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ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES OF THE CLASSES $H^{\omega} \cap V[v]$

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In this paper the absolute convergence of the Fourier series is studied for the class of the function f with the modulus of continuity and the modulus of variation satisfying the conditions $\omega(\delta, f) = O(\omega(\delta))$ and $\upsilon(n, f) = O(\upsilon(n))$ respectively, where the modulus of continuity $\omega(\delta)$ and the modulus of variation $\upsilon(n)$ are given. In terms of these properties the sufficient conditions of the absolute convergence are established. We prove that these conditions are unimprovable in certain sense.

1. Let f be a 2π periodic continuous function and let

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos ntdt\,, \qquad n = 0, 1, \cdots$$

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin ntdt\,, \qquad n = 1, 2, \cdots$$

be the sequence of its Fourier coefficients.

The paper is devoted to the conditions of convergence of the series

$$(1) \qquad \sum_{n=1}^{\infty}(|a_n(f)| + |b_n(f)|)$$

or of the series

$$(2) \qquad \sum_{n=-\infty}^{\infty} |c_n(f)|$$

where

$$c_0 = \frac{a_0}{2}\,, \qquad c_n(f) = \frac{a_n - ib_n}{2}\,, \qquad c_{-n} = \frac{a_n + ib_n}{2}\,,$$

if the complex form of the Fourier series is used.

The class of functions for which the series (1) (or (2)) converges is denoted by A.

The problem in question has a long history (see monographs [1], Chapter IX, [14], Chapter VI, [8]).

Let us introduce the classes which well be used in what follows.

If $f \in C(0, 2\pi)$ then the function

$$\omega(\delta, f) = \max_{\substack{x, y \in [0\ 2\pi] \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

is called the modulus of continuity of the function f.

The modulus of continuity of an arbitrary function $f \in C(0, 2\pi)$ has the following properties:

(1) $\omega(0) = 0$,

(2) $\omega(\delta)$ is nondecreasing on δ,

(3) $\omega(\delta)$ is continuous on $[0, \pi]$,

(4) $\omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$ for $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq \pi$.

An arbitrary function $\omega(\delta)$ which is defined on $[0, \pi]$ and has the properties (1)-(4) is called the modulus of continuity.

If the modulus of continuity $\omega(\delta)$ is given then H^ω denoted the class of functions $f \in C(0, 2\pi)$ for which $\omega(\delta, f) = O(\omega(\delta))$ when $\delta \to 0$ is denoted.

S. N. Bernstein has proved ([1], p. 608), that if $f \in H^\omega$ and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega\left(\frac{1}{n}\right) = \infty$$

then $f \in A$.

Bernstein's theorem is best possible in the sense that if the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega\left(\frac{1}{n}\right) < \infty ,$$

then there exists the function $f_0 \overline{\in} A$ in the class H^ω.

This was proved by Bernstein for the class H^ω under the condition that there exists $\varepsilon > 0$ such that $\delta^{\varepsilon-1}\omega(\delta)$ is decreasing. S. B. Stechkin proved the same for arbitrary classes ([1], p. 625).

If the function f has bounded variation i.e., belongs to the class V, then lesser smoothness of its modulus of continuity may be required. That follows from the theorem of A. Zygmund [15]: If $f \in V \cap H^\omega$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{\omega\left(\frac{1}{n}\right)} < \infty \tag{3}$$

then $f \in A$.

In particular, the absolute convergence occurs when

$$\omega(\delta) = O\left(\left(\ln \frac{1}{\delta}\right)^{-\eta}\right), \qquad \eta > 2 . \tag{4}$$

Zygmund has pointed out that the latter statement is wrong for $\eta \leqq 1$, because the function

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sin nx$$

is absolutely continuous and has the modulus of continuity of order $O((\ln 1/\delta)^{-1})$. He has posed the question whether the absolute convergence is true for $1 < \eta \leqq 2$.

R. Salem [9] has proved that if for any $\varepsilon > 0$ the modulus of continuity $\omega(\delta)$ satisfies the condition

$$\sum \frac{1}{n}\left[\omega\left(\frac{1}{n}\right)\right]^{1/2+\varepsilon} = \infty ,$$

then there exists the function $f_0 \in V \cap H^\omega$ which does not belong to A. This implies that absolute convergence does not hold for the class H^ω, $\omega(\delta) = O((\ln 1/\delta)^{-\eta})$, $\eta < 2$.

J.-P. Kahane ([8], p, 24) has sharpened Salem's theorem in the following way: if the modulus of continuity satisfies the condition

$$\overline{\lim_{n\to\infty}}\, n^2\omega(2^{-n}) = \infty$$

then there exists the function $f_0 \in V \cap H^\omega$ which does not belong to A.

But neither Kahan's theorem gives the answere in the logarithmic scale for $\eta = 2$ and this question has remained open until recently.

Only in 1972 I. Wik [12] proved that there exists the function of bounded variation satisfying the condition (4) for $\eta = 2$ for which the absolute convergence does not hold.

As regards the general modulus of continuity the final answer to the question was obtained by S. V. Bochkarev [3]. He has proved that the condition (3) is necessary for the absolute convergence of all Fourier series of the class $V \cap H^\omega$.

In what follows we shall use the notation of the modulus of variation of a function introduced by us in 1973 [4].

DEFINITION. Let f be a bounded 2π-periodic function. The modulus of variation $\upsilon(n, f)$ of the function f is defined for nonnegative integers n as follows:

$$\upsilon(0, f) = O$$

and for $n \geqq 1$

$$\upsilon(n, f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(t_{2k+1}) - f(t_{2k})| ,$$

where Π_n is an arbitrary system of n disjoint intervals (t_{2k}, t_{2k+1}), $k = 0, 1, \cdots, n-1$ i.e., $0 \leqq t_0 < t_1 \leqq t_2 < \cdots \leqq t_{2k-2} < t_{2k-1} \leqq 2\pi$.

