PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 1, 1981

THE HAUSDORFF DIMENSION OF A SET OF NORMAL NUMBERS

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Suppose that numbers $2, 3, \cdots$ are partitioned into two disjoint classes R, S so that rational powers lie in the same class. In this paper we prove that the set of numbers ξ which are normal to every base from R and to no base from S has Hausdorff dimension 1. The existence of such numbers was first shown by W. M. Schmidt.

1. Introduction. We call two natural numbers r, s equivalent and write $r \sim s$, when each is a rational power of the other.

Schmidt [2] has shown that normality to base r implies normality to base s precisely when s is a rational power of r and also [3] that, given any partition of the numbers $2, 3, \cdots$ into two disjoint classes R, S so that equivalent numbers fall in the same class, there are real numbers normal to every base from R and to no base from S.

In this paper we prove the following.

THEOREM 1. *Given any partition of the numbers $2, 3, \cdots$ into two disjoint classes R, S so that equivalent numbers fall in the same class, the set, $\mathscr{N}$, of numbers which are normal to every base from R and to no base from S has Hausdorff dimension* 1.

If R is empty then $\mathscr{N}$ consists of those numbers which are not normal to any integer base. In this case Theorem 1 is already known, see for example Schmidt [4]. If S is empty then $\mathscr{N}$ consists of those numbers which are normal to all integers bases. This set contains almost all numbers, in the sense of Lebesgue's measure, and Theorem 1 is obvious. We will therefore restrict our attention to the case when $R = \{r_1, r_2, \cdots\}$ and $S = \{s_1, s_2, \cdots\}$ are both nonempty.

After some preliminaries, and given a certain parameter A, a nested sequence

$$J_0 = [0, 1] \supset J_1 \supset \cdots$$

of sets is constructed, where each set J_i is a union of closed intervals. It is then shown that a number

$$\xi \in \bigcap_{i=1}^{\infty} J_i$$

is nonnormal to each base $s_1, s_2, \cdots$. Then a new sequence of sets

$$K_0 = [0, 1] \supset K_1 \supset \cdots$$

is constructed, where each $K_i \subseteqq J_i$, and it is shown that a number

$$\xi \in \bigcap_{i=1}^{\infty} K_i$$

is normal to each base $r_1, r_2, \cdots$. For this, estimates of exponential sums and two lemmas of Schmidt [3] are required. Finally, a theorem of Eggleston [1] is used to show that $\bigcap_{i=1}^{\infty} K_i$ has Hausdorff dimension at least $\log(A-1)/\log A$. Since A can be chosen arbitrarily large, the desired conclusion follows.

We will require the following lemma, due to Schmidt [3], which is the cornerstone of his proof that $\mathscr{N}$ is nonempty.

LEMMA 1. *Let K, l, r, s be natural numbers with $l \geqq s^K$ and $r \not\sim s$. Then*

$$(1) \qquad \sum_{n=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n l/s^k)| \leqq 2N^{1-\alpha(r,s)} \qquad where \quad \alpha(r,s) > 0 .$$

The following result implies Theorem 1.

THEOREM 2. *Let $A > 2$ be a natural number. Let R, S be two subsets of $\{A, A+1, \cdots\}$ such that if $r \in R$ and $s \in S$ then $r \not\sim s$. Then the set $\mathscr{N}_A$ of numbers which are normal to every base from R and to no base from S has Hausdorff dimension at least $\log(A-1)/\log A$.*

2. Deduction of Theorem 1 from Theorem 2. Suppose that we are given a partition of the natural numbers R, S as in Theorem 1. Let $R_A = R \cap \{A, A+1, \cdots\}$, $S_A = S \cap \{A, A+1, \cdots\}$.

We apply Theorem 2 for R_A, S_A. Then $\mathscr{N}_A = \mathscr{N}$. For suppose $r \in R$ and $x \in \mathscr{N}_A$. Then clearly if $r \geqq A$ then x is normal to base r, if $r < A$, then $r^A > A$ and also $r^A \in R$ since rational powers lie in the same class. Hence x is normal to base r^A. But then x is also normal to base r. Similarly x is nonnormal to base s for any $s \in S$.

