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A NOTE ON FR-PERFECT MODULES

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This paper defines and characterizes projective FR-perfect modules which is a generalization of semiperfect modules. Using these some characterizations of semiperfect modules are obtained.

Introduction. Let R be a ring with identity. All modules we consider are unitary left R-modules. A submodule $N \subseteqq M$ is said to be *small* in M if $N + T = M$ implies $T = M$. An epimorphism $P \xrightarrow{f} M$ is called *minimal* if $\mathrm{Ker}(f)$ is small in P. A minimal epimorphism $P \xrightarrow{f} M$, where P is projective, is called a *projective cover* of M. We denote by J, the Jacobson radical of R. By $J(M)$ we mean the radical (intersection of all maximal submodules) of a module M. If M is projective $J(M) = JM$. We call a module N *M-finitely related* (M-FR) if $N \cong M^n/B$, where B is finitely generated. A module M is called FR-*perfect* if every M-FR module has a projective cover. Similarly we define finitely presented perfect (FR-perfect) modules.

Our aim is to characterise M-FR perfect projective modules. So we would like to find out equivalent conditions for a module M/A to have a projective cover, where M is projective. In §1 we do this when either A is finitely generated or JM is small in M. In particular, we show that for a projective module M, M/A has a projective cover if and only if $f(A)$ is a direct summand of M/JM and any direct decomposition $f(A) \oplus B$ of M/JM can be lifted up, where the summand of M corresponding to $f(A)$ is finitely generated and $f: M \to M/JM$ is the natural projection. So for a projective FR-perfect module every finitely generated submodule of M/JM is a direct summand. That is M/JM is a regular R/J-module (R. Ware [8]). We prove some properties of regular modules which are used later in proving that direct sum of FR-perfect projective modules is FR-perfect projective if and only if each summand is so.

In §2, we give several characterizations of FR-perfect projective modules. We prove that the following conditions are equivalent for a projective R-module M (i) M is FR-perfect (ii) M/JM is a regular R/J module and any direct decomposition $A \oplus B$ of M/JM can be lifted up whenever A is cyclic (finitely generated) and the summand of M corresponding to A is finitely generated and (iii) M/U has a projective cover whenever U is cyclic (finitely generated). If further JM is small in M then the above conditions are equivalent to (iv)

M/JM is a regular R/J module and every direct decomposition of M/JM can be lifted up and (*v*) M/JM is a regular R/J-module and every direct summand of M/JM has a projective cover. Using these we give some characterizations of semiperfect ring. We prove that a projective module M is semiperfect if and only if JM is small in M and M/U has a projective cover whenever U is countably generated. We also show that a module M is semiperfect if and only if M is FR-perfect projective, JM is small in M and M/JM is semisimple.

In §3, we define *essentially finitely related perfect* (EFR-perfect) modules. We prove that if the singular submodule of R/J is zero, then a projective R-module is EFR-perfect if and only if it is FR-perfect.

1. Preliminaries. This section contains properties fo projective covers needed for our purpose. Proposition 1.7 is an important step for proving our main theorem. This proposition may have its own value. In a later part of this section we mention some results about regular modules (R. Ware [8]) which will be used in §2.

PROPOSITION 1.1 (*Exercise* 15(2), P. 203 [1]). *Let A, B be two R-modules. If A and $A \oplus B$ have projective covers, then B also has a projective cover.*

Since R/J is Jacobson semisimple, only projective modules over R/J can have projective covers. Hence

PROPOSITION 1.2. *If an R/J-module N has a projective cover as an R-module, then N is projective as an R/J module.*

PROPOSITION 1.3. *Let N be a small submodule of an R-module M. Let U be any submodule of M. Then M/U has a projective cover if and only if $M/(U+N)$ has a projective cover.*

Proof. Since the natural map $M/U \to M/(U+N)$ is minimal, the proposition follows.

