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BROWNIAN MOTION AND SETS OF HARMONIC MEASURE ZERO

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Using Brownian motion the following results are established:

(1) Harmonic measure and Keldysh measure are always singular with respect to area measure in the plane. More generally, this holds for the distribution of the first exit point for Brownian motion of a given Borel set.

(2) If U is open and $K \subset \partial U$ is compact, then K has harmonic measure 0 w.r.t. U if ∂U satisfies a certain metric density condition at each point of K and, in addition, K satisfies one of the following two conditions:

(i) K has zero length and is lying on a straight line or

(ii) K has α-dimensional Hausdorff measure zero, for some $\alpha < 1/2$.

1. **Introduction.** Let U be a connected open set in the complex plane C whose complement has positive logarithmic capacity. If $a \in U$ we let λ_a^U denote the harmonic measure at a with respect to U. What are the metric properties of λ_a? In particular, what can be said about sets of harmonic measure zero?

In this paper we use the Brownian motion characterization of harmonic measure to give some answers to these questions. If $b_\omega^a(t)$ denotes the two-dimensional Brownian motion starting at a (i.e., $b_\omega^a(0) = a$), let $T_U = T_U^a(\omega) = \inf\{t > 0; b_\omega^a(t) \notin U\}$ be the first exit time for b_ω^a in U. Then for Borel sets $G \subset \partial U$, the topological boundary of U, we have

$$\lambda_a(G) = P^a(b_\omega(T_U) \in G) ,$$

where P^a is the probability measure of the Brownian motion starting at a. (See for example [10], p. 264.) In other words, $\lambda_a(G)$ is the probability that $b_\omega^a(t)$ hits G before it hits any other part of ∂U.

In [16] (Corollary 1.5) it is proved that harmonic measure is always singular with respect to area measure, using methods based on analytic capacity and function algebras. In § 2 we prove a result which implies this, using Brownian motion. The same proof applies to the hitting distribution of $b_\omega^a(t)$ on any Borel measurable set, in particular to the Keldysh measure.

If U is a Jordan domain with rectifiable boundary, a classic theorem due to F. and M. Riesz (see [4], Theorem 3.3) states that λ_a is equivalent to arc length on ∂U. However, for non-rectifiable boundaries it is not true in general that harmonic measure is equivalet to 1-dimensional Hausdorff measure on ∂U, even if U is simply connected. Lavrentiev [12] was the first to give an example of a Jordan domain U with a subset E of ∂U of zero length and $\lambda_a(E) > 0$. A simpler example can be found in McMillan and Piranian [15]. And Lohwater and Seidel [13] constructed a Jordan domain whose boundary meets a line segment in a set of positive length and harmonic measure zero with respect to the domain. In § 3 it is proved that if $K \subset \partial U$ is a compact set of zero length and K is lying on a straight line, then $\lambda_a(K) = 0$, provided ∂U satisfies a certain density condition at each point of K. (This density condition is trivially satisfied if U is simply connected, for example.)

In § 4 we consider the general case when K is a compact subset of ∂U, not necessarily linear. For the case when U is simply connected, Carleson [3] has proved that there exists a constant $\beta > 1/2$ (which does not depend on U) such that λ_a is absolutely continuous with respect to β-dimensional Hausdorff measure on ∂U. For general sets U we prove that if K has r-dimensional Hausdorff measure zero for some $r < 1/2$, then $\lambda_a(K) = 0$, provided ∂U satisfies a density condition at each point of K.

It seems clear that all—or almost all—the arguments involving Brownian motion in this paper can be translated into the language of classical potential theory. Our main reason for preferring the Brownian motion version is that it brings more intuition into the subject, which again makes it easier to find the necessary arguments.

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2. First exit distribution is singular with respect to area. We introduce the following notation:

If $x \in \partial U$, $r > 0$ let $L(x, r) = m_1(U \cap \{|z - x| = r\})$ and $A(x, r) = m_2(U \cap \Delta(x, r))$, where m_1, m_2 denote 1- and 2-dimensional Lebesgue measure, respectively. Here and later $\Delta(x, r)$ denotes the open disc $\{|z - x| < r\}$.

THEOREM 1. *Let U be an open set, $a \in U$. There exists $\varepsilon > 0$ (independent of U and a) such that if we define*

$$E = \left\{x \in \partial U; \limsup_{n\to\infty} \left(\inf_{2^{-n-1} \leq r \leq 2^{-n}} \frac{L(x, r)}{2\pi r} \right) < \varepsilon \right\}$$

then $\lambda_a(E) = 0$.

In particular, λ_a is singular with respect to area measure.