The modulus of variation of any function is nondecreasing and upwards convex. Functions of an integral argument with such properties will be said to be modulus of variation. If the modulus of variation $\upsilon(n)$ is given, then by $V[\upsilon]$ we denote the class of 2π-periodic functions for which $\upsilon(n, f) = O(\upsilon(n))$ when $n \to \infty$.

Note that the Jordan class $V = V[1]$ and the class of 2π-periodic bounded functions $M(0, 2\pi) = V[n]$; if $0 \leqq \alpha \leqq \beta \leqq 1$, then $V[1] \subset V[n^\alpha] \subset V[n^\beta] \subset M(0, 2\pi)$. In general, if $\upsilon_1(n) \leqq \upsilon_2(n)$, $n = 0, 1, \cdots$, then $V[\upsilon_1] \subset V[\upsilon_2]$.

In [4] we have extended Zygmund's theorem to the wider classes $V[n^\alpha] \cap H^\omega$ and in [5] under some restrictions on ω the necessity of the obtained condition has been proved.

2. In the present paper we extend the Zygmund and Bochkarev theorems to the classes $V[\upsilon] \cap H^{\omega}$[1].

The following theorem is valid.

THEOREM 1. *Let* $f \in H^\omega \cap V[\upsilon]$, $\upsilon(n) = o(n)$[2], $\upsilon(1) = \omega(1) = 1$,

$$\varphi(n) = \max \left\{m; \frac{\upsilon(m)}{m} \geqq \omega\left(\frac{1}{n}\right)\right\} .$$

If

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=\varphi(n)}^{n+\varphi(n)} \frac{\upsilon^2(k)}{k^2}\right)^{1/2} < \infty \tag{6}$$

then $f \in A$.

We shall deduce this theorem from the following theorem of O. Szász ([1], p. 609). If f satisfies the condition

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\omega_2\left(\frac{1}{n}, f\right) < \infty ,$$

where $\omega_2(\delta, f) = \max_{0<h\leqq\delta} \left\{\int_0^{2\pi} |f(x + h) - f(x)|\, dx\right\}^{1/2}$, then $f \in A$.

Although Theorem 1 is a corollary of the Szasz theorem it seems that our theorem is expressed in terms which are more applicable in the field under consideration than those of the Szász theorem.

In order to prove Theorem 1 we need the following lemma.

LEMMA 1. *Let* $\upsilon(n)$ *be a modulus of variation and* $\upsilon(n) = o(n)$, $\upsilon(1) > 0$. *Then there exists a natural number* n_0 *such that* $\upsilon(n)/n$

[1] Several results of this paper were published in [**6**] without proofs.

[2] This condition is natural since for continuous f $\upsilon(n, f) = o(n)$ (see [**4**]).

is decreasing for $n \geqq n_0$.

Proof. Since $\upsilon(n)$ is upwards convex, for any $n \geqq 1$

$$\upsilon(n) - \upsilon(n-1) \geqq \upsilon(n+1) - \upsilon(n) \text{ ,}$$

therefore

$$\upsilon(n) = \sum_{k=1}^{n} [\upsilon(k) - \upsilon(k-1)] \geqq n[\upsilon(n) - \upsilon(n-1)]$$

i.e.,

$$\frac{\upsilon(n)}{n} \geqq \upsilon(n) - \upsilon(n-1)$$

or

$$\frac{\upsilon(n)}{n} \geqq \frac{\upsilon(n-1)}{n-1} \text{ .}$$

So we have proved, that $\upsilon(n)/n$ does not increase. Now it is necessary to prove that $\upsilon(n)/n$ is decreasing beginning with a certain n_0. Assume the contrary. Then there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\frac{\upsilon(n_k)}{n_k} = \frac{\upsilon(n_k+1)}{n_k+1} \text{ ,} \qquad k = 1, 2, \cdots$$

or

$$\frac{\upsilon(n_k)}{n_k} = \upsilon(n_k+1) - \upsilon(n_k) \text{ ,} \qquad k = 1, 2, \cdots$$

i.e.,

$$\frac{1}{n_k} \sum_{m=1}^{n_k} [\upsilon(m) - \upsilon(m-1)] = \upsilon(n_k+1) - \upsilon(n_k) \text{ .}$$

But as $\upsilon(n) - \upsilon(n-1)$ does not increase it follows from the last equality that for $n = 1, 2, \cdots, n_k + 1$ we have

$$\upsilon(n) - \upsilon(n-1) = \upsilon(1)$$

therefore

$$\upsilon(n) = n\upsilon(1) \text{ ,} \qquad 1 \leqq n \leqq n_k + 1 \text{ .}$$

Since $\{n_k\}$ is an infinite sequence the last equality is valid for all n, i.e., $\upsilon(n) \neq o(n)$. This contradicts the conditions of the lemma.

Proof of Theorem 1. Let

$$c_1 = \sup_{0<\delta\leqq\pi} \frac{\omega(\delta, f)}{\omega(\delta)} ,$$

$$c_2 = \sup_{n\geqq 1} \frac{\upsilon(n, f)}{\upsilon(n)} ,$$

and $c_0 = \max\{c_1, c_2\}$.

Then $\omega(\delta, f) \leqq c_0\omega(\delta)$, $\delta \in [0, \pi]$ and $\upsilon(n, f) \leqq c_0\upsilon(n)$, $n \geqq 0$.

We have

$$\omega_2^2\left(\frac{1}{n}, f\right) = \max_{0<h\leqq 1/n} \int_0^{2\pi} |f(x+h) - f(x)|^2 dx = \int_0^{2\pi} |f(x+h_n) - f(x)|^2 dx ,$$

where $0 < h_n \leqq 1/n$.