PROPOSITION 1.4. *Let $A = \bigoplus_{s \in S} A_s$ be an R-module such that $f_s\colon P_s \to A_s$ is a projective cover of A_s for every $s \in S$. If $h\colon P \to A$ is a projective cover of A, then there exists an isomorphism $g\colon \bigoplus P_s \to P$ such that $(h \circ g)|P_s = f_s$.*

Proof. Define $f\colon \bigoplus P_s \to \bigoplus A_s$ by $f|P_s = f_s$ for every $s \in S$. As $\bigoplus P_s$ is projective, there exists a homomorphism $g\colon \bigoplus P_s \to P$ such that $h \circ g = f$. As h is a minimal epimorphism and f is onto, g is onto. As P is projective, g splits and $\ker(g)$ is a direct summand of $\bigoplus P_s$. Now $\ker(g) \subseteqq \ker(f) = \bigoplus \ker(f_s) \subseteqq \bigoplus JP_s \subseteqq J(\bigoplus P_s)$; since $J(\bigoplus P_s)$ cannot contain a direct summand of $\bigoplus P_s$, $\ker(g) = 0$.

PROPOSITION 1.5 (cf. Nicholson [5, Lemma 1.6]). *Let N be a direct summand of a projective module P. Let M' be a submodule of P such that $P = N + M'$. Then $P = N \bigoplus M$ for some submodule $M \subseteqq M'$.*

COROLLARY 1.6. *Let M be any projective module. Then decompositions modulo JM can be lifted if and only if for any direct summand D of M/JM there is a direct summand B of M such that $f(B) = D$, where $f: M \to M/JM$ is the natural map.*

Proof. The only if part is trivial. Assume that the condition is satisfied. Suppose $M/JM = C \bigoplus D$. Let B be a direct summand of M such that $f(B) = D$. Now clearly $M = f^{-1}(C) + B$. Then by Proposition 1.5, we can write $M = A \bigoplus B$ where $A \subseteqq f^{-1}(C)$. Then $M/JM = f(A) + D$. But $f(A) \subseteqq f(f^{-1}(C)) \subseteqq C$. Since $M/JM = C \bigoplus D$, $f(A) = C$. This completes the proof.

The following proposition is an important step towards proving the main theorem. It may also be of some independent interest.

PROPOSITION 1.7. *Let M be a projective R-module, $f: M \to M/JM$ be the natural map and A be any submodule of M. Assume either* (a) *A is finitely generated or* (b) *JM is small in M. Then the following conditions are equivalent:*

(1) *M/A has a projective cover.*

(2) *$M = C \bigoplus T$ such that $C \subseteqq A$ and $A \cap T$ is small in M.*

(3) *$f(A)$ is a summand of M/JM and $f(A)$ has a projective cover as an R-module.*

(4) *There is a summand C of M (which is finitely generated if A is finitely generated) such that $f(C) = f(A)$.*

(5) *$f(A)$ is a summand of M/JM and for any decomposition $M/JM = f(A) \bigoplus B$ there is a decomposition $M = C \bigoplus D$ such that $f(C) = f(A)$, $f(D) = B$ (where C is finitely generated if A is finitely generated).*

Proof. The equivalence of conditions (1) and (2) is well known without (a) or (b). We prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3). Since $M = C \bigoplus T$, $M/JM = f(C) \bigoplus f(T)$. $C \subseteqq A$ implies $A = C \bigoplus (A \cap T)$. This together with $A \cap T$ is small in M shows that $f(A) = f(C)$. Hence $f(A)$ is a summand of M/JM. Consider the epimorphism $g = f|C: C \to f(A)$; $\ker(g) = C \cap JM = JC$, since C

is direct summand of M. If A is finitely generated or JM is small in M, then JC is small in C. In any case, $g\colon C \to f(A)$ is a projective cover.

$(3) \Rightarrow (4)$. Let $M/JM = f(A) \oplus T$ and $g\colon M/JM \to f(A)$ be the corresponding projection map. Then $g \circ f\colon M \to f(A)$ is an epimorphism and further $g \circ f(A) = f(A)$. Let $h\colon P \to f(A)$ be a projective cover. Then there exists $h'\colon M \to P$ such that $g \circ f = h \circ h'$. Since $f(A) = g \circ f(A)$ and h is minimal $h'(A) = P$. This means $M = \ker h' + A$ and $\ker h'$ is a summand of M, and so $M = \ker h' \oplus A'$ with $A' \subseteq A$ by Proposition 1.5. Hence there is a splitting monomorphism $j\colon P \to M$ such that $j(P) \subseteq A$. Let $C = j(P)$. Then $(g \circ f)|C\colon C \to f(A)$ is a projective cover. Then clearly $f(C) = f(A)$. If A is finitely generated, then so is $f(A)$ and consequently its projective cover C is finitely generated.