Hence $\mathscr{N}_A \subset \mathscr{N}$ and clearly $\mathscr{N} \subset \mathscr{N}_A$. Thus

$$\bigcup_{A=3}^{\infty} \mathscr{N}_A = \mathscr{N} .$$

But

$$\dim\left(\bigcup_{A=3}^{\infty} \mathscr{N}_A\right) \geqq \frac{\log(A-1)}{\log A} \qquad A = 3, 4, \cdots .$$

Thus $\dim \mathscr{N} = 1$ which proves Theorem 1.

We now construct a subset of $\mathcal{N}_A$ to show that

$$\dim \mathcal{N}_A \geqq \frac{\log (A-1)}{\log A} .$$

Suppose $R = \{r_1, r_2, \cdots\}$ and $S = \{s_1, s_2, \cdots\}$ are given as in Theorem 2. It is sufficient to construct a set of numbers ξ such that ξ is normal to each of the bases $r_1, r_2, \cdots$ but not normal to the bases $s_1, s_2, \cdots$.

3. Preliminaries. Let

$$\beta_{ij} = \alpha(r_i, s_j) \qquad (i, j = 1, 2, \cdots)$$

where $\alpha(r, s)$ is the constant in Lemma 1.

Put

$$\beta_k = \min_{1 \leqq i,j \leqq k} \beta_{i,j}$$

and

$$\gamma_k = \max (r_1, \cdots, r_k, s_1, \cdots s_k) .$$

We may assume $\beta_k < 1/2$. Put $\phi(1) = 1$ and let $\phi(k)$ be the largest natural number ϕ which satisfies

$$\phi \leqq \phi(k-1) + 1 , \quad \beta_\phi \geqq \beta_1 k^{-1/4} , \quad \gamma_\phi \leqq \gamma_1 k .$$

Then $\phi(1), \phi(2), \cdots$ is a nondecreasing sequence of natural numbers; in which every natural number occurs. We let $r'_i = r_{\phi(i)}$, $s'_i = s_{\phi(i)}$, then $\{r'_i\}$ and $\{s'_i\}$ have the same properties as $\{r_i\}$ and $\{s_i\}$ but further

$$\beta'_k \geqq \beta'_1 k^{-1/4} \quad \text{and} \quad \gamma'_k \leqq \gamma'_1 k .$$

Therefore we may assume that the original sequence satisfies

$$\beta_k \geqq \beta_1 k^{-1/4} , \qquad \gamma_k \leqq \gamma_1 k . \tag{2}$$

We write $h(m)$ for the least number h, such that

$$m \not\equiv 0 (\bmod 2^h) .$$

Put $s(m) = s_{h(m)}$. Then every term s_i occurs infinitely many times in the sequence $s(m)$.

Let $\delta_1, \delta_2, \cdots$ denote absolute constants.

4. Construction of a set of nonnormal numbers. We construct sets

$$J_0 = [0, 1] \supset J_1 \supset J_2 \supset \cdots \tag{3}$$

(each the union of closed intervals) as follows:

Let

$$f(m) = e^{\sqrt{m}} + 2s_1 m^3 .$$

Put

$$\langle m \rangle = \lceil f(m) \rceil , \quad \langle m; x \rangle = \lceil \langle m \rangle / \log x \rceil ,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x,

$$b_m = \langle m + 1; s(m) \rangle \tag{4}$$

$$a_{m+1} = \left[\frac{b_m \log s(m)}{\log s(m+1)} \right] + 2 . \tag{5}$$

Then

$$\frac{\langle m+1 \rangle}{\log s(m+1)} + 2 \leqq a_{m+1} \leqq \frac{\langle m+1 \rangle}{\log s(m+1)} + \log \log m + 3 \tag{6}$$

and

$$e^{\langle m \rangle} s(m)^2 \leqq s(m)^{a_m} \leqq e^{\langle m \rangle} s(m)^{\log \log m + 3} . \tag{7}$$

The numbers a_m and b_m, defined in (4) and (5), are chosen so that

$$s(1)^{b_1} < s(2)^{a_2} < s(2)^{b_2} < s(3)^{a_3} < s(3)^{b_3} < \cdots .$$

Let J_1 be the union of the intervals I, each of length $s(1)^{-b_1}$, whose left end points are of the form

$$\xi_1 = \frac{\varepsilon_1}{s(1)} + \frac{\varepsilon_2}{s(1)^2} + \cdots + \frac{\varepsilon_{b_1}}{s(1)^{b_1}} \tag{8}$$

where ε_i range over $0, 1, \cdots, s(1) - 2$ if $s(1)$ is odd, and over $0, 1, \cdots, s(1) - 3$ if $s(1)$ is even.