Let $l_n = [2\pi/h_n]$, then since f is periodic

$$\omega_2^2\left(\frac{1}{n}, f\right) = \frac{1}{l_n} \int_0^{2\pi} \sum_{k=1}^{l_n} [f(x+kh_n) - f(x+(k-1)h_n)]^2 dx . \tag{7}$$

Consider the sum

$$\xi_n(x) = \sum_{k=1}^{l_n} [f(x+kh_n) - f(x+(k-1)h_n)]^2 , \qquad 0 \leqq x \leqq 2\pi .$$

Assume that $x_0 \in [0, \pi]$ is the point of maximum of the function ξ_n.

Consider the sets

$$S_\infty = \{k \in N;\ |f(x_0+kh_n) - f(x_0+(k-1)h_n)| = 0\} ,$$

$$S_m = \left\{k \in N;\ c_0\frac{\upsilon(2^{m+1})}{2^{m+1}} < f(x_0+kh_n) - f(x_0+(k-1)h_n)| \leqq c_0\frac{\upsilon(2^m)}{2^m}\right\} ,$$
$$m \geqq 0 .$$

Let us show that for $m < m_0(n) = [\log_2 \varphi(n)]$ the sets S_m are empty.

In fact, for any $k \in \{1, \cdots, l_n\}$ according to the definition of $\varphi(n)$ and the monotonicity of $\upsilon(n)/n$ we have

$$|f(x_0+kh_n) - f(x_0+(k-1)h_n)| \leqq \omega(h_n, f) \leqq \omega\left(\frac{1}{n}, f\right) \leqq c_0\omega\left(\frac{1}{n}\right)$$
$$\leqq c_0\frac{\upsilon(\varphi(n))}{\varphi(n)} \leqq c_0\frac{\upsilon(2^{m_0})}{2^{m_0}} .$$

So for $0 \leqq m \leqq m_0(n) - 1$, $S_m = \varnothing$.

Denote by σ_m the number of elements of S_m. It is obvious that

$$\sigma_\infty + \sum_{m=m_0}^{\infty} \sigma_m = l_n .$$

Because of

$$\sum_{k \in S_m} |f(x_0 + kh_n) - f(x_0 + (k-1)h_n)| \leqq \upsilon(\sigma_m, f) \leqq c_0\upsilon(\sigma_m) ,$$

and the definition of the sets S_m, we obtain

$$c_0\upsilon(\sigma_m) \geqq \sigma_m \cdot c_0 \frac{\upsilon(2^{m+1})}{2^{m+1}}$$

or

$$\frac{\upsilon(\sigma_m)}{\sigma_m} \geqq \frac{\upsilon(2^{m+1})}{2^{m+1}} . \tag{8}$$

But according to Lemma 1 $\upsilon(n)/n$ is decreasing beginning with a certain N, so it follows from (8) that for $m \geqq [\log_2 N] + 1$

$$\sigma_m \leqq 2^{m+1} . \tag{9}$$

If n is also large that $2^{m_0(n)} \geqq N$, then using (9) we have

$$\begin{aligned} \xi_n(x_0) &\leqq \sum_{m=m_0}^{\infty} \sum_{k \in S_m} |f(x_0 + kh_n) - f(x_0 + (k-1)h_n)|^2 \\ &\leqq c_0^2 \sum_{m=m_0}^{\infty} \sigma_m \frac{\upsilon^2(2^m)}{2^{2m}} . \end{aligned} \tag{10}$$

Let M be defined according to the condition

$$\sum_{m=m_0}^{M-1} 2^{m+1} < l_n \leqq \sum_{m=m_0}^{M} 2^{m+1} .$$

Then for $m_0(n) \geqq \log_2 N$, using (10), we obtain

$$\begin{aligned} \xi_n(x_0) &\leqq c_0^2 \left\{ \sum_{m=m_0}^{M-1} 2^{m+1} \cdot \frac{\upsilon^2(2^m)}{2^{2m}} + \left(l_n - \sum_{m=m_0}^{M-1} 2^{m+1}\right) \frac{\upsilon^2(2^{M+1})}{2^{2(M+1)}} \right\} \\ &\leqq c^3 \left\{ \sum_{m=m_0}^{M-1} \sum_{k=2^m+1}^{2^{m+1}} \frac{\upsilon^2(2^{m+1})}{2^{2(m+1)}} + \left(l_n - \sum_{m=m_0}^{M-1} 2^{m+1}\right) \frac{\upsilon^2(2^{M+1})}{2^{2(M+1)}} \right\} \\ &\leqq c \left\{ \sum_{k=2^{m_0+1}}^{2^{m_0+1}+l_n} \frac{\upsilon^2(k)}{k^2} + \sum_{k=2^{m_0}}^{2^{m_0+1}-1} \frac{\upsilon^2(k)}{k^2} \right\} \leqq c \sum_{k=2^{m_0+1}}^{2^{m_0+1}+l_n} \frac{\upsilon^2(k)}{k^2} . \end{aligned}$$

From the last estimation and from (7) by the definition of $m_0(n)$ we have

$$\omega_2^2\left(\frac{1}{n}, f\right) \leqq \frac{1}{l_n} \cdot c \sum_{k=2^{m_0+1}}^{2^{m_0+1}+l_n} \frac{\upsilon^2(k)}{k^2} \leqq cn_n \sum_{k=\varphi(n)}^{\varphi(n)+l_n} \frac{\upsilon^2(k)}{k^2} . \tag{11}$$

Now let us show that

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[3] Here and in what follows by c we denote absolute positive constants which are, in general, distinct in different formulas.