$(4) \Rightarrow (5)$. Assume (4). $f(A) = f(C)$ is a direct summand of M/JM, since C is a direct summd of M. Let $M/JM = f(A) \oplus B$ be any decomposition. We have $M/JM = f(C) \oplus B$; consequently $M = C + f^{-1}(B)$. By Proposition 1.5, $M = C \oplus D$, where $D \subseteq f^{-1}(B)$. Clearly this is the required decomposition.

$(5) \Rightarrow (1)$. Assume (5). Write $M = C \oplus S$ such that $f(C) = f(A)$. Let $\{a_t\}$ be a set of generators for A. Write $a_t = c_t + s_t$, $c_t \in C$, $s_t \in S$ for each t. Then $f(a_t) = f(c_t) + f(s_t)$. But $M/JM = f(C) \oplus f(S)$ and $f(A) = f(C)$ together imply $f(s_t) =$ zero i.e., $s_t \in JM$. Let $C' = \Sigma R c_t$, We have

$$f(C') = f(\Sigma R c_t) = \Sigma f(R c_t) = \Sigma f(R a_t) = f(\Sigma R a_t) = f(A) = f(C)\ .$$

Consequently, $C' + JM = C + JM$. Since C is direct summand of M, $C' + JC = C$. If A is finitely generated so is C by assumption and therefore JC is small in C. If JM is small in M, then JC is small in C. In any case we have $C' = C$. From $a_t = c_t + s_t$ for each t we see that $A + S' = C' + S' = C + S'$ where $S' = \Sigma R s_t$. Since each $s_t \in JM$, we see that S' is small in M (if the indexing set of t is finite or JM small in M). By repeated use of Proposition 1.3, we see that M/A has a projective cover if and only if $M/(A + S') = M/(C + S')$ has a projective cover. This is so if and only if $M/C \cong S$ has a projective cover. But S is itself projective. This completes the proof.

A module M is called *regular* if M is projective and every cyclic submodule of M is a direct summand of M (R. Ware [8, Definition 2.3]). R. Ware has proved that if M is a regular R-module, then every finitely generated submodule of M is a direct summand [8, Proposition 2.1] and $\bigoplus_s P_s$ is regular if and only if each P_s is regular [8, Page 239].

We need the following propositions to prove our main theorem.

PROPOSITION 1.8 (R. Ware [8, Theorem 2.12]). *An R-module P is regular if and only if P is isomorphic to a direct sum of principal left ideals of R, each of which is a regular module.*

PROPOSITION 1.9. *Let U be a direct summand of* $\bigoplus_{s \in S} M_s$, *where each* M_s *is a regular R-module. Then* $U \cong \bigoplus_{t \in T} Re_t$ *where each* Re_t *is a direct summand of* M_s *for some* $s \in S$.

Proof. By Proposition 1.8 it is enough to prove the result when U is cyclic and hence when the set S is finite. If the cardinality of S is 1, the result is obvious. Assume the result whenever the cardinality of $S \leqq n-1$. Let $S = \{s_1, \cdots, s_n\}$ and U be a direct summand of $\bigoplus_{s \in S} M_s$. Suppose $f: M_{s_1} \oplus \cdots \oplus M_{s_n} \to M_{s_n}$ is the natural projection and $g = f \mid U$. As $g(U)$ is a direct summand of M_{s_n}, it is projective and $U \cong \ker(g) \oplus g(U)$. Since $\ker(g)$ is a direct summand of $M_{s_1} \oplus \cdots \oplus M_{s_{n-1}}$, the result follows from the induction hypothesis.

2. FR-Perfect modules. In this section we define the concept of FR-perfect modules and obtain our main theorem. Using this some characterizations of semiperfect rings and modules are obtained.

A module N is said to be *M-finitely related* (M-FR) if $N \cong M^n/B$, where B is finitely generated. A module N is called *M-finitely presented* (M-FP) if $N \cong P/U$, where P is a direct sum of copies of M and U is a finitely generated submodule of P. A module M is called FR-*perfect* (FP-*perfect*) if every M-FR(M-FP) module has a projective cover. M is called *semiperfect* if it is projective and every factor module of M has a projective cover.

