PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 2, 1981

THE SPACE OF REAL PARTS OF ALGEBRAS OF FOURIER TRANSFORMS

SUNGWOO SUH

Let G be a locally compact abelian group with dual group Γ, and let A denote a closed subalgebra of $A(\Gamma)$, the algebra of all Fourier transforms of functions in $L^1(G)$, which separates the points of Γ, and whose members do not all vanish at any one point on Γ. Then $\mathrm{Re}\, A \cdot \mathrm{Re}\, A \subset \mathrm{Re}\, A$ implies $A = A(\Gamma)$ if Γ is totally disconnected.

1. Introduction. Let B be a semi-simple commutative Banach algebra with maximal ideal space ΣB. We regard B as an algebra of functions on ΣB. We consider a closed subalgebra A of B which separates points on ΣB, and whose members do not all vanish at any one point on ΣB.

Let $\mathrm{Re}\, A$ denote the space of real functions μ on ΣB such that for some f in A, $\mu = \mathrm{Re} f$.

Let G be a locally compact abelian group with dual group Γ. We denote by $A(\Gamma)$ the algebra of all Fourier transforms of functions in $L^1(G)$. That is,

$$f \in A(\Gamma) \Longleftrightarrow f(x) = \hat{F}(x) = \int_G F(y)(-y, x)dy$$
$$x \in \Gamma, \ y \in G, \ F \in L^1(G)$$

with norm $\|f\| = \|F\|L_1$. $A(\Gamma)$ is a semi-simple commutative Banach algebra and Γ is its maximal ideal space. We consider, also, a closed subalgebra A of $A(\Gamma)$ which separates the points on Γ, and whose members do not all vanish at any one point of Γ.

In [4] and [5] the following theorems were proved by Rudin and Katznelson: (cf. [5], p. 239–244).

THEOREM 1.1. *If B is spanned by its set of idempotents, then self-adjointness of A implies $A = B$.*

THEOREM 1.2. *If Γ is totally disconnected, then $A(\Gamma)$ is spanned by its set of idempotents.*

THEOREM 1.2. *If Γ is not totally disconnected, then there is an A which is proper and self-adjoint. That is: self-adjointness of A implies $A = A(\Gamma)''$ if and only if Γ is totally disconnected.*

Here we shall show that, for B as in Theorem 1.1 and for any subalgebra C of B, B-norm denseness and uniform denseness are exactly equivalent. That is:

THEOREM 1.4. *If B is spanned by its set of idempotents, then, for a subalgebra C of B, C is B-norm dense in B if and only if C is uniformly dense in B; more strongly, if and only if* $\operatorname{Re} C$ *is uniformly dense in* $\operatorname{Re} B$.

In the case of Theorem 1.1, by the Stone-Weierstrass theorem A is uniformly dense in $C_0(\Sigma B)$, and so in B, so that A is B-norm dense in B, and hence $A = B$.

As an immediate consequence of Theorem 1.4, we have

THEOREM 1.5. *If B is spanned by its set of idempotents, then* $\operatorname{Re} A \cdot \operatorname{Re} A \subset \operatorname{Re} A$ *implies* $A = B$.

In this case, $\operatorname{Re} A$ is uniformly dense in $\operatorname{Re} B$ by the Stone-Weierstrass theorem. In the case that $B = C(X)$ and A is a function algebra on X, the last statement of the above theorem holds for any compact $X(=\Sigma B)$ [8]. In the case of $B = A(\Gamma)$, we rewrite the above theorem:

THEOREM 1.6. *If Γ is totally disconnected, then*

$$\operatorname{Re} A \cdot \operatorname{Re} A \subset \operatorname{Re} A \text{ implies } A = A(\Gamma) .$$

Now we conclude that "$\operatorname{Re} A \cdot \operatorname{Re} A \subset \operatorname{Re} A$ implies $A = A(\Gamma)$" if and only if Γ is totally disconnected.

