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ON THE THEOREM OF S. KAKUTANI-M. NAGUMO AND J. L. WALSH FOR THE MEAN VALUE PROPERTY OF HARMONIC AND COMPLEX POLYNOMIALS

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Let K be either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$. Let $f, f_j: C \to K$ for $j = 0, \cdots, n$. Define Λ_n and Ω_n by

$$\Lambda_n(x, y) = n^{-1}\left[\sum_{j=0}^{n-1} f(x + \theta^j y)\right] - f(x) ,$$

$$\Omega_n(x, y) = n^{-1}\left[\sum_{j=0}^{n-1} f_j(x + \theta^j y)\right] - f_n(x) ,$$

for all $x, y \in C$. Our main result is the following. If $(n+1)$ unknown functions $f_j: C \to K$ for $j = 0, 1, \cdots, n$ satisfy the quasi mean value property $\Omega_n(x, y) = 0$ for all $x, y \in C$, then $(n+1)$ unknown functions f_j satisfy the difference functional equation $\Delta_u^n f_j(x) = 0$ for all $u, x \in C$ and for each $j = 0, 1, \cdots, n$, where the usual difference operator Δ_u is defined by $\Delta_u f(x) = f(x+u) - f(x)$. By using this result we prove somewhat stronger results than the theorem of S. Kakutani-M. Nagumo (Zenkoku, Sūgaku Danwakai, 66 (1935), 10–12) and J. L. Walsh (Bull. Amer. Math. Soc., 42 (1936), 923–930) for the mean value property $\Lambda_n(x, y) = 0$ of harmonic and complex polynomials.

1. Introduction. Throughout this note K denotes either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$. Let $f, f_\nu: C \to K$ for $\nu = 0, 1, \cdots, n$. Define $\Lambda_n(x, y)$ and $\Omega_n(x, y)$ by

$$\Lambda_n(x, y) = n^{-1}\left[\sum_{\nu=0}^{n-1} f(x + \theta^\nu y)\right] - f(x) ,$$

$$\Omega_n(x, y) = n^{-1}\left[\sum_{\nu=0}^{n-1} f_\nu(x + \theta^\nu y)\right] - f_n(x)$$

for all $x, y \in C$. A function $f: C \to K$ is said to have the mean value property for polynomials if f satisfies the equation

$$\Lambda_n(x, y) = 0 \qquad \text{for all} \quad x, y \in C ,$$

while, as a generalization of the mean value property, $n+1$ functions $f_\nu: C \to K$ are said to have the quasi mean value property for polynomials if f_ν satisfy the equation

$$\Omega_n(x, y) = 0 \qquad \text{for all} \quad x, y \in C .$$

In 1935 S. Kakutani and M. Nagumo [19], and independently, in 1936 J. L. Walsh [29] proved the following theorems concerning the mean value property of harmonic and complex polynomials.

THEOREM A. (*Kakutani-Nagumo-Walsh.*) *If $f: C \to R$ is continuous, the mean value property $\Lambda_n(x, y) = 0$ holds for all $x, y \in C$ if, and only if, $f(x)$ is a harmonic polynomial of degree at most $n - 1$.*

THEOREM B. *An entire function f satisfies the mean value property $\Lambda_n(x, y) = 0$ for all $x, y \in C$ if and only if f is given by a complex polynomial of degree at most $n - 1$.*

The above Theorem A and Theorem B are direct or indirect motivations for the generalizations and applications of J. Aczél, H. Haruki, M. A. McKiernan and G. N. Sakovič [2], E. F. Beckenbach and M. Reade [3], [4], A. K. Bose [5], L. Flatto [7], [8], [9], A. Friedman and W. Littman [10], A. Garsia [11], H. Haruki [13], [14], S. Haruki [15], [16], [17], J. H. B. Kemperman and D. Girod [21], M. A. McKiernan [25], M. O. Reade [27]. For more details of functional equations of type $\Lambda_n(x, y) = 0$, see M. A. McKiernan [26], and for the relation to Gauss' mean value theorem, harmonic functions and differential equations, see L. Zalcman [30].