THEOREM 2.1. *Let R be a ring and M be any projective R-module. Let* $f: M \to M/JM$ *be the natural map. Then the following are equivalent.*

(i) M/JM *is a regular* R/J*-module and for every direct decomposition* $A \oplus B$ *of* M/JM, *where A is cyclic, there exists a decomposition* $M = C \oplus D$, *where* $f(C) = A$, $f(D) = B$ *and C is cyclic.*

(ii) M/JM *is a regular* R/J*-module and for every direct decomposition* $A \oplus B$ *of* M/JM, *where A is finitely generated, there exists a direct decomposition* $M = C \oplus D$, *where* $f(C) = A$, $f(D) = B$ *and C is finitely generated.*

(iii) M/JM *is a regular* R/J*-module and every cyclic (finitely generated) submodule of* M/JM *has a projective cover as an R-module.*

(iv) *M is* FR-*perfect.*

(v) M/U *has a projective cover for ever finitely generated submodule U of M.*

(vi) *M/U has a projective cover for every cyclic submodule U of M.*

Proof. (i) $\Rightarrow$ (iii). Let A be a cyclic submodule of M/JM. Since M/JM is regular, we can write $M/JM = A \oplus B$. By (i) $M = C \oplus D$ where $f(C) = A$. Since $M/C \cong D$ is projective, by (1) $\Rightarrow$ (3) of Proposition 1.7, we see that $A = f(C)$ has a projective cover. Since any finitely generated submodule of M/JM is a finite direct sum of cyclic submodules of M/JM, we get the result.

(iii) $\Rightarrow$ (iv). In view of Proposition 1.9 any direct sum of copies of M also satisfies condition (iiii). Thus it is enough to prove that M/U has a projective cover whenever U is finitely generated. But this is clear in view of (3) $\Rightarrow$ (1) of Proposition 1.7.

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (ii). Using (1) $\Rightarrow$ (5) of Proposition 1.7 we get this.

(ii) $\Rightarrow$ (vi). Using (5) $\Rightarrow$ (1) of Proposition 1.7 we get this.

(vi) $\Rightarrow$ (i). This follows from (1) $\Rightarrow$ (5) of Proposition 1.7 and the fact that the projective cover of a cyclic module is cyclic.

COROLLARY 2.2. *Let M be a projective module such that JM is small in M. Let $f: M \to M/JM$ be the natural map. Then the following are equivalent.*

(a) *M is* FR-*perfect.*

(b) *M/JM is a regular R/J-module and any direct decomposition of M/JM can be lifted to a direct decomposition of M.*

(c) *M/U has a projective cover whenever $f(U)$ is finitely generated.*

(d) *M/JM is a regular R/J-module and every direct summand of M/JM has a projective cover as an R-module.*

Proof. (a) $\Rightarrow$ (b). As M/JM is a regular R/J-module it is enough to prove that a direct decomposition $\bigoplus_{s \in S} V_s$ of M/JM, where each V_s is cyclic, can be lifted up. Let $f_s: P_s - V_s$ be projective covers for every $s \in S$. Since JM is small in M, by Proposition 1.4, $M = \bigoplus P'_s$, where $f(P'_s) = V_s$ for every $s \in S$.

(b) $\Rightarrow$ (c). Let $U \subseteq M$ be a submodule such that $f(U)$ is finitely generated. Then $f(U)$ is a direct summand of M/JM. By assumption we can write $M = C \oplus T$ such that $f(C) = f(U)$. Then since JM is small in M, by (4) $\Rightarrow$ (1) of Proposition 1.7, M/U has a projective cover.

(c) $\Rightarrow$ (a) is obvious in view of Theorem 2.1.

(b) $\Rightarrow$ (d). Let $M/JM = V \oplus W$. Then there exits a decomposition $A \oplus B$ of M such that $f(A) = V$ and $f(B) = W$. Let $g = f \mid A$. Then $\ker(g) = JA$ is small in A since JM is small in M. So $g: A \to V$ is a projective cover of V.

(d) $\Rightarrow$ (a). This is clear by Theorem 2.1 (iii) $\Rightarrow$ (iv).

COROLLARY 2.3. *Let M be a projective R-module and JM be small in M. If M is* FR-*perfect then $M \cong \bigoplus_{s\in S} Re_s$, where each e_s is an idempotent in R.*

PROPOSITION 2.4. *Let $M = \bigoplus_{s\in S} M_s$, be a a projective R-module. M is* FR-*perfect if and only if each M_s is* FR-*perfect.*

Proof. The only if part is obvious. For the converse, it is sufficient to prove for finite direct sums in view of Theorem 2.1 (iii). This can be easily seen by using Proposition 1.9 and Theorem 2.1 (iii).