$$h_n \sum_{k=\varphi(n)}^{\varphi(n)+l_n} \frac{\upsilon^2(k)}{k^2} \leqq c\frac{1}{n} \sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{\upsilon^2(k)}{k^2}\,. \tag{12}$$

Recally, as $\upsilon(n)/n \downarrow$

$$h_n \sum_{k=\varphi(n)+n+1}^{p(n)+l_n} \frac{\upsilon^2(k)}{k^?} \leqq h_n\left(\frac{\upsilon(\varphi(n)+n)}{\varphi(n)+n}\right)^2 (l_n - n) \leqq c\left(\frac{v(\varphi(n)+n)}{\varphi(n)+n}\right)^2$$
$$\leqq c\frac{1}{n} \sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\,,$$

which since $h_n \leqq 1/n$ implies

$$h_n \sum_{k=\varphi(n)}^{\varphi(n)+l_n} \frac{v^2(k)}{k^2} = h_n\left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2} + \sum_{k=\varphi(n)+n+1}^{\varphi(n)+l_n} \frac{v^2(k)}{k^2}\right)$$
$$\leqq \frac{c}{n} \sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\,.$$

It follows from (11) and (12) that

$$\omega_2\left(\frac{1}{n}, f\right) \leqq c\frac{1}{\sqrt{n}}\left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\right)^{1/2}.$$

According to the convergence of the series (6) from the previous estimate we obtain that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2\left(\frac{1}{n}, f\right)$$

is also convergent and by the theorem of O. Szász $f \in A$. This completes the proof.

COROLLARY 1 (*S. N. Bernstein*, [1] *p.* 608). *If* $f \in H^\omega$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega\left(\frac{1}{n}\right) < \infty \tag{13}$$

then $f \in A$.

Proof. Using the monotonicity of $v(n)/n$ and the definition of $\varphi(n)$ we have

$$\left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\right)^{1/2} \leqq \left(\frac{v^2(\varphi(n))}{\varphi^2(n)} + \frac{v^2(\varphi(n)+1)}{(\varphi(n)+1)^2}\cdot n\right)^{1/2}$$
$$\leqq \frac{v(\varphi(n))}{\varphi(n)} + \sqrt{n}\,\omega\left(\frac{1}{n}\right) \leqq \frac{v(\varphi(n)+1)}{\varphi(n)+1}\cdot\frac{\varphi(n)+1}{\varphi(n)} + \sqrt{n}\,\omega\left(\frac{1}{n}\right)$$
$$\leqq c\sqrt{n}\cdot\omega\left(\frac{1}{n}\right)$$

i.e.,

$$\frac{1}{n}\left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{v^2(k)}{k^2}\right)^{1/2} \leqq c\frac{1}{\sqrt{n}}\omega\left(\frac{1}{n}\right).$$

Hence by (13) and Theorem 1 $f \in A$.

COROLLARY 2. *Let* $f \in H^\omega \cap V[n^\alpha]$, $0 \leqq \alpha < 1/2$ *and*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left[\omega\left(\frac{1}{n}\right)\right]^{1-2\alpha/2(1-\alpha)} < \infty , \tag{14}$$

then $f \in A$.

Proof. It is easy to calculate that in this case

$$\varphi(n) = \left[\omega^{-1/1-\alpha}\left(\frac{1}{n}\right)\right].$$

Then

$$\left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{v^2(k)}{k^2}\right)^{1/2} \leqq \left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{1}{k^{2-2\alpha}}\right)^{1/2} \leqq c\frac{1}{(\varphi(n))^{1-2\alpha/2}}$$
$$\leqq c\left(\omega\left(\frac{1}{n}\right)\right)^{1-2\alpha/2(1-\alpha)} .$$

From this estimate, (14) and Theorem 1 it follows that $f \in A$.

In particular, for $\alpha = 0$ Corollary 2 implies Zygmund's theorem.

COROLLARY 2′. *If* $f \in H^\omega \cap V[n^\alpha]$, *where* $0 \leqq \alpha < 1/2$ *and* $\omega(\delta) = (\ln 1/\delta)^{-(2(1-\alpha)/1-2\alpha)-\varepsilon}$, $\varepsilon > 0$ *then* $f \in A$.

COROLLARY 3. *If* $f \in H^\omega \cap V[\ln^\beta(n+1)]$, *where* $0 < \beta < \infty$ *and*

$$\sum_{n=1}^{\infty}\frac{1}{n}\omega^{1/2}\left(\frac{1}{n}\right)\ln^{\beta/2}\frac{1}{\omega(1/n)} < \infty \tag{15}$$

then $f \in A$.

Proof. It is easy to calculate that for sufficiently large n

$$\omega^{-1}\left(\frac{1}{n}\right)\ln^\beta\frac{1}{\omega(1/n)} \leqq \varphi(n) \leqq 2\omega^{-1}\left(\frac{1}{n}\right)\ln^\beta\frac{1}{\omega(1/n)} .$$

Then

$$\left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{v^2(k)}{k^2}\right)^{1/2} = \left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{\ln^{2\beta}(k+1)}{k^2}\right)^{1/2} \leqq c\frac{\ln^\beta\varphi(n)}{\sqrt{\varphi(n)}}$$
$$\leqq c\omega^{1/2}\left(\frac{1}{n}\right)\ln^{\beta/2}\frac{1}{\omega(1/n)} .$$

This estimate, (15) and Theorem 1 imply that $f \in A$.

COROLLARY 3′. *If* $f \in H^{\omega} \cap V[\ln^{\beta}(n+1)]$, *where* $0 < \beta < \infty$ *and*

$$\omega(\delta) = \left(\ln \frac{1}{\delta}\right)^{-2}\left(\ln \ln \frac{1}{\delta}\right)^{-(\beta+2+\varepsilon)}, \qquad \varepsilon > 0$$

then $f \in A$.