Put

$$\begin{aligned} \delta(i) &= 2 \quad \text{if} \quad s(i) \quad \text{is odd} \\ &= 3 \quad \text{if} \quad s(i) \quad \text{is even} . \end{aligned}$$

There are $(s(1) - \delta(1))^{b_1}$ such intervals I of J_1.

Suppose that J_k has been constructed and that I_k is an interval of J_k of length $s(k)^{-b_k}$.

By (5)

$$s(k+1)^{-a_{k+1}+2} \leqq s(k)^{-b_k} .$$

Thus in each interval I_k there are at least

$$\left[\frac{s(k+1)^{a_{k+1}}}{s(k)^{b_k}}\right] - 2 \text{ intervals } I'_k \text{ of length}$$

$s(k+1)^{-a_{k+1}}$ whose left end points are finite "decimals" of length a_{k+1} in base $s(k+1)$.

To construct J_{k+1} we proceed as follows:

Let ρ_k be the left end point of an interval I'_k. We construct subintervals of I'_k of length $s(k+1)^{-b_{k+1}}$ whose left end points are of the form

$$\xi_{k+1} = \rho_k + \frac{\varepsilon_1}{s(k+1)^{a_{k+1}+1}} + \cdots + \frac{\varepsilon_{t_{k+1}}}{s(k+1)^{b_{k+1}}} \tag{9}$$

where $t_k = b_k - a_k$ and $\varepsilon_1, \cdots, \varepsilon_{t_{k+1}}$ can range over $0, 1, \cdots, s(k+1) - \delta(k+1)$.

In each interval I'_k there are $(s(k+1) - \delta(k+1) + 1)^{t_{k+1}}$ such intervals. Let J_{k+1} be the union of all such intervals taken over all I'_k. Then J_{k+1} is the union of at least

$$\left(\left[\frac{s(k+1)^{a_{k+1}}}{s(k)^{b_k}}\right] - 2\right)(s(k+1) - \delta(k+1) + 1)^{t_{k+1}}$$

intervals of length $s(k+1)^{-b_{k+1}}$. This completes the construction of the sequence of sets $J_0 \supset J_1 \supset \cdots$.

LEMMA 2. *If* $\xi \in \bigcap_{i=1}^{\infty} J_i$ *then* ξ *is nonnormal to each base* $s_1, s_2, \cdots$.

Proof. Fix h and let $s = s_h$. Let q be so large that

$$\left(\frac{s-1}{s}\right)^q < 2^{-h}\ . \tag{10}$$

For a number M with $h(M) = h$ there are at least

$$\sum_{\substack{m \leq M \\ h(m)=h}} (t_m - 1 - q) \tag{11}$$

q-blocks $\varepsilon_{i+1}, \cdots, \varepsilon_{i+q}$, consisting of the digits $0, 1, \cdots, s-2$ in the expansion of ξ, such that $i + q \leqq b_M$. Now $h(m) = h$ precisely if $m \equiv 2^{h-1} (\operatorname{mod} 2^h)$. If $h(m) = h$ and $m > 2^{h-1}$, then, by (6),

$$t_m - 1 - q \geqq 2^{-h} \sum_{j=m-2^h+1}^{m} [(\langle j+1; s\rangle - \langle j; s\rangle) - \log\log m - 5 - q]$$

since $t_m = b_m - a_m$ and $\langle m+1; s\rangle - \langle m; s\rangle$ is a nondecreasing function of m.