COROLLARY 2.5. *A projective module is* FR-*perfect if and only if it is* FP-*perfect.*

REMARK. W. K. Nicholson [5, Page 1107] calls a module M *semiregular* if for every $m \in M$, there exists a decomposition $M = P \oplus Q$ where $P \subseteqq Rm$, P is projective and $Rm \cap Q$ is small in M. If M is projective, then M is semiregular if and only if M/Rm has a projective cover for every $m \in M$. By Theorem 2.1, a projective module is semiregular if and only if M is FR-perfect. Theorem 2.1 (ii) $\Leftrightarrow$ (iv) is an improvement of Proposition 1.17 of [5]. We do not know whether if M is a projective R-module and M/JM is a finitely generated R/J-regular module, then M is finitely generated. If this is so then the finitely generated condition on C in Theorem 2.1 (ii) can be removed. Also Proposition 1.17 of [5] will be then be true without JM being small in M. For a projective module M to be FR-perfect, JM need not be small in M. Let R be a ring which is semiperfect but not perfect. Then M, a countable direct sum of copies of R, is FR-perfect but JM is not small in M. We do not know whether every FR-perfect projective module is a direct sum of cyclics.

A module Q is called *Quasi-projective* if for every exact sequence $Q \to A \to 0$, the induced sequence $\mathrm{Hom}(Q, Q) \to \mathrm{Hom}(Q, A) \to 0$ is exact. Let M be an R-module. A minimal epimorphism $f: Q \to M$ is called a *quasi-projective cover* of M if Q is quasi-projective and Q/T is not quasi-projective whenever $T \subseteqq \ker(f)$.

LEMMA 2.6 ([6, Lemma 3.2]). *A module A is projective if and only if there exists an epimorphism $P \to A$ with P projective and $A \oplus P$ quasi-projective.*

LEMMA 2.7. *If $P \oplus A$ has a quasi-projective cover where P is* *projective and A is an epimorphic image of P, then A has a projective cover.*

Proof. Let $g: Q \to P \oplus A$ be a quasi-projective cover. As P is projective Q splits. We can write $Q = P_1 \oplus g^{-1}(A)$ where $P_1 \cong P$. Then there is an epimorphism $f: P_1 \to A$. We note that $g|g^{-1}(A)$: $g^{-1}(A) \to A$ is a minimal epimorphism. Hence there is an epimorphism from P_1 onto $g^{-1}(A)$. Then Lemma 2.6 shows that $g^{-1}(A)$ is projective. Hence A has a projective cover.

COROLLARY 2.8. *For a projective module M conditions in Theorem* 2.1 *are equivalent to*:

(vii) *every M-*FR *module has a quasi-projective cover.*

(viii) *$(M/U) \oplus M$ has a quasi-projective cover whenever U is finitely generated.*

(ix) *$(M/U) \oplus M$ has a quasi-projective cover whenever U is cyclic.*

In particular by taking $M = R$ we get the following result. The equivalence of (1) and (2) is already proved in [7, Proposition 5].

COROLLARY 2.9. *The following are equivalent for any ring R*

(i) *R is left* FR*-perfect.*

(ii) *R/J is Von-Neumann regular and idempotents can be lifted modulo J.*

(iii) *R/J is Von-Neumann regular and every cyclic (finitely generated) left ideal of R/J has a projective cover as an R-module.*

(iv) *R/U has a projective cover whenever U is finitely generated left ideal of R.*

(v) *R/U has a projective cover whenever U is a principal left ideal of R.*

(vi) *R is right* FR*-perfect.*

(vii) *R is right* (left) FR*-quasi-perfect.*

Theorem 19.27 of [3] tells us that there exists plenty of FR-perfect rings. We rephrase it in our notation.

PROPOSITION 2.10 (C. Faith [3, Theorem 19.27]). *If Q is a quasi-injective module, then the endomorphism ring of Q is self injective* FR*-perfect ring.*

COROLLARY 2.11. *Any self injective ring is a* FR*-perfect ring.*

From the proof of Theorem 5.6 of [4] we get, THEOREM 2.12. *If M is an* FR-*perfect module and if P is a projective cover of M, then P is* FR-*perfect.*

A factor module of an FR-perfect module need not be FR-perfect. Let R be a Von-Neumann regular ring which is not semisimple. Then R as a module over itself is an example. But if M is an FR-perfect module, then M/U is FR-perfect whenever U is finitely generated.