The main purpose of this note is to study some more generalizations of Theorem A and Theorem B from the standpoint of the theory of finite difference functional equations.

2. P-additive symmetrical mappings, generalized polynomials and $\Delta_y^n f(x) = 0$. In this section we present some notation, definitions for p-additive symmetrical mappings, generalized polynomials and results of S. Mazur and W. Orlicz [23] for the finite difference functional equation $\Delta_y^n f(x) = 0$.

DEFINITION. A mapping $Q^p: C \to K$ is called a homogeneous polynomial of degree p if and only if there exists a p-additive symmetrical mapping $Q_p: C^p \to K$; that is, $Q_p(x_1, \cdots, x_p) = Q_p(x_{i_1}, \cdots, x_{i_p})$ for all $(x_1, \cdots, x_p) \in C$ and for all permutations $(i_1, \cdots, i_p)$ of the sequence $(1, \cdots, p)$ and Q_p is an additive function in each x_q, $1 \leqq q \leqq p$, such that $Q^p(x) = Q_p(x, \cdots, x)$ for all $x \in C$. We say that Q_p is associated with Q^p or that Q_p generates Q^p.

We agree that for $p = 0$ a homogeneous polynomial of degree zero is a constant. If p is a fixed positive integer, then $\pi_p: C \to C^p$ will denote the diagonal mapping given by $\pi_p(x) = (x, \cdots, x)$. It is clear from the relation $Q^p(x) = Q_p(x, \cdots, x)$ that $Q^p: C \to K$ is the composition of two mappings

$$C \xrightarrow{\pi_p} C^p \xrightarrow{Q_p} K \quad \text{and} \quad Q^p = Q_p \circ \pi_p \,.$$

If $Q^p: C \to K$ is a homogeneous polynomial of degree p, one obtains $Q^p(\lambda x) = \lambda^p Q^p(x)$ for any rational number λ. Indeed, the relation $Q^p = Q_p$ yields $Q^p(\lambda x) = Q_p(\lambda x, \cdots, \lambda x) = \lambda^p Q_p(x, \cdots, x) = \lambda^p Q^p(x)$ for all $x \in C$ and for any rational number λ.

DEFINITION. Let β be any nonnegative integer. If $f: C \to K$ is a finite sum $f = Q^0 + Q^1 + \cdots + Q^\beta$ of homogeneous polynomials, then f is called a generalized polynomial of degree at most β.

For $f: C \to K$ and for $y \in C$ we define the usual difference operator Δ_y by $\Delta_y f(x) = f(x+y) - f(x)$. For $y_i \in C$, $i = 1, 2, \cdots, n$, we inductively define the nth order difference operator $\Delta^n_{y_1, \cdots, y_n}$ by

$$\Delta^n_{y_1 \cdots y_n} f(x) = (\Delta^{n-1}_{y_1 \cdots y_{n-1}}) \Delta_{y_n} f(x) \,.$$

Notice that the ring of operators generated by this family of operators is commutative and distributive.

The following general theorem of S. Mazur and W. Orlicz [23] in the theory of finite difference functional equations plays a fundamental role in our study.

Fundamental theorem. Let M, N be fixed integers $\geqq 0$. Let X be an Abelian additive semigroup with unit element 0 and $lx = x + x + \cdots + x$ for integer $l > 0$, $x \in X$, and let F be an Abelian group and $ly = y + y + \cdots + y$ for integer $l > 0$, $y \in F$. Let $f: X \to F$. The following three statements are equivalent if $M^N \neq 0$ in F:

(a) $\Delta^{N+1}_y f(x) = 0$ for all $x, y \in X$,

(b) $\Delta^{N+1}_{y_1 \cdots y_{N+1}} f(x) = 0$ for all $x, y_1, \cdots, y_{N+1} \in X$,

(c) f is a generalized polynomial of degree at most N, that is, $f(x) = Q^0 + Q^1(x) + \cdots + Q^N(x)$ for all $x \in X$, where $Q^p: X \to F$ for $p = 0, 1, \cdots, N$ are homogeneous polynomials.