Thus (11) is at least

$$\sum_{\substack{m \leqq M \\ h(m)=h}} 2^{-h} \sum_{j=m-2^h+1}^{m} ((\langle j+1; s\rangle) - (\langle j; s\rangle) - \log\log m - 5 - q)$$
$$\geqq 2^{-h}(\langle M+1; s\rangle - \langle 1; s\rangle) - M(\log\log M + 5 + q)$$
$$= 2^{-h} b_M(1 + 0(1)) .$$

If ξ were normal to the base $s = s_h$, the number of q-blocks with digits $0, 1, \cdots, s-2$ and indices smaller than b_M would be asymptotic to $((s-1)/s)^q b_M$. By (10) this is clearly not the case and Lemma 2 is proved.

5. Construction of a set of normal numbers. We also have to ensure that the numbers we have constructed are also all normal to every base from R. To do this we will modify our construction by discarding certain of intervals of J_i at each stage, to obtain a new sequence, $K_1 \supset K_2 \supset \cdots$, with $K_i \subset J_i$.

Consider the intervals I'_{m-1}. In each such interval there are $(s(m) - \delta(m) + 1)^{t_m}$ intervals of J_m whose left end points we denote by ξ_m.

Let

$$A_m(x) = \sum_{\substack{t=-m \\ t \neq 0}}^{m} \sum_{i=1}^{m} \left| \sum_{j=\langle m; r_i\rangle+1}^{\langle m+1; r_i\rangle} e(r_i^j t x) \right|^2 ,$$

where $e(x)$ denotes $e^{2\pi i x}$.

LEMMA 3. *If $m \geqq \delta_1$ there are at least $(s(m) - 3)^{t_m}$ numbers $\xi_m \in I'_{m-1}$ for which*

$$A_m(\xi_m) \leqq \delta_2 m^2 (\langle m+1\rangle - \langle m\rangle)^{2-\beta m/2} .$$

Here δ_1 and δ_2 are absolute constants.

Proof. Now

$$\sum_{\xi_m \in I'_{m-1}} A_m(\xi_m) = \sum_{\substack{t=-m \\ t \neq 0}}^{m} \sum_{i=1}^{m} \sum_{\xi_m \in I'_{m-1}} \left| \sum_{j=\langle m, r_i\rangle+1}^{\langle m+1; r_i\rangle} e(r_i^j t \xi_m) \right|^2$$

and the inner sum,

$$\sum_{\xi_m \in I'_{m-1}} = \sum_{\xi_m} \sum_{j=\langle m; r_i\rangle+1}^{\langle m+1; r_i\rangle} \sum_{g=\langle m; r_i\rangle+1}^{\langle m+1; r_i\rangle} e((r_i^j - r_i^g) t \xi_m)$$
$$= \sum_j \sum_g \sum_{\xi_m} e((r_i^j - r_i^g) t \xi_m) .$$

Thus

$$\left|\sum_{\xi_m \in I'_{m-1}}\right|$$
$$\leqq \sum_j \sum_g \prod_{k=a_m+1}^{b_m} \left| 1 + e\left(\frac{t(r_i^j - r_i^g)}{s(m)^k}\right) + \cdots + e\left(\frac{t(r_i^j - r_i^g)(s(m) - \delta(m))}{s(m)^k}\right) \right| .$$

Thus

$$|\sum_{\xi_m \in I'_{m-1}} A_m(\xi_m)| \leqq \sum_t \sum_i \sum_j \sum_g \prod_{k=a_m+1}^{b_m} |1 + \cdots| . \tag{12}$$

We write $B_m(x)$ for that part of $A_m(x)$ for which either $|j - g| < m$ or g is at least $\langle m + 1; r_i \rangle - m$ and we write $C_m(x)$ for the remaining part.

Then

$$A_m(x) = B_m(x) + C_m(x) . \tag{13}$$

We have the following trivial estimate.

$$\begin{aligned} B_m(x) &\leqq 10m^2 \sum_{i=1}^{m} (\langle m + 1; r_i \rangle - \langle m; r_i \rangle) \\ &\leqq \delta_3 m^3(\langle m + 1 \rangle - \langle m \rangle) \\ &\leqq \delta_4 m^2(\langle m + 1 \rangle - \langle m \rangle)^{2-\beta_m} . \end{aligned}$$

Thus

$$\sum_{\xi_m} B_m(\xi_m) \leqq \delta_4 m^2(\langle m + 1 \rangle - \langle m \rangle)^{2-\beta_m}(s(m) - \delta(m) + 1)^{t_m} .$$

Here the δ_i are absolute constants.