Note. Self-adjointness of A always implies $\operatorname{Re} A \cdot \operatorname{Re} A \subset \operatorname{Re} A$, but the latter condition does notimply self-adjointness in general.

2. Idempotent elements in a Banach algebra. To prove Theorem 1.4, we obtain a general version of this theorem in a complex commutative Banach algebra B, without assuming semi-simplicity of B, after an investigation of the behavior of idempotent elements in B.

Let the set $\operatorname{Sp}(B, b)$ be the spectrum of an element b of B, and let $\partial \operatorname{Sp}(B, b)$ denote the topological boundary of $\operatorname{Sp}(B, b)$. The following lemma is well known (cf. [1], p. 25):

LEMMA 2.1. *Suppose B has a unit, and let C be a closed subalgebra of B (C with or without unit).*
For $a \in C$,

(i) $\mathrm{Sp}\,(B, a) \subset \mathrm{Sp}\,(C, a) \cup \{0\}$, *and*

(ii) $\partial\,\mathrm{Sp}\,(C, a) \subset \partial\,\mathrm{Sp}\,(B, a)$.

Using the unitization of B, it is easy to check that Lemma 2.1 is valid in an algebra B without unit.

We then have the following:

THEOREM 2.2. *Let C be a closed subalgebra of B such that* $\mathrm{Re}\,\hat{C}$ *is uniformly dense in* $\mathrm{Re}\,\hat{B}$. *Then every idempotent element in $\hat{B}$ is contained in $\hat{C}$. More strongly, every idempotent element in B is contained in C.*

Here $\hat{C}$ and $\hat{B}$ denote the spaces of all Gelfand transforms of elements of C and B, respectively.

Proof. First we note that uniform denseness of $\mathrm{Re}\,\hat{C}$ implies that for every nonzero complex homomorphism on B, $\phi \in \Sigma B$, its restriction $\phi | C$ to C is a nonzero complex homomorphism on C. That is

(1) $$\forall\, \phi \in \Sigma B,\ \phi | C \in \Sigma C\ .$$

Let e be an idempotent element in $\hat{B}$, then e is 0 or 1 on ΣB. Choose a in C so that $|\mathrm{Re}\,\hat{a} - e| < 1/3$. Then we have, for all $\phi \in \Sigma B$,

(2) $$-1/3 < \mathrm{Re}\,\hat{a}(\phi) < 1/3 \text{ or } 2/3 < \mathrm{Re}\,\hat{a}(\phi) < 4/3.$$

Let $r(a)$ be the spectral radius of a,

$$r(a) = \mathrm{Max}\,\{|\lambda|\colon \lambda \in \mathrm{Sp}\,(B, a)\}\ .$$

Let

$$K_1 = \{z \in \boldsymbol{C}\colon |z| \leqq r(a)\} \cap \left\{z \in \boldsymbol{C}\colon -\frac{1}{3} \leqq \mathrm{Re}\,z \leqq \frac{1}{3}\right\}\ ,$$

$$K_2 = \{z \in \boldsymbol{C}\colon |z| \leqq r(a)\} \cap \left\{z \in \boldsymbol{C}\colon \frac{2}{3} \leqq \mathrm{Re}\,z \leqq \frac{4}{3}\right\}\ .$$

Then K_1, K_2 are disjoint compact subsets of $\boldsymbol{C}$, and since $0 \in K_1$ and by (2) we have, whether or not B has a unit,

(3) $$\mathrm{Sp}\,(B, a) \subset K_1 \cup K_2\ .$$

Since $\partial\,\mathrm{Sp}\,(B, a) \subset K_1 \cup K_2$, by Lemma 2.1 we have

(4) $$\partial\,\mathrm{Sp}\,(C, a) \subset K_1 \cup K_2\ .$$

Since $\mathrm{Sp}\,(C, a)$ is a nonvoid compact subset of $\boldsymbol{C}$,

$$(5) \qquad \mathrm{Sp}\,(C, a) \subset K_1 \cup K_2 \,.$$

Now, by applying standard functional calculus and by (1), we shall show that $e \in \hat{C}$. To see this, let V_1, V_2 be disjoint open neighborhoods of K_1, K_2 respectively. Define $f(z) = 0$ if $z \in V_1$, $f(z) = 1$ if $z \in V_2$, then f is a function analytic on $V_1 \cup V_2$ and $f(0) = 0$. Choose $\hat{c}$ in $\hat{C}$ such that $\hat{c} = f \circ \hat{a}$ on ΣC (this is possible because of (5)), then we have,