E. A. Mares [4, Theorems 3.3, 3.5, 4.3 and 5.1] has proved that a projective module M is semiperfect if and only if it has the following properties (1) M/JM is semisimple (that is every submodule is a summand), (2) Every direct decomposition of M/JM can be lifted to a direct decomposition of M and (3) JM is small in M. We give below some characterizations of a projective semiperfect module.

It can be seen from Proposition 1.7 that a projective module M is semiperfect if and only if JM is small in M, M/JM is semisimple and every submodule of M/JM has a projective cover. From Corollary 2.2 we get

PROPOSITION 2.13. *A projective module M is semiperfect if and only if it is* FR-*perfect, JM is small in M and M/JM is semisimple.*

A module is called finite dimensional if it contains no infinite direct sum of nonzero submodules.

THEOREM 2.14. *Let M be a projective* FR-*perfect module. If M is finite dimensional, then M is semiperfect.*

Proof. Let U be a submodule of M/JM. We show that U is finitely generated. Suppose not. Let $m_1 \in U$. Then $M/JM = Rm_1 \oplus N_1$ and this decomposition be lifted to a decomposition $M = T_1 \oplus L_1$. There exists $m_2 \in U$ such that $m_2 \notin Rm_1$. Let $m_2 = rm_1 + n_2$, $n_2 \in N_1$. Then $N_1 = L_1/JL_1 = Rn_2 \oplus N_2$ and this decomposition can be lifted to a decomposition $L_1 = T_2 \oplus L_2$. Hence if U is not finitely generated, then M is not finite dimensional. Thus M/JM is a Noetherian and regular module and so is a finite direct sum of simple modules. Hence M is a semiperfect.

THEOREM 2.15. *A projective module M is semiperfect if and only if M/U has a projective cover for any countably generated submodule U of M and JM is small in M.*

Proof. The 'only if' part follows from Mares' result [4, Theorem 3.3]. Conversely suppose that the conditions are satisfied. By

Theorem 2.1, (v) $\Rightarrow$ (iv), M is FR-perfect. Then by Corollary 2.3, M is a direct sum of cyclics. Clearly any direct summand of M also satisfies the conditions. Hence by [4, Theorem 5.2], it is enough to prove the result when M is cyclic. Let $f: M \to M/JM$ be the natural map. Let A be a countably generated submodule of M/JM and let U be a countably generated submodule of M such that $f(U) = A$. Since M/U has a projective cover, $M = K \oplus P$, where $K \subseteq U$ and $P \cap U$ is small in M. As K is cyclic, $A = f(U) = f(K)$ is cyclic. Hence M/JM is Noetherian. As M/JM is regular, it is semisimple. Using Corollary 2.13 we see that M is semiperfect.

COROLLARY 2.16 (*Nicholson* [5, *Corollary* 2.10]). *Let R be a ring. Then R is semiperfect if and only if R/U has a projective cover whenever U is a countably generated left ideal.*

Let R be a ring which has ascending chain condition on left ideals Re, $e^2 = e$. It is easy to see that such a ring R also satisfies descending chain condition on left ideals Re, $e^2 = e$. We show that for such a ring R any projective FR-perfect module M is semiperfect if and only if JM is small in M.

THEOREM 2.17. *Let R be a ring with ascending chain condition on left ideals Re, $e^2 = e$. Then a projective* FR-*perfect module is semiperfect if and only if JM is small in M.*

Proof. Let A be a cyclic summand of M/JM. Then $M = C \oplus D$, $f|C: C \to C/JC = A$ is a projective cover, where $f: M \to M/JM$ is the natural map. As A is cyclic, C is cyclic and hence $C \cong Re$, $e^2 = e$. From the given condition on R we see that C is a finite direct sum of indecomposable modules. Let $C = \bigoplus_{i=1}^{k} C_i$, where each C_i is a indecomposable cyclic projective module. Then

$$A = \bigoplus_{i=1}^{k} A_i,\ f(C_i) = C_i/JC_i = A_i\ .$$

As A_i is regular, if it is not simple, then A_i is decomposable. As any direct decomposition of A_i can be lifted to a direct decomposition of C_i, this would imply that C_i is decomposable. Hence A is a direct sum of simple modules. As M/JM is a regular module, it is a semisimple module. If JM is small in M, then Proposition 2.13 shows that M is semiperfect.