We now estimate $\sum_{\xi_m} C_m(\xi_m)$.

That part of the sum (12) corresponding to $C_m(\xi_m)$ is at most

$$2 \sum_t \sum_i \sum_{g=\langle m; r_i \rangle + 1}^{\langle m+1; r_i \rangle} \sum_{j=g+m}^{\langle m+1; r_i \rangle - m} \prod_k \left| \sum_{l=0}^{s(m)-\delta(m)} (e(ltr_i^g(r_i^{j-g} - 1)s(m)^{-k})) \right| ,$$

since $|\sum_x e(x)| = |\sum_x e(-x)|$. By making a change of variable we obtain

$$|\sum_{\xi_m} C_m(\xi_m)| \leqq 2 \sum_{\substack{t=-m \\ t \neq 0}}^{m} \sum_{i=1}^{m} \sum_{g=m}^{\alpha_m} \sum_{j=1}^{\alpha_m - g} \prod_{k=a_m+1}^{b_m} |D(m, t, i, g, j, k)| , \tag{14}$$

where

$$\alpha_m = \langle m + 1; r_i \rangle - \langle m; r_i \rangle - m$$

and

$$|D| = \left| \sum_{l=0}^{s(m)-\delta(m)} e(t(r_i^g - 1)r_i^{\langle m, r_i \rangle} r_i^j l s(m)^{-k}) \right|$$

$$\leqq \frac{1}{2}(s(m) - \delta(m) + 1)|1 + e(t(r_i^g - 1)r_i^{\langle m, r_i \rangle} r^j s(m)^{-k})|$$
$$= (s(m) - \delta(m) + 1)|\cos(\pi L_i r_i^j s(m))^{-k}|$$

where $L_i = (r_i^g - 1)r_i^{\langle m; r_i \rangle} t$.

Fix $L = L_i$, t, $r = r_i$, $s = s(m)$, $\delta = \delta(s)$ and g. Then the inner sum in (14) is

$$\leqq \sum_{j=1}^{\langle m+1, r\rangle - \langle m, r\rangle - m - g} \prod_{k=a_m+1}^{b_m} |\cos(\pi L r^j s^{-k})| . \tag{15}$$

Now

$$\begin{aligned} L r^j s^{-b_m} &\leqq r^{\langle m+1; r\rangle - \langle m; r\rangle - m - g} m r^{\langle m, r\rangle} r^g s^{-b_m} \\ &= r^{\langle m+1; r\rangle} r^{-m} m s^{-\langle m+1; s\rangle} \\ &\leqq r^{\langle m+1\rangle/\log r} r^{1-m} m s^{-\langle m+1\rangle/\log s} \\ &= m r^{1-m} \leqq 1/2 \quad \text{(provided } m > 1,\ r \geqq 4) . \end{aligned}$$

Thus

$$\prod_{k=b_k+1}^{\infty} |\cos(\pi L r^j s^{-k})| \geqq \prod_{k=1}^{\infty} |\cos(\pi/2^{k+1})| = \delta_5 > 0 .$$

The sum (15) is at most equal to

$$\delta_6 \sum_{j=1}^{\langle m+1; r\rangle - \langle m; r\rangle - m - g} \prod_{k=a_m+1}^{\infty} |\cos(\pi L r^j / s^k)| .$$

