases imply that each pair of $[x]$, $[y]$, $[z]$ differ by at least 2. If each pair differ by at least 3, then some two of x, y, z differ by at least 5, which implies that $\varphi \geqq 25 > 14t$ since $t^2 \leqq 1.9$. Therefore, we may assume from now on that some pair of $[x]$, $[y]$, $[z]$ differ by exactly 2. The symmetry of x and y yields three cases.

Case 3. $-2 < x < -1$, $0 < y < 1$, $2 < z$.

We have $f(1) \leqq -t^2$, $f(0) \leqq -t^2$, $f(-1) \geqq 1$ and $f(-2) \geqq 1$, i.e.,

$$\zeta \leqq -1 - t^2 - \eta - \xi \tag{4}$$

$$\xi \leqq -t^2 \tag{5}$$

$$\zeta \geqq 2 + \eta - \xi \tag{6}$$

$$4\zeta \geqq 9 + 2\eta - \xi \, . \tag{7}$$

Inequalities (4) and (6) imply that

$$\eta \leqq -\frac{1}{2}(t^2 + 3) \tag{8}$$

whereas (4) and (7) yield

$$\eta \leqq -\frac{1}{6}(13 + 4t^2 + 3\xi) \, . \tag{9}$$

Assume first that

$$2\eta - 3\xi \geqq 1 \, , \tag{10}$$

so that (8) and (10) give

$$\xi \leqq -\frac{1}{3}(t^2 + 4) \, . \tag{11}$$

By (6) and (11),

$$\zeta \geqq \frac{1}{3}(t^2 + 10) + \eta \, . \tag{12}$$

Now if $\eta \leqq -1/3(t^2 + 10)$, then

$$\frac{1}{2}\varphi = \zeta^2 - 3\eta \geqq t^2 + 10 > 11 > 7t \, .$$

Therefore we may assume that

$$\eta > -\frac{1}{3}(t^2 + 10) \, . \tag{13}$$

Then (12) and (13) imply that

$$\begin{aligned}\zeta^2 - 3\eta &\geqq \left(\eta + \frac{1}{3}(t^2 + 10)\right)^2 - 3\eta \\ &> 7t\end{aligned}$$

provided that the quadratic in η,

$$\left(\eta + \frac{1}{3}(t^2 + 10)\right)^2 - 3\eta - 7t ,$$

has nonreal roots, i.e., provided that $4t^2 - 28t + 31 > 0$. This inequality holds if $t < 1/2(7 - 3\sqrt{2})$, which is true since $t^2 < 1.9$. Hence we may suppose that (10) is false, i.e.,

$$\eta < \frac{1}{2}(1 + 3\xi) . \tag{14}$$

We may further assume that

$$9 + 2\eta - \xi > 0 ,$$

for otherwise, by (5),

$$2\eta \leqq \xi - 9 \leqq -t^2 - 9 < -10 ,$$

and therefore also

$$\zeta^2 - 3\eta > 15 > 7t .$$

Thus, by (7),

$$\zeta^2 - 3\eta \geqq \frac{1}{16}(9 + 2\eta - \xi)^2 - 3\eta = g(\eta) , \quad \text{say.}$$

The quadratic $g(\eta)$ attains its minimum value at

$$\eta = \frac{1}{2}(\xi + 3) > \frac{1}{2}(1 + 3\xi) \quad \text{by (5)} .$$

Hence, by (14),

$$g(\eta) \geqq \frac{1}{16}(10 + 2\xi)^2 - \frac{3}{2}(1 + 3\xi) = h(\xi) , \quad \text{say.}$$

The quadratic $h(\xi)$ attains its minimum value at $\xi = 4$. Suppose first that $\xi \leqq -1/3(4 + t^2)$. Then

$$g(\eta) \geqq h(\xi) \geqq \frac{1}{36}(11 - t^2)^2 + \frac{1}{2}(9 + 3t^3) > 7t$$

since

$$t^4 + 32t^2 - 252t + 283 > 0$$

when

$$t^2 < 1.9 .$$

Thus we may assume that

$$\xi > -\frac{1}{3}(4 + t^2) . \tag{15}$$

As $g(\eta)$ is decreasing $\eta \leqq 1/2(\xi + 3)$, and (15) shows that

$$-\frac{1}{6}(13 + 4t^2 + 3\xi) < \frac{1}{2}(\xi + 3) ,$$

so (9) implies that

$$g(\eta) \geqq \frac{1}{36}(7 - 2t^2 - 3\xi)^2 + \frac{1}{2}(13 + 4t^2 + 3\xi) = j(\xi) , \quad \text{say.}$$

But $j(\xi)$ has the minimum value $31/4 + t^2$. Hence

$$g(\eta) \geqq \frac{31}{4} + t^2 > 7t ,$$

since $4t^2 - 28t + 31 > 0$, as we have already seen. This completes the proof for Case 3.