Note. The last statement follows from the preceding since clearly $\{x \in \partial U; \lim_{r\to 0} A(x, r)/\pi r^2 = 0\} \subset E$ for all $\varepsilon > 0$, so that all the points of density of ∂U w.r.t. m_2 is included in E for all $\varepsilon > 0$.

Proof of Theorem 1. Choose $\varepsilon > 0$, to be determined later. For $N = 1, 2, \cdots$ let

$$E_N = \left\{x \in \partial U; \inf_{2^{-n-1} \leqq r \leqq 2^{-n}} \frac{L(x, r)}{2\pi r} < \varepsilon \text{ for all } n \geqq N\right\} .$$

Then $E = \bigcup_{N=1}^{\infty} E_N$, so it suffices to prove that $\lambda_a(E_N) = 0$. Fix $x \in E_N$.

For each $n \geqq N$ choose a circle $\Gamma_n = \{|z - x| = r_n\}$, s.t. $2^{-n-1} \leqq r_n \leqq 2^{-n}$ and

$$\frac{L(x, r_n)}{2\pi r_n} < \varepsilon .$$

Choose $0 < \rho < 2^{-N-1}$ and put $\Delta = \Delta(x, \rho)$. Let

$$C_k = \Gamma_{N+2k}$$
$$T_k = \inf\{t; b_\omega^a(t) \in U \backslash \overline{\Delta(x, r_{N+2k})}\}$$

and

$$T = \inf\{t; b_\omega^a(t) \notin U\} , \quad k \geqq 0 .$$

Then

$$T_k \leqq T_{k+1} \leqq T \quad \text{for all} \quad k \geqq 0 .$$

Using conditional expectation we get

$$(1) \qquad P^a(b_\omega(T) \in \bar{\Delta}) = \int_{C_0 \cap U} P^a(b_\omega(T) \in \bar{\Delta} \,|\, b_\omega(T_0) = x) d\mu_0(x) ,$$

where μ_0 is the distribution of $b_\omega(T_0)$ on $C_0 \cap U$ ($\mu_0(H) = P^a(b_\omega(T_0) \in H)$ for Borel sets $H \subset C_0 \cap U$). By the strong Markov property,

$$(2) \qquad P^a(b_\omega(T) \in \bar{\Delta} \,|\, b_\omega(T_0) = x) = P^x(b_\omega(T) \in \bar{\Delta}) .$$

So

$$(3) \qquad P^a(b_\omega(T) \in \bar{\Delta}) = \int_{C_0 \cap U} P^x(b_\omega(T) \in \bar{\Delta}) d\mu_0(x) .$$

Repeating the argument (1)-(3) on the integrand, we obtain

$$P^a(b_\omega(T) \in \bar{\Delta}) = \int_{C_0 \cap U} \left(\int_{C_1 \cap U} P^{x_1}(b_{(T)} \in \bar{\Delta}) d\mu_1(x_1)\right) d\mu_0(x) ,$$

where μ_1 is the distribution of $b_\omega(T_1)$ on $C_1 \cap U$. Repeating this k times, where $\rho < 2^{-N-2k-1}$, we have

$$(4) \quad P^a(b_\omega(T) \in \bar{\Delta}) = \int_{C_0 \cap U} \left(\cdots \left(\int_{C_k \cap U} P^{x_k}(b(T) \in \bar{\Delta}) d\mu_k(x_k)\right) \cdots \right) d\mu_0(x) ,$$

where μ_j is the distribution of $b_\omega(T_j)$ on $C_j \cap U$; $1 \leqq j \leqq k$. Since $C_j \subset \{z; 2^{-N-2j-1} \leqq |z - x| \leqq 2^{-N-2j}\}$, the ratio of the radii of C_{j+1} and C_j is at most 1/2. Therefore there exists a universal constant M such that

$$(5) \qquad P^{x_{j-1}}(b(T_j) \in C_j \cap U) \leqq M \frac{m_1(C_j \cap U)}{2\pi r_{N+2j}} < M\varepsilon\ ; \quad 1 \leqq j \leqq k\ .$$

This gives

$$(6) \qquad P^a(b(T) \in \bar{\varDelta}) \leqq (M\varepsilon)^k \quad \text{for} \quad \rho < 2^{-N-2k-1}\ .$$

If we choose k so large that $2^{-N-2k-3} \leqq \rho$ we have

$$(7) \qquad k \geqq \frac{1}{2}\left(\log\left(\frac{1}{\rho}\right) - N - 3\right),$$

where the log is taken with base 2.