Now

$$\begin{aligned} |L| &\geqq (r^m - 1) r^{\langle m; r\rangle} \geqq (r^m - 1) e^{\langle m\rangle} \\ &\geqq (r^m - 1) s(m)^{a_m} s(m)^{-\log\log m - 3} \qquad \text{by (6)} \\ &\geqq s(m)^{a_m+1} \end{aligned}$$

provided

$$r^m \geqq s(m)^{\log\log m + 4} + 1 ,$$

which holds for m sufficiently large, by (2). Hence from $m \geqq \delta_4$ we may apply Lemma 1 and see that (15) is at most

$$2\delta_6(\langle m+1; r\rangle - \langle m; r\rangle)^{1-\alpha(r,s)} .$$

Thus we have

$$\Big|\sum_{\xi_m \in I'_{m-1}} C_m(\xi_m)\Big| \leqq \delta_7 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m}(s - \delta + 1)^{t_m} .$$

Combining this with the estimate for $|\sum B_m(\xi_m)|$ we have

$$\Big|\sum_{\xi_m \in I'_{m-1}} A_m(\xi_m)\Big| \leqq \delta_2 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m}(s - \delta + 1) .$$

Hence the number of $\xi_m \in I'_{m-1}$ for which

$$A_m(\xi_m) > \delta_2 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2}$$

is at most

$$(\langle m+1\rangle - \langle m\rangle)^{-\beta_m/2}(s-\delta+1)^{t_m} .$$

But

$$\beta_m \geqq \beta_1 m^{-1/4} \quad \text{and} \quad (\langle m+1\rangle - \langle m\rangle) \geqq \frac{e^{\sqrt{m}}}{2\sqrt{m+1}}$$

and so

$$\begin{aligned}(\langle m+1\rangle - \langle m\rangle)^{-\beta_m/2} &\leqq \left(\frac{2\sqrt{m+1}}{e^{\sqrt{m}}}\right)^{\beta_1 m^{-1/4}/2} \\ &= [(2\sqrt{m+1})^{m^{-1/4}} e^{m^{1/4}}]^{\beta_1/2} \\ &< 1/2 \quad \text{for} \quad m > \delta_4 .\end{aligned}$$

Hence there are at least $\frac{1}{2}(s-\delta+1)^{t_m}$ numbers $\xi_m \in I'_{m-1}$ for which

$$A_m(\xi_m) \leqq \delta_2 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2} .$$

For $m \geqq \delta_1$ $(s-3)^{t_m} < \frac{1}{2}(s-\delta+1)^{t_m}$ and the proof of Lemma 3 is complete.

We construct a sequence of sets $K_1 \supset K_2 \supset \cdots$ in the same way as $J_1 \supset J_2 \supset \cdots$ was constructed. But at each stage in our construction of $\{K_m\}$ we use only the $(s(m)-3)^{t_m}$ points ξ_m satisfying Lemma 3.

LEMMA 4. *If* $\xi \in \bigcap_{m=1}^{\infty} K_m$ *then*

$$A_m(\xi) \leqq \delta_8 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2} .$$

Proof. Clearly

$$\begin{aligned}&A_m(\xi) = A_m(\xi) - A_m(\xi_m) + A_m(\xi_m) \quad \text{and} \\ &A_m(\xi) - A_m(\xi_m) = C_m(\xi) - C_m(\xi_m) + B_m(\xi) - B_m(\xi_m) .\end{aligned}$$

We estimate $B_m(\xi) - B_m(\xi_m)$ as we did for $B_m(x)$ above.

Put $L_g = (r^g - 1)r^{\langle m+1;r\rangle - m - g}t(\xi - \xi_m)$. Then $|L_g| \leqq 1/2$ for $m \geqq \delta_1$. The part of the expression for $|C_m(\xi) - C_m(\xi_m)|$ for which t and $r = r_i$ remain fixed is at most equal to

$$2\sum_{g=1}^{\alpha_m}\sum_{j=1}^{\alpha_m - g} |e(L_g r^{-j}) - 1|$$

$$\leqq 2 \sum_{g=1}^{\langle m+1;r\rangle-\langle m;r\rangle-m} \sum_{j=1}^{\infty} r^{-j}$$
$$< 2(\langle m+1; r\rangle - \langle m; r\rangle) .$$

Thus

$$|C_m(\xi) - C_m(\xi_m)| \leqq \delta_9 m^3(\langle m+1\rangle - \langle m\rangle)$$
$$\leqq \delta_{10} m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2} .$$

Thus

$$|A_m(\xi) - A_m(\xi_m)| \leqq \delta_{11} m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2}$$

and so, combining this with Lemma 3,

$$A_m(\xi) \leqq \delta_8 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2} .$$

We now apply the following lemma, Hilfsatz 8, of Schmidt [3] to show that ξ is normal to every base from R.