Note that the above Fundamental theorem clearly holds for the case $X = C$ and $F = K$.

Notation. We denote $Q^p_\nu(x) = Q_{\nu,p}(x, \cdots, x)$ for $\nu = 0, 1, \cdots, n$, where $Q^p_\nu: C \to K$ are homogeneous polynomials of degree p for $\nu = 0, 1, \cdots, n$.

Notation. Let $Q_{(n-r,r)}(x; y)$ denote the value of $Q_n(x_1, \cdots, x_n)$ for $x_i = x$, $i = 1, \cdots, n-r$ and $x_i = y$, $i = n-r+1, \cdots, n$. In particular $Q_{(0,n)}(y; x) = Q_{(n,0)}(x; y) = Q^n(x)$.

3. The quasi mean value property $\Omega_n(x, y) = 0$. Our first result is the following:

THEOREM 3.1. *If $n+1$ unknown functions $f_\nu: C \to K$ for $\nu = 0, 1, \cdots, n$ satisfy the quasi mean value property $\Omega_n(x, y) = 0$ for all $x, y \in C$, then there exist generalized polynomials of degree at most $n-1$ such that*

$$f_\nu(x) = Q_\nu^0 + Q_\nu^1(x) + \cdots + Q_\nu^{n-1}(x)$$

for all $x \in C$ and for each $\nu = 0, 1, \cdots, n$.

The proof of Theorem 3.1 is based on the Lemma 3.1 below. Let G and H be additive Abelian groups. Let S be any field and G, H be a unital S-modules. Let $f: G \to H$ satisfy the equation

$$\sum_{i=0}^{n} \gamma_i f(x + \alpha_i y) = 0 \qquad \text{for all} \quad x, y \in G\,,$$

where $n > 2$ is a given integer, $\gamma_i \neq 0$, $\alpha_i \neq 0 (= \alpha_0)$ for $i = 0, 1, \cdots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. The above equation is a generalization of the difference functional equation (cf. J. Aczél [1], D. Ž. Djoković [6], D. Girod and J. H. B. Kemperman [12], M. H. Ingraham [18], J. H. B. Kemperman [20], [22], G. van der Lijn [28], S. Mazur and W. Orlicz [23], M. A. McKiernan [24], [26])

$$\Delta_y^n f(x) = 0\,, \qquad \text{i.e.,} \quad \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + iy) = 0$$

for all $x, y \in G$. More generally we have

LEMMA 3.1. *Let $f_i: G \to H$ for $i = 0, 1, \cdots, n$ satisfy the equation*

$$\sum_{i=0}^{n} f_i(x + \alpha_i y) = 0 \qquad \textit{for all} \quad x, y \in G\,, \tag{3.1}$$

where $\alpha_i \neq 0$ for $i = 0, 1, \cdots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. Then equation (3.1) *implies*

$$\Delta_u^n f_i(x) = 0 \qquad \textit{for each} \quad i = 0, 1, \cdots, n \textit{ and for all } x, u \in G\,. \tag{3.2}$$

Proof of Lemma 3.1. In view of equation (3.1) one can observe the following property.

(3.3) To eliminate the kth term f_k, $0 \leqq k \leqq n$, we replace x by $x - \alpha_k z_k$ and y by $y + z_k$ in (3.1).