Combining (6) and (7) we get

$$(8) \qquad P^a(b(T) \in \bar{\varDelta}) \leqq (M\varepsilon)^{(-N-3)/2} \cdot \rho^{(1/2)\log(1/M\varepsilon)}\ .$$

Now choose ε so small that

$$(9) \qquad \frac{1}{2}\log\left(\frac{1}{M\varepsilon}\right) \geqq 3\ .$$

Then

$$(10) \qquad P^a(b(T) \in \bar{\varDelta}) \leqq M_1\rho^3\ ,$$

where M_1 does not depend on ρ or x.

To complete the proof, choose $\eta > 0$ arbitrary, cover E_N by discs $\varDelta(x_1, \rho_1), \cdots, \varDelta(x_n, \rho_n)$ with $\rho_k < 2^{-N-1}$ and

$$\sum_{k=1}^{n} \rho_k^3 < \eta\ .$$

Then by (10)

$$P^a(b_\omega(T) \in E_N) \leqq \sum_{k=1}^{n} M_1\rho_k^3 < M_1\eta\ .$$

Since η was arbitrary the proof is complete.

Let $E \subseteqq C$ be Borel measurable with $\operatorname{cap}(C\backslash E) > 0$, where cap denotes logarithmic capacity. For a fixed $a \in \bar{E}$ we define the first exit time of E

$$T_E = \inf\{t > 0; b_\omega^a \notin E\}\ ,$$

and the "first exit distribution"

$$\mu_a^E(G) = P^a(b(T_E) \in G)\ , \quad G \text{ Borel measurable.}$$

μ_a^E is a probability measure supported on ∂E.

If E is open, μ_a^E coincides with harmonic measure λ_a^E. If E is compact, μ_a^E coincides with the *Keldysh measure* for a with respect to E. This is proved in [5].

For more information about Keldysh measures, see also [7] and [8]. The proof of Theorem 1 also applies to the Keldysh measure. More generally, the proof gives:

COROLLARY 1. *The first exit distribution μ_a^E is singular with respect to area, for any Borel measurable set E.*

It seems reasonable to conjecture that μ_a^E is singular with respect to α-dimensional Hausdorff measure, for any $\alpha > 1$. (See § 4 for definition of Hausdorff measure.)

3. Linear zero sets. In this section we consider linear sets, i.e., sets lying on straight lines. If K is a compact, linear set of zero length, it need not have harmonic measure zero in general, but the next result shows that the harmonic measure of such a set is zero if ∂U satisfies a density condition at each point of the compact.

The *circular projection* of a plane set E about a point x_0 is defined as follows:

$$E^*(x_0) = \{|z - x_0|; z \in E\} .$$

THEOREM 2. *Let K be a compact subset of ∂U, assume that K is lying on a straight line segment γ and has zero length. Then if*

$$(*) \qquad \liminf_{t \to 0} \frac{m_1((\partial U)^*(x) \cap [0, t])}{t} > 0 \quad \textit{for all} \quad x \in K ,$$

K has harmonic mearsure zero with respect to U.

An immediate consequence is

COROLLARY 2. *Assume U is simply connected and $K \subset \partial U$ is a compact, linear set of zero length. Then K has harmonic measure zero with respect to U.*

Therefore the examples of Lavrentiev and MacMillan/Piranian mentioned in the introduction, must be nonlinear sets.

REMARKS. (i) If U is simply connected, a shorter and more direct proof can be given. See [17].

(ii) We conjecture that Theorem 2 holds for all rectifiable arcs γ. This would constitute a nice generalization of (one half of) the

F. and M. Riesz theorem stated in the introduction.

Before we give the proof of Theorem 2 let us illustrate the result by an example.

EXAMPLE 1. Consider the following linear Cantor sets: Let $p_1, p_2, \cdots$ be numbers greater than 1. Start with the interval $C_0 = [0, 1]$. The first step is to remove the middle interval of length $1 - 1/p_1$. The remaining part C_1 consists of 2 intervals, each of length $1/2p_1$. In step 2 we remove from each of these 2 intervals the middle interval of length $(1 - 1/p_2)(1/2p_2)$. After n steps we are left with a set C_n consisting of 2^n intervals, each of length $2^{-n} \prod_{k=1}^{n} p_k^{-1}$. Put

$$C = \bigcap_{n=1}^{\infty} C_n .$$

Then C has positive length iff $\sum_{n=1}^{\infty} \log p_n < \infty$.

Therefore we see that if we let X be such a linear Cantor set of positive length and put $U = C\backslash X$, then the density condition (*) in Theorem 2 is satisfied at each point of X. In fact, the density defined in (*) is equal to 1 for all $x \in X$. We conclude that harmonic measure is absolutely continuous with respect to 1-dimensional Lebesgue measure on X in this case.