COROLLARY 4. *Let* $f \in H^{\omega} \cap V[n^{1/2} \ln^{-\beta}(n+1)]$, *where* $3/2 < \beta < \infty$ *and*

$$\sum \frac{1}{n} \ln^{1/2-\beta}\left(\frac{1}{\omega(1/n)} + 1\right) < \infty \tag{16}$$

then $f \in A$.

Proof. It may be calculated that for sufficiently large n

$$\varphi(n) \sim \omega^{-2}\left(\frac{1}{n}\right) \ln^{-2\beta} \frac{1}{\omega(1/n)}\,.$$[4]

Then

$$\left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\right)^{1/2} = \left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{1}{k \ln^{2\beta}(k+1)}\right)^{1/2} \leqq \frac{c}{\ln^{\beta-1/2} \varphi(n)}$$
$$\leqq c \ln^{1/2-\beta}\left(\frac{1}{\omega(1/n)} + 1\right).$$

This by (16) and Theorem 1 imply $f \in A$.

COROLLARY 4′. *If* $f \in \operatorname{Lip} \alpha \cap V[\sqrt{n} \ln^{-\beta}(n+1)]$, *where* $\alpha > 0$ *and* $\beta \in (3/2, \infty)$, *then* $f \in A$.

REMARK. Theorem 1 implies that for any specific ω for which the series (3) converges Zygmund's theorem may be improved. E.g., if $f \in H^{\omega}$, where $\omega(\delta) = (\ln 1/\delta)^{-3}$ then the function f may have the modulus of variation of order $n^{1/4-\varepsilon}$, $\varepsilon > 0$ but the Fourier series of f is still absolutely convergent; or if $f \in H^{\omega}$ where $\omega(\delta) \sim \ln^{-2} 1/\delta (\ln \ln 1/\delta)^{-3}$ in a neighborhood of $\delta = 0$, then the function f may have the modulus of variation of order $\ln^{1-\varepsilon} n$, $\varepsilon > 0$ to provide $f \in A$.

3. Now we shall show that the condition (6) is necessary for

[4] We write $\upsilon_1(n) \sim \upsilon_2(n)$ if there exist positive constants A and B such that
$$A\upsilon_1(n) \leqq \upsilon_2(n) \leqq B\upsilon_1(n)$$
for all n.

absolute convergence of Fourier series of class $H^{\omega} \cap V[v]$ in a sufficiently wide class of moduluses of variation and for arbitrary moduluses of continuity. But first we formulate same subsidiary statements which will be needed below.

LEMMA 2 (*Gauss identity* [11], *p.* 81). *If k is a natural number and n is an integer, then*

$$\left|\sum_{\tau=0}^{2k} e^{2\pi i}\right| = \sqrt{2k+1}\ .$$

LEMMA 3 (*Cauchy* [7], *p.* 290). *If the sequence $\{a_n\}$ is almost decreasing*[5], *then the series $\sum a_n$ is convergent or divergent, together with the series $\sum p^n a_{p^n}$ where $p > 1$ is a natural number.*

LEMMA 4 (*N. K. Bari, S. B. Stechkin* [2]). *The following statements are equivalent for a positive sequence $\{a_n\}$:*

(a) $\exists \varepsilon > 0$ *such that $n^{\varepsilon} a_n$ is almost decreasing* $\Leftrightarrow$

$$\sum_{k=n+1}^{\infty} \frac{1}{k} a_k = O(a_n) \qquad for \quad n \longrightarrow \infty\ ;$$

(b) $\exists \varepsilon > 0$ *such that $n^{1-\varepsilon} a_n$ is almost increasing*[6] $\Leftrightarrow$

$$\sum_{k=1}^{n} a_k = O(n a_n) \qquad for \quad n \longrightarrow \infty\ .$$

LEMMA 5 (*I. Wik* [13]). *Let a positive sequence $\{a_n\}$ be bounded and $\sum a_n = \infty$. Then for any $\alpha \in (0, 1)$ and $\beta > 1$ there exists a sequence of natural numbers $\{q_\nu\}$ such that*

$$\sum_{\nu=1}^{\infty} a_{q_\nu} = \infty$$

and

$$\alpha^{q_{\nu+1}-q_\nu} \leqq \frac{a_{q_\nu}}{a_{q_{\nu+1}}} \leqq \beta^{q_{\nu+1}-q_\nu}\ , \qquad \nu = 1, 2, \cdots\ .$$

LEMMA 6 ([10], *p.* 111). *For any modulus of continuity $\omega(\delta)$*

$$\frac{\omega(\delta_1)}{\delta_1} \leqq 2\frac{\omega(\delta_2)}{\delta_2}$$

for $0 < \delta_2 < \delta_1$.

[5] i.e., there exists a constant $c > 1$ such that for all n and $m > n$ $a_n \geqq (1/c)a_m$.

[6] a sequence $\{a_n\}$ is almost increasing, if there exists a constant $c > 1$ such that for any n and $m > n$ $a_n \leqq c a_m$.