Indeed, for $k = j$ we have

$$f_0(x - \alpha_j z_j + \alpha_0 y + \alpha_0 z_j) + \cdots + f_j(x + \alpha_j y) + \cdots + f_n(x - \alpha_j z_j + \alpha_n y + \alpha_n z_j) = 0$$

for all $x, y, z_j \in G$. Take the difference between (3.1) and the above equation to obtain

$$\Delta_{(\alpha_0 - \alpha_j)z_j} f_0(x + \alpha_0 y) + \cdots + 0 + \cdots + \Delta_{(\alpha_n - \alpha_j)z_j} f_n(x + \alpha_n y) = 0 \tag{3.4}$$

for all $x, y, z_j \in G$, since $f_j(x + \alpha_j y)$ is unchanged. Thus f_j is eliminated. If the same argument (3.3) is repeated $(n - 1)$ times, then (3.4) yields

$$\Delta_{(\alpha_0 - \alpha_j)z_j} \Delta_{\beta_1 z_1} \cdots \Delta_{\beta_n z_n} f_0(x + \alpha_0 y) = 0 \tag{3.5}$$

for all $x, y, z_1, \cdots, z_n \in G$, where $\beta_l = \alpha_0 - \alpha_l$ for $l = 1, 2, \cdots, n$ and $l \neq j$. In (3.5), replace $x + \alpha_0 y$ by x and set $u = (\alpha_0 - \alpha_j)z_j = \beta_1 z_1 = \cdots = \beta_n z_n$. Then (3.5) becomes

$$\Delta_u^n f_0(x) = 0 \qquad \text{for all} \quad x, u \in G .$$

It is clear that an obvious modification can be applied for the terms $f_k(x + \alpha_k y)$ for $k = 1, 2, \cdots, n$ to obtain

$$\Delta_u^n f_k(x) = 0 \qquad \text{for each} \quad k = 1, 2, \cdots, n \quad \text{and for all} \quad x, u \in G .$$

Thus (3.1) implies (3.2). The Lemma 3.1 is proved.

Proof of Theorem 3.1. Observe that without loss of generality we may assume one of $\alpha_i = 0$, i.e., $\alpha_i \neq 0 = \alpha_n$, $i = 0, 1, \cdots, n - 1$, in Lemma 3.1 in order to obtain the same conclusion. The proof now immediately follows from Lemma 3.1 and the Fundamental theorem with $G = X = C$ and $F = S = H = K$.

4. The mean valued property $A_n(x, y) = 0$.

We first determine the general solution of the mean value property under no regularity assumptions. Then we prove somewhat stronger results than that of Theorem A and Theorem B, when some weak regularity assumptions are imposed on f.

THEOREM 4.1. *A function $f: C \to K$ satisfies the mean value property $A_n(x, y) = 0$ for all $x, y \in C$ if and only if there exists a generalized polynomial of degree at most $n - 1$ such that*

$$f(x) = Q^0 + Q^1(x) + \cdots + Q^{n-1}(x) \qquad \text{for all} \quad x \in C , \tag{4.1}$$

where the homogeneous polynomials $Q^p: C \to K$ for $p = 1, \cdots, n - 1$ must satisfy the equation

$$\sum_{\nu=0}^{n-1}\sum_{\delta=1}^{n-1}\sum_{\sigma=1}^{\delta}\binom{\delta}{\sigma}Q_{(\delta-\sigma,\sigma)}(x;\theta^{\nu}y)=0 \qquad \textit{for all} \quad x, y\in C\,. \tag{4.2}$$

Proof of Theorem 4.1. If $f: C \to K$ satisfies $\Lambda_n(x, y) = 0$ for all $x, y \in C$, then (4.1) immediately follows from Theorem 3.1. To show the converse, substitute (4.1) into $\Lambda_n(x, y) = 0$ to obtain

$$\begin{aligned}\sum_{\nu=0}^{n-1}(Q^0 + Q^1(x+\theta^{\nu}y) + \cdots + Q^{n-1}(x+\theta^{\nu}y))\\ = n(Q^0 + Q^1(x) + \cdots + Q^{n-1}(x))\,,\end{aligned}$$

which implies, since $Q^{n-1}(x+\theta^{\nu}y) = \sum_{\sigma=0}^{n-1}\binom{n-1}{\sigma}Q_{(n-1-\sigma,\sigma)}(x;\theta^{\nu}y)$,