Proof of Theorem 2. We may assume $K \subset [0, 1]$ and U bounded. Fix $a \in U$. For $n = 2, 3, \cdots$ let

$$K_n = \left\{x \in K; \liminf_{t \to 0} \frac{m_1((\partial U)^*(x) \cap [0, t])}{t} > \frac{1}{n}\right\} .$$

Then $K = \bigcup_{n=1}^{\infty} K_n$, so it is enough to prove the result when there exists $\eta > 0$ such that

$$\liminf_{t \to 0} \frac{m_1((\partial U)^*(x) \cap [0, t])}{t} > \eta \quad \text{for all} \quad x \in K .$$

Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive numbers decreasing to zero. Put

$$f_n(x) = \frac{m_1((\partial U)^*(x) \cap [0, \delta_n])}{\delta_n} ; \quad x \in K ,$$

$G_n = \{x \in K; f_n(x) < \eta\}$ and $E_N = \bigcup_{n \geq N} G_n$; $N = 1, 2, \cdots$. Then $\bigcap_{N=1}^{\infty} E_N = \emptyset$, so $\lambda_a(E_N) \to 0$ as $N \to \infty$.

Therefore, if $\varepsilon > 0$ is given, there exists N such that if we put

$$H_N = \{x \in K; f_n(x) \geqq \eta \text{ for } n \geqq N\}$$

then we have

$$\lambda_a(K\backslash H_N) < \varepsilon .$$

We conclude that it is enough to prove the result for the case when

$$(1) \qquad \frac{m_1((\partial U)^*(x) \cap [0, \delta_n])}{\delta_n} \geqq \eta \quad \text{for} \quad x \in K\,, \quad n \geqq N\,.$$

We will choose $\delta_n = (\eta/2)^n$; $n = 1, 2, \cdots$.

Let $U_1 = U \cap \{z; \operatorname{Im} z > 0\}$. We may assume $a \in U_1$.

Let $T_1 = T_{U_1}$ and $T = T_U$ be the first exit times for $b(t)$ of U_1 and U respectively. Then clearly $T_1 \leqq T$ and since harmonic measure for the half-plane is absolutely continuous, we have $\lambda_a^{U_1}(K) = 0$, and therefore

$$(2) \qquad P^a(b(T) \in K) = \int_J P^x(b(T) \in K) d\mu_a(x)\,,$$

where $J = U \cap \boldsymbol{R}$ and μ_a is the distribution of $b(T_1)$.

Let $I_1, I_2, \cdots$ be the complementary intervals of $K \cap \boldsymbol{R}$ in $[-R, R]$, where R is chosen so large that $\bar{U} \subset \Delta(0, R)$. Set

$$(3) \qquad V_k = U \backslash (J \backslash I_k)\,,$$

and let T_k be the first exit time for $b(t)$ of V_k, $k = 1, 2, \cdots$. Then we claim that there exists a constant $c > 0$ independent of x and k such that

$$(4) \qquad P^x(b(T_k) \in \partial U) \geqq c \quad \text{for all} \quad x \in I_k\,, \quad k = 1, 2, \cdots\,.$$

Let us complete the proof under the assumption that (4) holds. By (2) and the F. and M. Riesz theorem we get

$$(5) \qquad P^a(b(T) \in K) = \int_J \left(\int_J P^{x_2}(b(T) \in K) d\nu_{x_1}(x_2) \right) d\mu_a(x_1)$$

where ν_x is the distribution of $b(T_k)$, when $x \in I_k$. Repeating this n times we get

$$(6) \qquad P^a(b(T) \in K) = \int_J \left(\int_J \left(\cdots \left(\int_J P^{x_n}(b(T) \in K) d\nu_{x_{n-1}}(x_n) \right) \cdots \right) \right) d\mu_a(x_1)\,.$$

By (4) we have

$$(7) \qquad \nu_{x_i}(J) \leqq 1 - c\,,$$

so by (6)

$$(8) \qquad P^a(b(T) \in K) \leqq (1 - c)^n\,,$$

and since n is arbitrary, the result follows.

It remains to prove the claim (4):

Let $W_k = C \backslash (\boldsymbol{R} \backslash I_k)$, $I_k = (x_k', x_k)$, $d_k = x_k - x_k'$. Put $\Delta = \Delta(x_k, d_k)$ and assume $x \in I_k$.