LEMMA 7 ([5]). *Let $\{a_n\}$ be almost decreasing and $\sum(1/n)a_n = \infty$. If*

$$b_n = \min\left\{a_n, \frac{1}{\ln(n+1)}\right\}, \qquad n = 1, 2, \cdots$$

then

$$\sum \frac{1}{n} b_n = \infty .$$

LEMMA 8. *Let $\lim_{m\to\infty} f_m(x) = f_0(x)$ for any $x \in [0, 2\pi]$. Then for any $n \geqq 1$*

$$v(n, f_0) \leqq \sup_m v(n, f_m) .$$

Proof. By the definition of $v(n, f)$ for any $\varepsilon > 0$ there exist $2n$ points $0 \leqq x_0^{(\varepsilon)} < x_1^{(\varepsilon)} \leqq x_2^{(\varepsilon)} < \cdots < x_{2n-1}^{(\varepsilon)} \leqq 2\pi$ such that

$$v(n, f_0) < \sum_{k=0}^{n-1} |f_0(x_{2k+1}^{(\varepsilon)}) - f_0(x_{2k}^{(\varepsilon)})| + \varepsilon .$$

As

$$f_m(x) \longrightarrow f_0(x)$$

for any $x \in [0, 2\pi]$ there exists m_0 such that for $m > m_0$

$$|f_m(x_k^{(\varepsilon)}) - f_0(x_k^{(\varepsilon)})| < \frac{\varepsilon}{2n} \qquad \text{for} \quad k = 0, 1, \cdots, 2n-1 .$$

Then

$$\begin{aligned}\sum_{k=0}^{n-1} |f_0(x_{2k+1}^{(\varepsilon)}) - f_0(x_{2k}^{(\varepsilon)})| &\leqq \sum_{k=0}^{n-1} |f_0(x_{2k+1}^{(\varepsilon)}) - f_m(x_{2k+1}^{(\varepsilon)})| \\ &\quad + \sum_{k=0}^{n-1} |f_m(x_{2k}^{(\varepsilon)}) - f_0(x_{2k}^{(\varepsilon)})| + \sum_{k=0}^{n-1} |f_m(x_{2k+1}^{(\varepsilon)}) - f_m(x_{2k}^{(\varepsilon)})| \\ &\leqq \frac{\varepsilon}{2n} n + \frac{\varepsilon}{2n} \cdot n + v(n, f_m) = \varepsilon + v(n, f_m)\end{aligned}$$

and so

$$v(n, f_0) \leqq 2\varepsilon + v(n, f_m)$$

for $m > m_0$. This completes the proof.

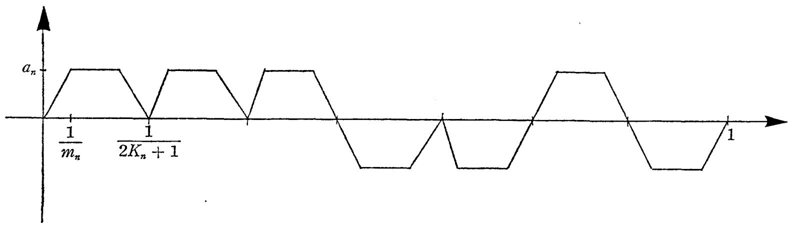
Now we shall prove

THEOREM 2. *If there exists $\varepsilon \in (0, 1/2)$ such that $n^{\varepsilon-1/2}v(n)$ is almost decreasing, $n^{-\varepsilon}v(n)$ is almost increasing and $\omega(\delta)$ satisfies the condition*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=\varphi(n)}^{\varphi(n)+n}\frac{v^2(k)}{k^2}\right)^{1/2}=\infty \tag{17}$$

then there exists a function in the class $H^\omega \cap V[v]$ *for which the series* (2) *is divergent.*

Before we prove this theorem let us explain how the function f_0 is constructed. The complex function of the real variable f_0 is the infinite sum of the functions with real part of type



(the imaginary part is analogous). The three parameters k_n, the altitude (a_n) and the slope (m_n) are selected in such a may that (1) $f_0 \in H^\omega$, (2) $f_0 \in V[v]$ and (3) $f_0 \overline{\in} A$.

REMARK. Note that according to Lemma 4 the first condition imposed on $v(n)$ is equivalent to the following one

$$\sum_{n=N}^{\infty}\frac{v^2(n)}{n^2}=O\left(\frac{v^2(N)}{N}\right),$$

and the second condition is equivalent to

$$\sum_{n=1}^{N}\frac{v(n)}{n}=O(v(N)).$$

Proving this theorem we use the concept of Wik to apply the Gauss identity.

For proving Theorem 2 we need some constructions.

Let m and k be natural numbers and $m>2(2k+1)$. Define the function $F_{m,k}$ as follows

$$F_{m,k}(x)=\begin{cases}\left.\begin{array}{ll}\dfrac{m}{2k+1}e^{i(2\pi/2k+1)\tau^2} & \text{for } \dfrac{\tau}{2k+1}<x<\dfrac{\tau}{2k+1}+\dfrac{1}{m}\\ -\dfrac{m}{2k+1}e^{i(2\pi/2k+1)\tau^2} & \text{for } \dfrac{\tau+1}{2k+1}-\dfrac{1}{m}<x<\dfrac{\tau+1}{2k+1}\end{array}\right\}\tau=0,1,\cdots,2k\\ 0 \text{ in other points of } [0,1],\\ \text{outside of } [0,1] \text{ continue periodically with the period of } 1.\end{cases}$$

Let

$$f_{m,k}(x) = \int_0^x F_{m,k}(t)dt \ .$$

As $f_{m,k}(1) = f_{m,k}(0) = 0$, $f_{m,k}$ is continuous and periodic with the period of 1.

We shall estimate the modulus of variation and the modulus of continuity of this function.

LEMMA 9. *The modulus of continuity of the function $f_{m,k}$ is subjected to the estimate*

$$\omega\left(\frac{1}{n}, f_{m,k}\right) = \begin{cases} \dfrac{2m}{2k+1} \cdot \dfrac{1}{n} & for \quad n > m \\ \dfrac{2}{2k+1} & for \quad 1 \leqq n \leqq m \ . \end{cases}$$

Proof. Since $\max_{0 \leqq x \leqq 1} |F_{m,k}(x)| = m/2k+1$ the estimate for $n > m$ follows from the finite increment formula.