$$\begin{aligned}\sum_{\nu=0}^{n-1}\Bigg(Q^0 + Q^1(x) + \cdots + Q^{n-1}(x) + Q^1(\theta^{\nu}y) + \sum_{\sigma=1}^{2}\binom{2}{\sigma}Q_{(2-\sigma,\sigma)}(x;\theta^{\nu}y)\\ + \cdots + \sum_{\sigma=1}^{n-1}\binom{n-\sigma}{\sigma}Q_{(n-1-\sigma,\sigma)}(x;\theta^{\nu}y)\Bigg)\\ = n(Q^0 + Q^1(x) + \cdots + Q^{n-1}(x))\,.\end{aligned} \tag{4.3}$$

But in order for (4.1) to be the general solution of $\Lambda_n(x, y) = 0$, the homogeneous polynomials Q^{δ}, $\delta = 1, 2, \cdots, n-1$, must satisfy equation (4.3). This case occurs only if

$$\begin{aligned}\sum_{\nu=0}^{n-1}\left\{Q^1(\theta^{\nu}y) + \sum_{\sigma=1}^{2}\binom{2}{\sigma}Q_{(2-\sigma,\sigma)}(x;\theta^{\nu}y) + \cdots + \sum_{\sigma=1}^{n-1}\binom{n-1}{\sigma}Q_{(n-1-\sigma,\sigma)}(x;\theta^{\nu}y)\right\}\\ = 0\,,\end{aligned}$$

which yields (4.2). This proves the Theorem 4.1.

THEOREM 4.2. *If a function $f: C \to R$ satisfies $\Lambda_n(x, y) = 0$ for all $x, y \in C$, then* (4.1) *holds for all $x \in C$, where $Q^p: C \to R$ for $p = 0, 1, \cdots, n-1$. Moreover, f is bounded on a set of positive Lebesgue measure if and only if f is given by a harmonic polynomial of degree at most $n-1$.*

LEMMA 4.1. *Let $f: C \to K$ be a generalized polynomial of degree at most $n-1$ such that*

$$f(x) = Q^0 + Q^1(x) + \cdots + Q^{n-1}(x) \tag{4.1}$$

for all $x \in C$, where $Q^p: C \to K$, $p = 0, 1, \cdots, n-1$, are homogeneous polynomials. If f is bounded on a set of positive Lebesgue measure, then Q^p for $p = 0, 1, \cdots, n-1$ are continuous everywhere and hence so is f.

Proof of Lemma 4.1. Replace x by Mx for each $M = 1, 2, \cdots, n$. Then

$$\begin{bmatrix} f(x) \\ f(2x) \\ \vdots \\ f(nx) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \cdots 1 \\ 1 & 2 & 2^2 \cdots 2^{n-1} \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 1 & n & n^2 \cdots n^{n-1} \end{bmatrix} \begin{bmatrix} Q^0 \\ Q^1(x) \\ \vdots \\ Q^{n-1}(x) \end{bmatrix}.$$

We briefly write this as $|F| = |V||Q|$. Observe that $|V|$ is the van der Monde determinant and is not zero. Therefore Q^p, $p = 0, 1, \cdots, n-1$, can be determined uniquely in terms of $f(Mx)$ for $M = 1, 2, \cdots, n$. Since f is bounded on a set of positive Lebesque measure, the $Q^p(x)$ for $p = 0, 1, \cdots, n-1$ are bounded on a set of positive Lebesgue measure for all x. On the other hand we have the basic identity

$$Q_{n-1}(x_1, \cdots, x_{n-1}) = (1/(n-1)!)\Delta_{x_1} \cdots \Delta_{x_{n-1}} Q^{n-1}(x)$$

for all $x, x_1, \cdots, x_{n-1}$. The right side is the sum of 2^{N-1} terms of the form