$$\forall \phi \in \Sigma B,\ \hat{c}(\phi) = \hat{c}(\phi | C) = f \circ \hat{a}(\phi | C) = f(\hat{a}(\phi)) \,.$$

Now, it is clear that $\hat{c} = e$ on ΣB and $e \in \hat{C}$.

The last conclusion is an immediate consequence of the Šilov idempotent theorem and related uniqueness property (cf. [2], p. 88) and the unitization of B if B has no unit.

Note. In Theorem 2.2, the mapping $\phi \to \phi | C$ from ΣB into ΣC is, in fact, one-to-one so that ΣB can be considered as a subset of ΣC. But the proof does not require this fact, which is stronger than (1) (cf. [5], p. 240).

Suppose that a closed subalgebra C of B has the property that members of $\hat{C}$ do not all vanish at any one point of ΣB. Then (1) also holds. Thus the argument in the proof of Theorem 2.2 implies the following:

COROLLARY 2.3. *Let C be a closed subalgebra of B with the above property. Then an idempotent element e of B such that $\hat{e}$ is a uniform limit of elements of $\hat{C}$ or of real parts of elements $\hat{C}$ is contained in C.*

Suppose B is a semi-simple commutative Banach algebra. We may regard B as $\hat{B}$. Then Theorem 2.2 can be restated as follows:

THEOREM 2.2′. *Let C be a closed subalgebra of B such that* $\mathrm{Re}\, C$ *is uniformly dense in* $\mathrm{Re}\, B$; *then C contains all idempotent elements of B.*

Now Theorem 1.4 is an immediate consequence of Theorem 2.2′.

REMARK. (1) Using Theorem 2.2 and adapting the original proof of Glicksberg's theorem in function algebras [3], it is easy to check that "$A|_E$ is closed in $A(\Gamma)|_E$ for every compact subset E of Γ'' implies $A = A(\Gamma)$ if Γ is totally disconnected, which is the analogue of [3]. (For the proof, see [6].) After this result, in his paper [7] B. B. Wells proved this result for any locally compact abelian group Γ.

(2) The stronger version of Theorem 2.2 was pointed out by the referee. We would like to thank the referee for his valuable suggestions and corrections, which makes this paper stronger and more comprehensive.

REFERENCES

1. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Band 80, 1973.
2. T. Gamelin, *Uniform Algebras*, Prentice-Hall, 1969.
3. I. Glicksberg, *Function algebras with closed restrictions*, Proc. Amer. Math. Soc., (1963), 158-161.
4. Y. Katznelson and W. Rudin, *The Stone-Weierstrass property in Banach algebras*, Pacific J. Math., **11** (1961), 253-265.
5. W. Rudin, *Fourier Analysis on Groups*, Interscience, 1962.
6. Sungwoo Suh, *Characterizations of $L^1(G)$ among its subalgebras*, Thesis, Univ. of Connecticut, 1978.
7. Benjamin B. Wells, Jr., *Algebras of Fourier transforms with closed restrictions*, preprint.
8. J. Wermer, *The space of real parts of a function algebra*, Pacific J. Math., **13** (1963), 1423-1426.

Received April 8, 1980 and in revised form August 4, 1980. This paper is a portion of the author's doctoral dissertation prepared at the University of Connecticut under the direction of Professor Stuart J. Sidney. The author wishes to extend his sincere gratitude to him for his guidance and advice. The author also wishes to thank Professor Jerome H. Neuwirth for suggesting the problems discussed here.

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70803