If $x \in [\tau/2k+1, \tau+1/2k+1]$, $\tau = 0, 1, \cdots, 2k$

$$\begin{aligned} |f_{m,k}(x)| &= \left|\int_0^x F_{m,k}(t)dt\right| = \left|\int_0^{\tau/2k+1+1/m} F_{m,k}(t) + \int_{\tau/2k+1}^x F_{m,k}(t)dt\right| \\ &= \left|\int_{\tau/2k+1}^x F_{m,k}(t)dt\right| \leqq \int_{\tau/2k+1}^{\tau/2k+1} |F_{m,k}(t)|dt = \frac{1}{2k+1} \ , \end{aligned} \tag{18}$$

i.e., $\max |f_{m,k}(x)| \leqq 1/2k+1$.

From this estimate for $n \leqq m$ it follows that

$$\omega\left(\frac{1}{n}, f_{m,k}\right) \leqq 2 \max |f_{m,k}| \leqq \frac{1}{2k+1} \ .$$

This completes the proof.

LEMMA 10. *The modulus of variation of the function $f_{m,k}$ is subjected to the estimate*

$$v(n, f_{m,k}) \leqq \begin{cases} \dfrac{2}{2k+1} n & for \quad 1 \leqq n \leqq 2k+1 \\ 2 & for \quad n > 2k+1 \ . \end{cases}$$

Proof. For $n = 1, \cdots, 2k+1$ using (18) we obtain

$$v(n, f_{m,k}) = \sup \sum |f_{m,k}(x_{2i+1}) - f_{m,k}(x_{2i})| \leqq 2n \max |f_{m,k}| \leqq \frac{2}{2k+1} n \ .$$

Now let $n > 2k+1$, then

$$v(n, f_{m,k}) \leqq Var f_{m,k} = \int_0^1 |F_{m,k}(t)|\, dt = \sum_{\tau=0}^{2k} 2 \int_{(\tau/2k+1)}^{(\tau/2k+1)+1/m} |F_{m,k}(t)|\, dt$$
$$= 2 \cdot \frac{m}{2k+1} \cdot \frac{1}{m} \cdot \left(2k+1\right) = 2\,.$$

Now we shall estimate the Fourier coefficients of the function $f_{m,k}$.

LEMMA 11. *The Fourier coefficients of the function $f_{m,k}$ are subjected to the relations*

$$(19) \qquad |c_n(f_{m,k})| \begin{cases} \leqq \dfrac{m}{\pi^2 n^2 \sqrt{2k+1}} & \text{for } \; n \geqq \dfrac{m}{2} \\ \leqq \dfrac{1}{\pi n \sqrt{2k+1}} \left| \sin \pi n \left(\dfrac{1}{2k+1} - \dfrac{1}{m} \right) \right| & \text{for } \; 0 < n < \dfrac{m}{2} \\ \geqq \dfrac{2}{\pi^2 n \sqrt{2k+1}} \left| \sin \pi n \left(\dfrac{1}{2k+1} - \dfrac{1}{m} \right) \right| & \text{for } \; 0 < n < \dfrac{m}{2}\,. \end{cases}$$

Proof. Since $f_{m,k}(0) = f_{m,k}(1) = 0$,

$$|c_n(f_{m,k})| = \left| \int_0^1 f_{m,k}(t) e^{2\pi i n t} dt \right| = \frac{1}{2\pi n} \left| \int_0^1 F_{m,k}(t) e^{2\pi i n t} dt \right|$$
$$= \frac{1}{2\pi n} \cdot \frac{m}{2k+1} \left| \sum_{\tau=0}^{2k} e^{2\pi i \tau^2/(2k+1)} \left[\int_{\tau/(2k+1)}^{\tau/(2k+1)+1/m} e^{2\pi i n t} dt - \int_{(\tau+1)/(2k+1)-1/m}^{(\tau+1)/(2k+1)} e^{2\pi i n t} dt \right] \right|$$
$$= \frac{1}{4\pi^2 n^2} \cdot \frac{m}{2k+1} \left| \sum_{\tau=0}^{2k} e^{2\pi i/(2k+1)(\tau^2+\tau n)} \right| \cdot \left| e^{2\pi i n/m} - 1 + e^{2\pi i n(1/(2k+1)-1/m)} \right.$$
$$\left. - \, e^{2\pi i n/(2k+1)} \right|.$$

Using the Gauss identity we obtain

$$(20) \qquad \begin{aligned} |c_n(f_{m,k})| &= \frac{1}{4\pi^2 n^2} \cdot \frac{m}{2k+1} \sqrt{2k+1}\, |e^{2\pi i n/m} - 1|\, |1 - e^{2\pi i n(1/2k+1)-1/m}| \\ &= \frac{1}{\pi^2 n^2} \cdot \frac{m}{\sqrt{2k+1}} \left| \sin \frac{\pi n}{m} \right| \cdot \left| \sin \pi n \left(\frac{1}{2k+1} - \frac{1}{m} \right) \right|. \end{aligned}$$

For $n \geqq m/2$ this implies the estimation (19) trivially. The upper bounds for $n < m/2$ may be obtained from (20) if use the relation

$$\left| \sin \frac{\pi n}{m} \right| \leqq \frac{\pi n}{m}$$

and the lower bound for $n < m/2$ follows from the relation

$$\left| \sin \frac{\pi n}{m} \right| \geqq \frac{2}{\pi} \cdot \frac{\pi n}{m} \qquad \text{for } \; 0 < n < \frac{m}{2}\,.$$

This completes the proof.

Besides, for proving Theorem 2 we need to construct a special sequence of natural numbers.