Let τ_k be the first exit time of $U \cap \Delta \backslash [x_k, \infty)$ and let σ_k be the first exit time of $W_k = \Delta \backslash [x_k, \infty)$. Then by the Hall projection theorem (see [6])

$$(9) \qquad P^x(b(T_k) \in \partial U) \geqq P^x(b(\tau_k) \in \partial U) \geqq c \cdot P^x(b(\sigma_k) \in F) ,$$

where $F = (\partial U)^*(x_k)$.

Put $A_n = \{z; \delta_{n+1} \leqq |z - x_k| \leqq \delta_n\}$, where $\delta_n = (\eta/2)^n$ as above.

We can write $d\lambda_x^{W_k}(t) = g_x(t)dt$ for $t \in [x_k, x_k + d_k]$, where $g_x(t) > 0$ and decreasing.

Let n_k be the smallest integer satisfying $n_k \geqq N$ and $\delta_{n_k} \leqq d_k$. Then we have, using (1):

$$\begin{aligned}(10) \qquad & P^x(b(\sigma_k) \in F) \\ &= \sum_{n=n_k}^{\infty} P^x(b(\sigma_k) \in F \cap A_n) \geqq \sum_{n=n_k}^{\infty} g_x(x_k + \delta_n) \cdot m_1(F \cap A_n) \\ &\geqq \sum_{n=n_k}^{\infty} g_x(x_k + \delta_n) \cdot \left(\frac{\eta}{2}\right)^{n+1} = \left(\frac{\eta}{2}\right)^2 \cdot \sum_{n_k-1}^{\infty} \left(\frac{\eta}{2}\right)^n g_x(x_k + \delta_{n+1}) \\ &\geqq \left(\frac{\eta}{2}\right)^2 \cdot \int_{x_k}^{x_k+\delta_{n_k-1}} g_x(t)\,dt \geqq \left(\frac{\eta}{2}\right)^2 \cdot P^x(b(\sigma_k) \in [x_k, x_k + \delta_{n_k-1}]) .\end{aligned}$$

We assert that

$$(11) \qquad P^x(b(\sigma_k) \in [x_k, x_k + \delta_{n_k-1}]) \text{ is bounded away from 0 for } x \in J .$$

To see this consider the two possible cases:

(i) $\delta_N \leqq d_k$: Then the assertion follows from the fact that U is bounded.

(ii) $\delta_N > d_k$: Then by minimality of n_k we have $d_k \leqq \delta_{n_k-1}$ and (11) follows.

We now combine (9), (10) and (11) and obtain the claim (4).

That completes the proof of Theorem 2.

4. Connection with Hausdorff measures. Let $h(t)$ be a continuous increasing function on $[0, \infty)$ such that $h(0) = 0$. Let E be a bounded, plane set. For $\delta > 0$ we consider all coverings of E with a countable number of discs Δ_j with radii $\rho_j \leqq \delta$, and define

$$\Lambda_h^\delta(E) = \inf \left\{\sum_j h(\rho_j)\right\} ,$$

the inf being taken over all such coverings. The limit

$$\Lambda_h(E) = \lim_{\delta \to 0} \Lambda_h^\delta(E)$$

is called the *Hausdorff measure* of E with respect to the measure function h. If $h(t) = t^\alpha$, for some $\alpha > 0$, Λ_h is called α-dimensional Hausdorff measure and denoted by Λ_α. For measurable subsets of rectifiable arcs Λ_1 is equivalent to arc length. See [2] and [9] for more information about Hausdorff measures.

For a general set U, λ_a need not be absolutely continuous with respect to Λ_α, for any $\alpha > 0$. However, in this section we prove that if ∂U satisfies a density condition at each point of a compact set $K \subset \partial U$, then $\lambda_a(K) = 0$ provided $\Lambda_\alpha(K) = 0$ for some $\alpha < 1/2$.

If the density condition is weakened, a similar connection can be established, but with lower values of α.

It is not clear to what extent these upper bounds for α can be improved.

We will need the following well known result (see for example [11], p. 366-367 for an explicit calculation).