LEMMA 5. *If* $A_m(\xi) \leqq \delta_8 m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m/2}$ *for* $m \geqq \delta_4$ *then* ξ *is normal to each base* $r_1, r_2, \cdots$.

Thus if $K = \bigcap_{m=1}^{\infty} K_m$, then K is a set of numbers normal to every base from R and to no base from S. It remains to estimate the Hausdorff dimension of K.

6. Estimation of the Hausdorff dimension of K. K_m is a linear set consisting of

$$N_m = \prod_{k=1}^{m} (s(k) - 3)^{b_k - a_k}\left(\left[\frac{s(k)^{a_k}}{s(k-1)^{b_{k-1}}}\right] - 2\right)$$

intervals of length $s(m)^{-b_m} = \delta_m$.

Hence

$$N_m > \prod_{k=1}^{m} (s(k) - 3)^{b_k - a_k} .$$

Now

$$(s-3)^n = s^{(\log(s-3)/\log s)\cdot n} \geqq s^{(\log(A-3)/\log A)\cdot n} , \qquad (\text{if } s \geqq A) ,$$
$$= e^{n(\log(A-3)/\log A)\cdot \log s} .$$

Thus

$$N_m > \exp\left[\frac{\log(A-3)}{\log A} \sum_{k=1}^{m} (b_k - a_k) \log s(k)\right]$$

$$\geqq \exp\left[\frac{\log(A-3)}{\log A}\sum_{k=1}^{m}\langle k+1\rangle - \langle k\rangle - (\log s(k))(\log\log k + 3)\right]$$
$$\geqq \exp\left[\frac{\log(A-3)}{\log A}\langle m+1\rangle(1+O(1))\right].$$

We also have

$$\frac{\delta_{m-1}}{\delta_m} = \frac{s(m)^{b_m}}{s(m-1)^{b_{m-1}}} \leqq s(m)e^{\langle m+1\rangle - \langle m\rangle} \leqq \exp\left(\frac{\langle m+1\rangle}{\sqrt{m}} + \log s(m)\right)$$

and

$$\delta_m^t = s(m)^{-b_m t} \geqq \exp(-t\langle m+1\rangle).$$

Thus

$$\sum_m \frac{\delta_{m-1}}{\delta_m}(N_m\delta_m^t)^{-1}$$
$$\leqq \sum_m \exp\left[\frac{\langle m+1\rangle}{\sqrt{m}} + \log s(m) - \frac{\log(A-3)}{\log A}\langle m+1\rangle(1+O(1)) + t\langle m+1\rangle\right]$$
$$= \sum_m \exp\left[\langle m+1\rangle\left(t - \frac{\log(A-3)}{\log A}\right)(1+O(1))\right].$$

This sum will certainly converge for all $t < \log(A-3)/\log A$.

We apply the following theorem of Eggleston, [1], to estimate the Hausdorff dimension of K.

THEOREM. *Suppose* K_k $(k = 1, 2, \cdots)$ *is a linear set consisting of* N_k *closed intervals each of length* δ_k. *Let each interval of* K_k *contain* $m_{k+1} > 0$ *disjoint intervals of* K_{k+1}.

Suppose that $0 < s_0 \leqq 1$ *and that for all* $s < s_0$ *the sum*

$$\sum_k \frac{\delta_{k-1}}{\delta_k}(N_k(\delta_k)^s)^{-1}$$

converges. Then $K = \bigcap_{k=1}^{\infty} K_k$ *has dimension greater than or equal to* s_0.

Clearly all the conditions necessary to apply Eggleston's theorem are satisfied where we may take $s_0 = \log(A-3)/\log A$. This proves Theorem 2.

The author would like to thank the referee for his careful reading of the original manuscript and for his many helpful suggestions for improving the presentation of this paper.

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Received November 30, 1979 and in revised form May 14, 1980.

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