COROLLARY 2.18. *A ring R is semiperfect if and only if R is* FR-*perfect and R satisfies ascending chain condition on left ideals Re, $e^2 = e$.*

THEOREM 2.19. *Let* $M = \bigoplus_{s \in S} M_s$ *be a* FR-*projective module. If* P *is a direct summand of* M *such that* JP *is small in* P, *then* P *is a direct sum of cyclic submodules which are isomorphic to direct summands of the modules* M_s, $s \in S$.

Proof. Let $f: M \to M/JM$ be the natural map. By Proposition 1.9, we see that $f(P) \cong \bigoplus_{i \in I} Q_i$, where each Q_i is a cyclic summand of $f(M_{s_i})$ for some $s_i \in S$. Since M_{s_i} is FR-perfect projective there exists a cyclic summand P_i of M_{s_i} such that $f(P_i) = Q_i$ [Theorem 2.1, (i)]. For each i, $f \mid P_i: P_i \to Q_i$ is a projective cover. As JP is small in P, by using Corollary 1.4, it is easy to see that $P \cong \bigoplus P_i$ (external direct sum).

As any semiperfect module is a direct sum of indecomposables [4, Corollary 4.4], we get the following theorem of E. A. Mares as a corollary.

COROLLARY 2.20 (E. A. Mares [4, Theorem 5.5]). *Let* P *be a direct summand in a direct sum of semiperfect modules,* $F = \bigoplus_{i \in I} M_i = P \oplus Q$, *and let* JP *be small in* P; *then* P *is a direct sum of indecompasable submodules which are isomorphie to direct summands of* M_i, $i \in I$.

3. EFR-Perfect modules. In this section we generalize the concept of FR-perfect modules. A submodule U of a module M is called *essential* in M if $K \cap U = 0$ implies $K = 0$ for any submodule K of M. A module U is called *essentially finitely generated* (EFG) if it contains a finitely generated essential submodule. A module N is called *M-essentially finitely related* (M-EFR) if $N \cong M^n/U$, where U is an EFG submodule of M^n. A module M is called EFR-*perfect* if every M-EFR module has a projective cover.

The following proposition gives a characterization of EFR-perfect modules.

PROPOSITION 3.1. *Let* M *be a projective* R-*module such that* JM *is small in* M. *Then the following conditions are equivalent*

(i) M is EFR-*perfect*.

(ii) M *is* FR-*perfect and for every* EFG *submodule* U *of* M^n, $(U + JM^n)/JM^n$ *is finitely generated for every* n.

(iii) M *is* EFR-*quasi-perfect*.

Proof. The equivalence of (i) and (iii) follows from Lemma 2.7.

(ii) $\Rightarrow$ (i). Since M^n also satisfies condition (ii) for every integer n, it is enough to prove M/U has a projective cover whenever U is an EFG submodule of M. By Corollary 2.2, M/U has a projective cover.

(i) $\Rightarrow$ (ii). Let U be an EFG submodule of M^n. Let $f_n: M^n \to M^n/JM^n$ be the natural map. As M^n/U has a projective cover, $M^n = K \oplus T$ where $K \subseteq U$ and $U \cap T$ is small in M^n. Suppose $f_n(K) = f_n(U)$ is infinitely generated. Then $f_n(K)$ is an infinite direct sum of cyclics and hence by Proposition 1.4, K is an infinite direct sum of cyclics and $U = K \oplus (U \cap T)$. So U cannot be an EFG module.

If M is an R-module, $Z(M)$ = the *singular submodule* of $M = \{m \in M \mid \mathrm{ann}_R(m)$ is essential in $R\}$.

LEMMA 3.2 [2, Proposition 1.1(v)]. *Let A and B two R-modules. If $f: A \to B$ is an epimorphism, $Z_R(B) = 0$ and C is essential submodule of A, then $f(C)$ is essential in B.*

PROPOSITION 3.3. *Let R be a ring such that $Z(R/J) = 0$. Then a projective R-module M is* EFR-*perfect if and only if it is* FR-*perfect.*

Proof. Any EFG submodule of a regular module is finitely generated. If U is an EFG submodule of M, then $(U + JM)/JM$ is an EFG submodule of M/JM.

COROLLARY 3.4. *Let R be a Von-Neumann regular ring. A projective R-module M is* FR-*perfect if and only if it is* EFR-*perfect.*

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