Case 4. $-2 < x < -1,\ 0 < z < 1,\ 2 < y$.

Here $f(-1) \geqq t^2$, $f(-2) \geqq t^2$, $f(1) \leqq -t^2$, $f(0) \leqq -t^2$ and these imply the four inequalities (4)-(7) of Case 3. Therefore the same argument applies here.

Case 5. $y < -1,\ 0 < x < 1,\ 2 < z < 3$.

Here $f(1) \leqq -t^2$, $f(0) \leqq -t^2$, $f(-1) \geqq 1$, $f(-2) \geqq 1$ which yield the four inequalities (4)-(7) of Case 3. Therefore the same argument applies here. This completes the proof of Lemma 2.

LEMMA 2. *With $g(n) = (x + n)(y + n)|z + n|$, suppose that $-t^2 < g(n) < 1$ has no solution in integers n. If, further, $-2 < z < -1 < x < 0$, $1 < y < 2$ then $t^2 \leqq 2$.*

Proof. We have $g(2) \geqq 1$, $g(1) \geqq 1$, $g(0) \leqq -t^2$, $g(-1) \leqq -t^2$ and $g(-2) \geqq 1$. Now

$$-3g(0) + 2g(1) + g(-2) \geqq 3(1 + t^2) ,$$

i.e.,

$$\zeta \leqq \frac{1}{2}(1 - t^2) .$$

Also

$$2g(1) - g(0) + g(2) \geqq 3 + t^2 ,$$

i.e.,

$$\zeta \geqq \frac{1}{2}(t^2 - 3) .$$

Hence $1/2(t^2 - 3) \leqq 1/2(1 - t^2)$ or $t^2 \leqq 2$, as required.

LEMMA 3. *With $g(n)$ as defined in Lemma 2, suppose that $-t^2 < g(n) < 1$ has no solution in integers n when $t^2 \geqq 1.9$. Then, with $X = x - z$ and $Y = y - z$, the point (X, Y) does not lie in the plane region given by the two inequalities*

$$XY > -2t^2 - \frac{1}{4} , \qquad |X + Y| < \delta ,$$

where $\delta = 5$ if $t^2 > 2$ and $\delta = 4.81$ if $1.9 \leqq t^2 \leqq 2$.

Proof. Determine an integer n_0 such that $[n_0 + z] = 0$ and put $\lambda = n_0 + z$, so that $0 < \lambda < 1$. Put $F(\lambda^1) = (X + \lambda^1)(Y + \lambda^1)|\lambda^1|$ so that the condition on $g(n)$ becomes

$$-t^2 < F(\lambda^1) < 1 \tag{16}$$

has no solutions in real numbers $\lambda^1 \equiv \lambda \pmod 1$.

Put $\zeta = XY$ and $\eta = X + Y$ and $\lambda^1 = \lambda,\ \lambda - 1$ successively in (16). It follows that the point (ζ, η) does not lie in either of the two strips given by

$$\frac{-t^2}{\lambda} < \zeta + \lambda\eta + \lambda^2 < \frac{1}{\lambda}$$

and

$$\frac{-t^2}{1-\lambda} < \zeta + (\lambda - 1)\eta + (\lambda - 1)^2 < \frac{1}{1-\lambda} .$$

Hence the point (ζ, η) lies in one of four regions, giving four cases, as follows.

Case a.

$$\zeta + \lambda\eta + \lambda^2 \leqq \frac{-t^2}{\lambda} \tag{ai}$$

(aii) $$\zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \leqq \frac{-t^2}{1-\lambda}.$$

Multiplying (ai) by $1 - \lambda$ and (aii) by λ and adding, we obtain

$$\zeta \leqq -t^2\left(\frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right) - \lambda + \lambda^2 .$$

Hence if

$$-t^2\left(\frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right) - \lambda + \lambda^2 \leqq -2t^2 - \frac{1}{4}$$

the lemma holds. But this inequality may be written in the form

$$\left(\lambda - \frac{1}{2}\right)^2(\lambda^2 - \lambda + 4t^2) \geqq 0 ,$$

which is true since $0 < \lambda < 1$ and $t > 1$.