$$((-1)^{n-1-q}/((n-1)!))Q^{n-1}(x_{i_1} + \cdots + x_{i_q})$$

with $x = 0$. But we have just proved that $Q^p(x)$ is bounded on a set of positive Lebesgue measure for $p = 0, 1, \cdots, n-1$ and for all x. Hence Q_p for $p = 0, 1, \cdots, n-1$ are also bounded on a set of positive Lebesgue measure for all $x_1, \cdots, x_{n-1}$. It is well-known (e.g., [20]) that an additive function $f: C \to K$ which is bounded on a set of positive measure is continuous everywhere. It follows from this theorem that a p-additive mapping which is bounded on a set of positive Lebesgue measure is continuous everywhere. Hence, Q^p for each $p = 0, 1, \cdots, n-1$ is continuous everywhere. Equation (4.1) now shows that f is continuous everywhere. This proves the Lemma 4.1.

Proof of Theorem 4.2. This is a consequence of Lemma 4.1 and Theorem A of Kakutani-Nagumo-Walsh.

For the case $K = C$ we have the following:

THEOREM 4.3. *If a function $f: C \to C$ satisfies $\Delta_n(x, y) = 0$ for all $x, y \in C$, then* (4.1) *holds for all $x \in C$. Further, f is bounded on a set of positive Lebesgue measure if and only if f is a complex polynomial of the form*

$$f(x) = \sum_{s=0}^{n-1} a_{0,s}x^s + \sum_{r=1}^{n-1} a_{r,r}\bar{x}^r \,, \tag{4.4}$$

where $\bar{x}$ denotes the conjugate of x.

LEMMA 4.2. *Let n be a given integer $\geqq 1$, and let $Q_n: C^n \to C$ be an n-additive symmetrical mapping and continuous everywhere. Then there exist complex constants $a_0, a_1, \cdots, a_n$ such that for all $x_1, \cdots, x_n \in C$,*

$$Q_n(x_1, \cdots, x_n) = \sum_{r=0}^{n}\left(a_r \sum_{\binom{n}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_n\right). \tag{4.5}$$

Proof of Lemma 4.2. For $n = 1$ we have

$$Q_1(x_1 + x_2) = Q_1(x_1) + Q_1(x_2) \qquad \text{for all} \quad x_1, x_2 \in C,$$

whose continuous solutions are well-known (e.g., see J. Azcél [1, p. 217]) to be of the form

$$Q_1(x) = Ax + B\bar{x}$$

where A and B are complex constants. We now assume that (4.5) is true for $n = m \geqq 1$. For $n = m + 1$ the continuous solution of the equation

$$Q_{m+1}(x_1, \cdots, x_m, y + z) = Q_{m+1}(x_1, \cdots, x_m, y) + Q_{m+1}(x_1, \cdots, x_m, z) \tag{4.6}$$

for all $x_1, \cdots, x_m, y, z \in C$ is given by

$$Q_{m+1}(x_1, \cdots, x_m, x_{m+1}) = \sum_{r=0}^{m}\left(A_r(x_{m+1}) \sum_{\binom{m}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_m\right). \tag{4.7}$$

Substitute (4.7) into (4.6) to obtain

$$\begin{aligned}\sum_{r=0}^{m}&\left(A_r(y + z) \sum_{\binom{m}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_m\right)\\ &= \sum_{r=0}^{m}\left(A_r(y) \sum_{\binom{m}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_m\right).\\ &\quad + \sum_{r=0}^{m}\left(A_r(z) \sum_{\binom{m}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_m\right).\end{aligned}$$

By the uniqueness theorem of polynomial coefficients we have

$$A_r(y + z) = A_r(y) + A_r(z) \qquad \text{for each} \quad r = 0, 1, \cdots, n$$

and $A_r(x) = \alpha_r x + \beta_r\bar{x}$ for each r, where α_r and β_r are complex constants. This solution in (4.7) implies

$$Q_{m+1} = \sum_{r=0}^{m}\left((\alpha_r x_{m+1} + \beta_r\bar{x}_{m+1}) \sum_{\binom{m}{r}} x_1x_2 \cdots x_r\bar{x}_{r+1}\bar{x}_{r+2} \cdots \bar{x}_m\right)$$

which shows that there exist complex constants $a_0, a_1, \cdots, a_{m+1}$ such that

$$Q_{m+1} = \sum_{r=0}^{m+1} \left(a_r \sum_{\binom{m+1}{r}} x_1 x_2 \cdots x_r \bar{x}_{r+1} \bar{x}_{r+2} \cdots \bar{x}_{m+1} \right),$$

yielding the Lemma 4.2.