LEMMA 1. *Let q be a point on the y-axis and put $V = \boldsymbol{C}\backslash\boldsymbol{R}$. Then for Borel subsets $G \subset \boldsymbol{R}$*

$$P^q(b(T_V) \in G) = \int_G \frac{dx}{\pi|b|(1 + (x/|b|)^2)} .$$

LEMMA 2. *Let a be a point on the x-axis. Put $V = \boldsymbol{C}\backslash i\boldsymbol{R}$, $W = \boldsymbol{C}\backslash i\boldsymbol{R}\backslash B$, where $B = \Delta(0, \rho)$. Then, if $|a| > \rho > 0$.*

$$P^a(b(T_W) \in B \cup (-2\rho i, 2\rho i)) \leqq 2\cdot P^a(b(T_V) \in (-2\rho i, 2\rho i)) .$$

Proof. Let $c > 1$ be a positive constant. Then

$$(1)\quad \begin{aligned} &P^a(b(T_W) \in B \cup (-c\rho i, c\rho i)) \\ &\quad = P^a(b(T_V) \in (-c\rho i, c\rho i), b(T_W) \in B \cup (-c\rho i, c\rho i)) \\ &\qquad + P^a(|b(T_V)| > c\rho, b(T_W) \in B \cup (-c\rho i, c\rho i)) \\ &\quad \leqq P^a(b(T_V) \in (-c\rho i, c\rho i)) + \int_{\partial B} P^z(|b(T_V)| > c\rho) d\mu(z) , \end{aligned}$$

where μ is the distribution of $b(T_W)$ on ∂B. By Lemma 1

$$(2)\quad \begin{aligned} P^z(|b(T_V)| > c\rho) &\leqq \int_{|y| \geqq (c-1)\rho} \frac{dy}{\pi|x|(1 + (y/|x|)^2)} \\ &\leqq 1 - \frac{2}{\pi} \operatorname{Arctan}(c-1) , \end{aligned}$$

where $x = \operatorname{re} z$, $z \in \partial B$.

Combining (1) and (2) we get

$$(3)\quad \begin{aligned} &P^a(b(T_W) \in B \cup (-c\rho i, c\rho i)) \\ &\quad \leqq \frac{\pi}{2} \frac{1}{\operatorname{Arctan}(c-1)} \cdot P^a(b(T_V) \in (-c\rho i, c\rho i)) . \end{aligned}$$

Therefore we obtain the result by choosing $c = 2$.

LEMMA 3. *Suppose* $y, \delta, \alpha > 0$. *Then*

$$\int_0^\infty \left(\frac{\delta}{x}\right)^\alpha \frac{dx}{\pi y(1 + (x/y)^2)} = \frac{1}{2\cdot\cos(\pi\alpha/2)}\cdot\left(\frac{\delta}{y}\right)^\alpha .$$

Proof. The substitution $u = (x/y)^\alpha$ transforms the integral to

$$\frac{1}{\pi\alpha}\cdot\left(\frac{\delta}{y}\right)^\alpha \int_0^\infty \frac{u^{(1/\alpha)-2}}{1 + 2^{2/\alpha}} du .$$

The value of this integral can be found in tables, and we get Lemma 3.

LEMMA 4. *Let U be open, $z_0 \in \delta U$. Let $0 < \varepsilon < 1/4$ and $\delta_n = 2^{-n}$; $n = 1, 2, \cdots$. Suppose*

$$\frac{m_1((\partial U)^*(z_0) \cap [0, \delta_n])}{\delta_n} \geqq 1 - \varepsilon \quad for\ all \quad n \geqq N .$$

Choose $\alpha > 0$ such that

$$\cos^2\left(\frac{\pi\alpha}{2}\right) > \frac{1}{2} + 2\varepsilon .$$

Then there exists a constant A depending only on ε and N such that

$$P^a(b(T_U) \in \Delta(z_0, \rho)) \leqq A\cdot\left(\frac{\rho}{|a|}\right)^\alpha$$

for all a with $|z_0 - a| \leqq \delta_N/2$; $\rho > 0$.

Proof. We may assume that $z_0 = 0$ and that a is a point on the negative x-axis, $|a| > \rho$. Put $B = \Delta(0, \rho)$, $F = (\partial U)^*(0)$ and let $D = \boldsymbol{C}\backslash F\backslash B$. Then by the Beurling projection theorem (see [1])

$$(1) \qquad P^a(b(T_U) \in B) \leqq P^a(b(T_D) \in B) .$$

Using Lemma 2 and its notation, we get, setting $\delta = 2\rho$,

$$\begin{aligned}(2) \quad & P^a(b(T_D) \in B) \\ & \leqq P^a(b(T_D) \in B, b(T_W) \in B \cup (-\delta i, \delta i)) + P^a(b(T_D) \in B, |b(T_W)| > \delta\} \\ & \leqq 2P^a(b(T_V) \in (-\delta i, \delta i)) + 2\int_\delta^\infty P^a(b(T_D) \in B | b(T_V) = y) d\nu(y) \\ & = 4\cdot\text{Arctan}\left(\frac{\delta}{|a|}\right) + 2\int_\delta^\infty P^y(b(T_D) \in B) d\nu(y)\end{aligned}$$

where $d\nu(y) = dy/(\pi|a|(1 + (y/|a|)^2))$ by Lemma 1.