Case b.

(bi) $$\zeta + \lambda\eta + \lambda^2 \leqq \frac{-t^2}{\lambda}$$

(bii) $$\zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \geqq \frac{1}{1-\lambda}.$$

Subtracting (bii) from (bi), we obtain

$$\eta \leqq +\frac{1}{1-\lambda} + \frac{t^2}{\lambda} + 2\lambda - 1 .$$

Hence the lemma holds if

$$\delta \leqq -\frac{1}{1-\lambda} - \frac{t^2}{\lambda} - 2\lambda + 1$$

i.e., if

(biii) $$2\lambda^3 - (3 + \delta)\lambda^2 + (t^2 + \delta)\lambda - t^2 < 0 .$$

In case $1.9 \leqq t^2 \leqq 2$ and $\delta = 4.81$, (biii) becomes

$$2\lambda^3 - 7.81\lambda^2 + 6.71\lambda - 1.9 < 0 ,$$

which is true for $0 < \lambda < 1$.

In case $t^2 > 2$ and $\delta = 5$, (biii) becomes

$$2\lambda^3 - 8\lambda^2 + 7\lambda - 2 < 0 ,$$

which also holds for $0 < \lambda < 1$. This takes care of Case b.

Case c.

(ci) $$\zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \leqq \frac{-t^2}{1-\lambda}$$

(cii) $$\zeta + \lambda\eta + \lambda^2 \geqq \frac{1}{\lambda}\,.$$

If we replace λ by $1-\lambda$ and η by $-\eta$ in (ci) and (cii), we obtain (bi) and (bii). Hence, by symmetry, $|\eta| > \delta$.

Case d.

(di) $$\zeta + \lambda\eta + \lambda^2 \geqq \frac{1}{\lambda}$$

(dii) $$\zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \geqq \frac{1}{1-\lambda}\,.$$

Multiplying (di) by $1-\lambda$ and (dii) by λ and adding, we obtain

$$\zeta \geqq \frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda} + \lambda(\lambda - 1) \geqq 1\,.$$

Hence $\zeta = XY > 0$ and X, Y have the same sign. If X, Y are both negative we may change them into $-X$, $-Y$ respectively, replace λ by $1-\lambda$ and η by $-\eta$ which leaves condition (16) unchanged and turns inequalities (di) and (dii) into each other. Therefore, there is no loss of generality in assuming that X, Y are both positive. Again by the symmetry of X, Y we may assume from now on that

$$0 < X \leqq Y.$$

If $X+\lambda \leqq Y+\lambda < 2$, then one of the values $F(\lambda)$, $F(\lambda-1)$ contradicts (16). Further, if $0 < X+\lambda < 1 < Y+\lambda$, then $F(\lambda-1) < 0$, contrary to (dii). Thus, we may assume from now on that $1 < X+\lambda$ and $2 < Y+\lambda$.

Assume first that $1 < X+\lambda < 2 < Y+\lambda$. Condition (16) with $\lambda^1 = \lambda - 2$ becomes

(diii) $$-\zeta - (\lambda - 2)\eta - (\lambda - 2)^2 \geqq \frac{t^2}{2-\lambda}\,.$$

Addition of this inequality to (dii) yields

(div) $$\begin{aligned}\eta &\geqq \frac{1}{1-\lambda} + \frac{t^2}{2-\lambda} + 3 - 2\lambda \\ &\geqq \frac{1}{1-\lambda} + \frac{1.9}{2-\lambda} + 3 - 2\lambda \\ &\geqq 4.81\end{aligned}$$

if $f(\lambda) = 2\lambda^3 - 4.19\lambda^2 + 1.47\lambda - .28 \leqq 0$. Now $f(\lambda)$ has a local maximum at λ_0 where $0 < \lambda_0 < 1$ and

$$f'(\lambda_0) = 6\lambda_0^2 - 8.38\lambda_0 + 1.47 = 0 .$$

Hence $3f(\lambda_0) - f'(\lambda_0) = -4.19\lambda_0^2 + 2.94\lambda_0 - .84 < 0$ since the discriminant is negative. Thus $f(\lambda_0) < 0$, and as $f(0) < 0$ and $f(1) < 0$, it follows that $f(\lambda) < 0$ and therefore also that $\eta \geqq 4.81$. Hence, if $1.9 \leqq t^2 \leqq 2$, the lemma holds. Now assume that $t^2 > 2$. Inequality (div) implies that