Note that in particular for the case $x_1 = x_2 = \cdots = x_r = \bar{x}_{r+1} = \bar{x}_{r+2} = \cdots = \bar{x}_m$, (4.5) becomes

$$Q^n(x) = \sum_{r=0}^{n} a_r x^{n-r} \bar{x}^r . \tag{4.8}$$

Proof of Theorem 4.3. By applying Lemma 4.1 with $K = C$ we obtain that Q^p is continuous for each $p = 0, 1, \cdots, n-1$. Hence, Lemma 4.2 with (4.8) yields

$$Q^p(x) = \sum_{r=0}^{p} a_r x^{p-r} \bar{x}^r \qquad \text{for each} \quad p = 0, 1, \cdots, n-1 .$$

Hence, by (4.1), we have

$$f(x) = \sum_{s=0}^{n-1} \sum_{r=0}^{s} a_{r,s} x^{s-r} \bar{x}^r . \tag{4.9}$$

Conversely, if (4.9) is substituted in the mean value property $\Lambda_n(x, y) = 0$, then we obtain

$$\begin{aligned} \sum_{\nu=0}^{n-1} \{ [a_{0,0}] &+ [a_{0,1}(x + \theta^\nu y) + a_{1,1}(\bar{x} + \bar{\theta}^\nu \bar{y})] \\ &+ [a_{0,2}(x + \theta^\nu y)^2 + a_{1,2}(x + \theta^\nu y)(\bar{x} + \bar{\theta}^\nu \bar{y}) + a_{2,2}(\bar{x} + \bar{\theta}^\nu \bar{y})^2] \\ &+ \cdots + [a_{0,n-1}(x + \theta^\nu y)^{n-1} + a_{1,n-1}(x + \theta^\nu y)^{n-2}(\bar{x} + \bar{\theta}^\nu \bar{y}) \\ &+ \cdots + a_{n-1,n-1}(\bar{x} + \bar{\theta}^\nu \bar{y})^{n-1}] \} \\ = n \sum_{s=0}^{n-1} \sum_{r=0}^{s} & a_{r,s} x^{s-r} \bar{x}^r . \end{aligned} \tag{4.10}$$

By expanding both sides of (4.10) and comparing coefficients $a_{r,s}$ one observes that (4.9) satisfies the mean value property $\Lambda_n(x, y) = 0$ if $a_{r,s} = 0$ for $r \neq s$, $r, s = 1, \cdots, n-1$, since the right side of (4.10) is independent of y and $\bar{y}$, and

$$\sum_{\nu=0}^{n-1} (\theta^\nu \bar{\theta}^\nu)^p = n \qquad \text{for} \quad p = 0, 1, \cdots, n-1 ,$$

$$\sum_{\nu=0}^{n-1} (\theta^\nu)^p = 0 \qquad \text{for} \quad p = 1, \cdots, n-1 ,$$

$$\sum_{\nu=0}^{n-1} (\bar{\theta}^\nu)^p = 0 \qquad \text{for} \quad p = 1, \cdots, n-1 ,$$

and

$$\sum_{\nu=0}^{n-1} (\theta^\nu)^j (\bar{\theta}^\nu)^l = 0 \qquad \text{for} \quad j \neq l,\ j, l = 1, \cdots, n-1 .$$

Therefore, we obtain

$$f(x) = \sum_{s=0}^{n-1} a_{0,s} x^s + \sum_{r=1}^{n-1} a_{r,r} \bar{x}^r . \tag{4.4}$$

This proves the Theorem 4.3.

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