Repeating this procedure, we get

$$(3)\qquad P^y(b(T_D)\in B)\leqq 4\,\mathrm{Arctan}\left(\frac{\delta}{|y|}\right)+\int_E P^x(b(T_D)\in B)d\mu_y(x)\ ,$$

where $E=(-\infty,\delta)\cup(\delta,\infty)\backslash F$ and μ_y is the distribution of $b^y(T_{C\backslash R})$ on $\boldsymbol{R}$:

$$d\mu_y(x)=\frac{dx}{\pi|y|(1+(x/|y|)^2)}\ .$$

Repeating (1)-(3) n times and combining, we obtain:

$$(4)\quad P^a(b(T_D)\in B)\leqq 4\cdot\mathrm{Arctan}\left(\frac{\delta}{|a|}\right)$$
$$+4\cdot\sum_{k=1}^{n}2^k\int_\delta^\infty\left(\int_E\left(\cdots\left(\int_E\left(\int_\delta^\infty\mathrm{Arctan}\left(\frac{\delta}{|y_k|}\right)d\nu_{x_{k-1}}(y_k)\right)d\mu_{y_{k-1}}(x_{k-1})\right)\cdots\right)\right)d\nu(y_1)$$
$$+4\cdot\sum_{k=1}^{n}2^k\int_\delta^\infty\left(\int_E\left(\cdots\left(\int_\delta^\infty\left(\int_E\mathrm{Arctan}\left(\frac{\delta}{|x_k|}\right)d\mu_{y_k}(x_k)\right)d\nu_{x_{k-1}}(y_k)\right)\cdots\right)\right)d\nu(y_1)$$
$$+4\cdot 2^n\int_\delta^\infty\left(\int_E\left(\cdots\left(\int_E P^{x_n}(b(T_D)\in B)d\mu_{y_n}(x_n)\right)\cdots\right)\right)d\nu(y_1)\ .$$

The last term is less than

$$4\cdot 2^n\cdot 2^{-n}\cdot c^n\ ,\quad\text{where}\quad c=\max_{|y|>\delta}\{\mu_y(E)\}<1\ ,$$

so it will tend to zero as $n\to\infty$.

Let $A_n=\{x\in R;\,2^{-n-1}\leqq x\leqq 2^{-n}\}$. For $n\geqq N$ we have, by hypothesis,

$$(5)\qquad m_1(E\cap A_n)\leqq\varepsilon\cdot m_1(A_n)\ .$$

Therefore, if $f(x)$ is positive and decreasing,

$$(6)\qquad \int_{E\cap[0,\delta_N]}f(x)d\mu_y(x)\leqq\sum_{k=n}^{\infty}f(\delta_{k+1})\frac{m_1(E\cap A_k)}{\pi|y|(1+(\delta_{k+1}/|y|)^2)}$$
$$\leqq 4\varepsilon\cdot\sum_{k=N}^{\infty}f(\delta_{k+1})\cdot\frac{m_1(A_{k+1})}{\pi|y|(1+(\delta_{k+1}/|y|)^2)}$$
$$\leqq 4\varepsilon\cdot\int_0^{\delta_{N+1}}f(x)d\mu_y(x)\ .$$

By (6) and Lemma 3 we get

$$(7)\qquad \int_E\left(\frac{\delta}{|x|}\right)^\alpha\frac{dx}{\pi y(1+(x/y)^2)}\leqq\int_{E\cap[0,\delta_N]}+\int_{R\backslash[0,\delta_N]}\leqq(1+4\varepsilon)\int_0^\infty+\int_{\delta_N}^\infty$$
$$\leqq\frac{1+4\varepsilon}{2\cdot\cos(\pi\alpha/2)}\left(\frac{\delta}{y}\right)^\alpha+\frac{1}{2}\left(\frac{\delta}{\delta_N}\right)^\alpha\ .$$

Since $\operatorname{Arctan}(\delta/|y|) \leqq \delta/|y| \leqq (\delta/|y|)^\alpha$ for $|y| > \delta$, we get by using (7) repeatedly:

$$(8)\quad \int_\delta^\infty\Big(\int_E\Big(\cdots\Big(\int_E\Big(\int_\delta^\infty \operatorname{Arctan}\Big(\frac{\delta}{|y_k|}\Big)d\nu_{x_{k-1}}(y_k)\Big)d\mu_{y_{k-1}}(x_{k-1})\Big)\cdots\Big)\Big)d\nu(y_1)$$
$$\leqq \Big[\frac{1+4\varepsilon}{4\cos^2(\pi\alpha/2)}\Big]^k \cdot \Big(\frac{\delta}{|a|}\Big)^\alpha + \frac{1}{2}\Big(\frac{\delta}{\delta_N}\Big)^\alpha \sum_j \Big[\frac{1+4\varepsilon}{4\cos^2(\pi\alpha/2)}\Big]^j\Big(\frac{c}{2}\Big)^{k-j}.$$