$$\begin{aligned}\eta &\geqq \frac{1}{1-\lambda} + \frac{2}{2-\lambda} - 2\lambda + 3 \\ &\geqq 5 \quad \text{if} \quad 2\lambda^3 - 4\lambda^2 + \lambda \leqq 0 ,\end{aligned}$$

which is true if $\lambda \geqq 1 - 1/\sqrt{2}$. Thus we may assume that $\lambda < 1 - 1/\sqrt{2}$. If $2 < Y + \lambda < 3$, inequality (diii) may be written in the form

$$(2-\lambda)(X + \lambda - 2)(Y + \lambda - 2) \leqq -t^2 ,$$

which is clearly false since $t^2 > 2$. If $3 < Y + \lambda < 4$ then, by Lemma 2, $t^2 > 2$. Therefore we may assume that $Y + \lambda > 4$. By (16) with $\lambda^1 = \lambda - 4$, it follows that

$$-\zeta - (\lambda - 4)\eta - (\lambda - 4)^2 \geqq \frac{t^2}{4-\lambda} .$$

Adding this inequality to (dii), we obtain

$$3\eta \geqq \frac{2}{4-\lambda} + \frac{1}{1-\lambda} + 15 - 6\lambda .$$

Hence

$$\eta \geqq 5 \quad \text{if} \quad \frac{2}{4-\lambda} + \frac{1}{1-\lambda} - 6\lambda \geqq 0$$

i.e., if

$$-2\lambda^3 + 10\lambda^2 - 9\lambda + 2 \geqq 0 .$$

The left hand side is monotone decreasing for $0 \leqq \lambda \leqq 1/3$ and has the value 1/27 at $\lambda = 1/3$. As $1/3 > 1 - 1/\sqrt{2}$, so $\eta \geqq 5$ if $\lambda \leqq 1 - 1/\sqrt{2}$. Therefore, the lemma is true if $1 < X + \lambda < 2$, and we may assume from now on that $X + \lambda > 2$.

Assume next that $2 < X + \lambda < 3$. In case $2 < Y + \lambda < 3$, condition (16) with λ^1 taken successively as $\lambda - 2$ and $\lambda - 3$ yields

$$(2-\lambda)(X + \lambda - 2)(Y + \lambda - 2) \geqq 1$$

and

$$(3 - \lambda)(X + \lambda - 3)(Y + \lambda - 3) \geqq 1 \, .$$

Multiplying these two inequalities together and observing that

$$-\frac{1}{4} \leqq (X + \lambda - 2)(X + \lambda - 3) \, , \qquad (Y + \lambda - 2)(Y + \lambda - 3) < 0 \, ,$$

we obtain a contradiction. Thus we may assume that $3 < Y + \lambda$. Again condition (16) with λ^1 taken as $\lambda - 2$ and $\lambda - 3$ yields

$$\zeta + (\lambda - 2)\eta + (\lambda - 2)^2 \geqq \frac{1}{2 - \lambda}$$

and

$$-\zeta - (\lambda - 3)\eta - (\lambda - 3)^2 \geqq \frac{t^2}{3 - \lambda} \, .$$

Adding these two inequalities together gives

$$\text{(dv)} \qquad \eta \geqq \frac{1}{2 - \lambda} + \frac{t^2}{3 - \lambda} + 5 - 2\lambda \, .$$

If $t^2 > 2$ then $\eta \geqq 5$ provided

$$\frac{1}{2 - \lambda} + \frac{2}{3 - \lambda} - 2\lambda \geqq 0$$

i.e.,

$$(1 - \lambda)(7 - 8\lambda + 2\lambda^2) \geqq 0 \, ,$$

which is true since $0 < \lambda < 1$. On the other hand, if $1.9 \leqq t^2 \leqq 2$, inequality (dv) implies $\eta \geqq 4.81$ provided

$$\frac{1}{2 - \lambda} + \frac{1.9}{3 - \lambda} + 5 - 2\lambda \geqq 4.81$$

i.e.,

$$-2\lambda^3 + 10.19\lambda^2 - 15.85\lambda + 7.94 \geqq 0 \, ,$$

which is true for $0 < \lambda < 1$, since the left hand side is monotone decreasing in this range.

We are left with the case $3 < X + \lambda$, $Y + \lambda$. Here, if $\eta < 5$, then

$$X + Y + 2\lambda < 7$$

so

$$\frac{(X + \lambda - 3) + (Y + \lambda - 3)}{2} < \frac{1}{2}$$

hence, by the arithmetic-geometric mean inequality,

$$(X + \lambda - 3)(Y + \lambda - 3) < \frac{1}{4}$$

and therefore also

$$(3 - \lambda)(X + \lambda - 3)(Y + \lambda - 3) < \frac{3}{4}$$

contrary to condition (16) with $\lambda^1 = \lambda - 3$. This proves Lemma 3.

Proof of the theorem. Denote by Λ^* the set of points of Λ other than 0. We may assume that $u < 1$, for otherwise, apply the transformation $T: x_1 \to -x_1$ so that, if $T(\Lambda^*)$ has a point in the region

$$-u \leqq x_1 x_2 |x_3| \leqq \frac{1}{u}$$

then Λ^* has a point in the region

$$-\frac{1}{u} \leqq x_1 x_2 |x_3| \leqq u \ .$$

Put $\mu = \inf x_1 x_2 |x_3|$ extended over all points (x_1, x_2, x_3) of Λ for which $x_1 x_2 |x_3| > 0$. Then, either the theorem is true, or $\mu \geqq u$. If $\mu \geqq 1$, the theorem follows immediately from Davenport's result. Hence, we may assume that $\mu < 1$ and that Λ^* has no point in the region given by

$$-\frac{1}{\mu} < x_1 x_2 |x_3| < \mu \ .$$

Put $\mu = \gamma^3$. By a classical argument, using Mahler's compactness theorem (5), there is no loss of generality in assuming that Λ^* contains the point (γ, γ, γ).

The projection of Λ^* onto the plane $x_1 + x_2 + x_3 = 0$, parallel to the vector (1, 1, 1) is a two-dimensional lattice, Λ' say, of determinant $d(\Lambda') = 7/\sqrt{3}\gamma$. By the classical theory of quadratic forms, there is a point of Λ', other than 0, within a euclidean distance $\sqrt{14/3\gamma}$ of 0. Hence there is a point (x, y, z) of Λ^*, linearly independent of (γ, γ, γ), such that

$$(x - y)^2 + (y - z)^2 + (z - x)^2 \leqq \frac{14}{\gamma} \ .$$

Taking $t = 1/\gamma^3$, if $1 < t^2 \leqq 1.9$, then by Lemma 1, there is an integer n such that

$$-t^2 < \left(n + \frac{x}{\gamma}\right)\left(n + \frac{y}{\gamma}\right)\left|n + \frac{z}{\gamma}\right| < 1\,,$$

i.e.

$$-\frac{1}{\mu} < (n\gamma + x)(n\gamma + y)\,|n\gamma + z| < \mu\,,$$

which proves the theorem for the case when $1 < t^2 \leqq 1.9$.

If $t^2 > 1.9$, the projection of Λ^* onto the plane $x_3 = 0$, parallel to the vector (1, 1, 1), is a two-dimensional lattice Λ'' of determinant $d(\Lambda'') = 7/\gamma$. Taking $\delta = 5$ if $t^2 > 2$, $\delta = 4.81$ if $1.9 < t^2 \leqq 2$, by Minkowski's theorem on linear forms, there is a point $(X, Y, 0)$ of Λ'', other than 0, such that

$$|X - Y| < 2\gamma\sqrt{2t^2 + 1/4}$$

and

$$|X + Y| < \delta\gamma\,,$$

since

$$49t^2 < \delta^2\left(2t^2 + \frac{1}{4}\right).$$

Therefore, by the arithmetic-geometric mean inequality, there is a point $(X, Y, 0)$ of Λ'', other than 0, such that

$$XY > -\gamma^2\left(2t^2 + \frac{1}{4}\right)$$

and

$$|X + Y| < \delta\gamma\,.$$

We have $X = x - z$, $Y = y - z$ for some point (x, y, z) of Λ^*, linearly independent of (γ, γ, γ). Applying Lemma 3, there is an integer n such that

$$-t^2 < \left(n + \frac{x}{\gamma}\right)\left(n + \frac{y}{\gamma}\right)\left|n + \frac{z}{\gamma}\right| < 1\,,$$

i.e.,

$$-\frac{1}{\mu} < (n\gamma + x)(n\gamma + y)\,|n\gamma + z| < \mu\,,$$

and the theorem is proved.

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