The terms in the other sum in (4) are estimated similarly. Therefore, combining (4) and (8) we get the estimate

$$(9)\quad P^a(b(T_D)\in B) \leqq \Big(\frac{\delta}{|a|}\Big)^\alpha c_1 \cdot \sum_{k=1}^\infty k\Big[\frac{1+4\varepsilon}{2\cos^2(\pi\alpha/2)}\Big]^k,$$

where c_1 is a constant.

By the choice of α this series converges, and Lemma 4 is proved.

We are now ready for the main result of this section:

THEOREM 3. *Let U be an open set, K a compact subset of ∂U such that*

$$(**)\quad \lim_{t\to 0}\frac{m_1((\partial U)^*(x)\cap[0,t])}{t} = 1 \quad \text{for all} \quad x\in K.$$

Then if $\Lambda_\alpha(K)=0$ for some $\alpha<1/2$, K has harmonic measure zero with respect to U.

Proof. Choose $\alpha < 1/2$ such that $\Lambda_\alpha(K)=0$. As in the proof of Theorem 5 we may assume that there exists $N<\infty$ s.t.

$$\frac{m_1((\partial U)^*(x)\cap[0,\delta_n])}{\delta_n} \geqq 1-\varepsilon$$

for all $n \geqq N$, where $\delta_n = 2^{-n}$, and $\varepsilon>0$ is chosen such that

$$\cos^2\Big(\frac{\pi\alpha}{2}\Big) > \frac{1}{2} + 2\varepsilon.$$

Let $D_k = \Delta(z_k, \delta_N/4)$; $k=1,\cdots,M$ be discs centered at K such that

$$K \subset \bigcup_{k=1}^M D_k.$$

Choose $a_k \in \partial D_k \cap U$ for $k=1,\cdots,M$. Choose $\eta>0$. Cover K by discs $\{\Delta(x_j,\rho_j)\}_{j=1}^m$ centered at K with radii $\rho_j<\delta_N$ such that

$$\sum_{j=1}^m \rho_j^\alpha < \eta.$$

Fix $a\in U$. Then there exists a constant C such that

$$\lambda_\alpha^U(K) \leqq C\lambda_{\alpha_k}^U(K) \quad \text{for} \quad 1 \leqq k \leqq M .$$

Then Lemma 4 gives

$$\begin{aligned}\lambda_\alpha^U(K) &\leqq \sum_{k=1}^{M} \lambda_\alpha^U(K \cap D_k) \leqq C \sum_{k=1}^{M} \lambda_{\alpha_k}^U(K \cap D_k) \\ &\leqq C \cdot \sum_{k=1}^{M} \sum_{x_j \in D_k} \lambda_{\alpha_k}^U(\Delta(x_j, \rho_j)) \leqq C \cdot \sum_{k=1}^{M} \sum_{x_j \in D_k} A\rho_j^\alpha \\ &\leqq CAM \sum_{j=1}^{m} \rho_j^\alpha < CAM\eta ,\end{aligned}$$

where A is a constant which does not depend on η. Since η was arbitrary, the proof is complete.

Note that the same argument also gives that if

$$\liminf_{t \to 0} \frac{m_1((\partial U)^*(x) \cap [0, t])}{t} \geqq 1 - \varepsilon \quad \text{for all} \quad x \in K ,$$

then K has harmonic measure zero with respect to U provided

$$\Lambda_\alpha(K) = 0 \quad \text{for some} \quad \alpha > 0 \quad \text{satisfying}$$
$$\cos^2\left(\frac{\pi\alpha}{2}\right) > \frac{1}{2} + 2\varepsilon .$$

We end this section by illustrating Theorem 3 with an example:

EXAMPLE 2. Let C be a linear Cantor set of positive length, as described in § 3.

Let $F \subset \boldsymbol{R}$ be any closed set. Put $X = C \times F$ and $U = \boldsymbol{C} \backslash X$. Then the condition (**) of Theorem 3 is satisfied at each point of X. Therefore

$$\lambda_\alpha^U \ll \Lambda_\alpha$$

for all $\alpha < 1/2$ in this case.

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