Since $n^{\varepsilon-1/2}v(n)$ is almost decreasing for a certain $\varepsilon > 0$ $n^{2\varepsilon-1}v^2(n)$ is also almost decreasing. Then according to Lemma 4

$$\sum_{k=n}^{\infty} \frac{v^2(k)}{k^2} = \sum_{k=n}^{\infty} \frac{1}{k} \frac{v^2(k)}{k} = O\left(\frac{v^2(n)}{n}\right).$$

Hence

$$\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2} \leqq c\frac{v^2(\varphi(n))}{\varphi(n)}.$$

This with the divergence of the series (17) implies

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{v(\varphi(n))}{\sqrt{\varphi(n)}} = \infty. \tag{21}$$

As $n^{-1/2+\varepsilon}v(n)$ is almost decreasing all the more $v(n)/\sqrt{n}$ is almost decreasing and since $\varphi(n)$ is increasing when $n \to \infty$ the sequence $v(\varphi(n))/\sqrt{\varphi(n)}$ is almost decreasing. Now using Lemma 7 and the divergence of the series (21) we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \gamma_n = \infty, \tag{22}$$

where

$$\gamma_n = \min\left\{\frac{v(\varphi(n))}{\sqrt{\varphi(n)}}, \frac{1}{\ln(n+1)}\right\}.$$

As far as $n^{-\varepsilon}v(n)$ is almost increasing and $n^{\varepsilon-1/2}v(n)$ is almost decreasing there exist a constant $c_1 \leqq 1$ such that for any k and $n > k$

$$\frac{v(n)}{n^{\varepsilon}} \geqq c_1 \frac{v(n)}{k^{\varepsilon}} \tag{23}$$

and

$$\frac{v(n)}{n^{1/2-\varepsilon}} \leqq \frac{1}{c_1} \frac{v(k)}{k^{1/2-\varepsilon}}. \tag{24}$$

Now set

$$c_0 = \left[25\left(\frac{2}{c_1}\right)^{1/2\varepsilon}\right] + 1.$$

Twice appling Lemma 3 to the series (22) at first with $p = 2$

and then with $p = c_0$ we get

$$\sum_{n=1}^{\infty} c_0^n \gamma_{2c_0^n} = \infty \ . \tag{25}$$

Since

$$c_0^n \gamma_{2c_0^n} \leqq c_0^n \frac{1}{\ln(2^{c_0^n}+1)} < c \ ,$$

the terms of the series (25) are uniformly bounded so we may apply Lemma 5. Set $\alpha = 4/5$ and $\beta = 6/5$. For these numbers there exists a sequence of natural numbers $\{q_n\}$ such that

$$\sum_{n=1}^{\infty} c_0^{q_n} \gamma_{2c_0^{q_n}} = \infty \tag{26}$$

and

$$\left(\frac{4}{5}\right)^{q_{n+1}-q_n} \leqq \frac{c_0^{q_n}\gamma_{2c_0^{q_n}}}{c_0^{q_{n+1}}\gamma_{2c_0^{q_{n+1}}}} \leqq \left(\frac{6}{5}\right)^{q_{n+1}-q_n} \ . \tag{27}$$

(27) implies

$$\left(\frac{4}{5}c_0\right)^{q_{n+1}-q_n} \leqq \frac{\gamma_{2c_0^{q_n}}}{\gamma_{2c_0^{q_{n+1}}}} \leqq \left(\frac{6}{5}c_0\right)^{q_{n+1}-q_n} \tag{28}$$

and as

$$\frac{4}{5}c_0 \geqq 20\left(\frac{2}{c_1}\right)^{1/2\varepsilon} > 1$$

the sequence $\{\gamma_{2c_0^{q_n}}\}$ is decreasing.

Define the sequence

$$\psi(n) = \max\left\{m;\ \frac{v(m)}{\sqrt{m}} \geqq \gamma_{2c_0^{q_n}}\right\} \ .$$

By the definition of γ_n we have

$$\frac{v(\varphi(2c_0^{q_n}))}{\sqrt{\varphi(2c_0^{q_n})}} \geqq \gamma_{2c_0^{q_n}} \ ,$$

which with the definition of $\psi(n)$ imply

$$\psi(n) \geqq \varphi(2c_0^{q_n}) \ .$$

Using $\gamma_{2c_0^{q_n}} \downarrow 0$ and $v(n)/\sqrt{n} \downarrow 0$ (this is true because $n^{-1/2+\varepsilon}v(n)$ is almost decreasing) we get $\psi(n) \uparrow \infty$ for $n \to \infty$.

According to the definition of $\varphi(n)$ and using the relation $v(n)/n \downarrow$ we have

$$\omega\Big(\frac{1}{2^{c_0^{q_n}}}\Big) > \frac{v(\varphi(2^{c_0^{q_n}})+1)}{\varphi(2^{c_0^{q_n}})+1} \geqq \frac{1}{2}\,\frac{v(\varphi(2^{c_0^{q_n}}))}{\varphi(2^{c_0^{q_n}})} \geqq \frac{1}{2}\,\frac{v(\psi(n))}{\psi(n)}\,. \tag{29}$$

Furthermore

$$\frac{v(\psi(n))}{\sqrt{\psi(n)}} \geqq \gamma_{2^{c_0^{q_n}}} > \frac{v(\psi(n)+1)}{\sqrt{\psi(n)+1}} > \frac{1}{2}\,\frac{v(\psi(n))}{\sqrt{\psi(n)}}\,. \tag{30}$$

The last relation with (28) implies

$$\frac{v(\psi(n))}{\sqrt{\psi(n)}} : 2\frac{v(\psi(n+1))}{\sqrt{\psi(n+1)}} \geqq \frac{\gamma_{2^{c_0^{q_n}}}}{\gamma_{2^{c_0 q_{n+1}}}} \geqq \Big(\frac{4}{5}c_0\Big)^{q_{n+1}-q_n} \tag{31}$$

or

$$\frac{\sqrt{\psi(n+1)}}{\sqrt{\psi(n)}} \geqq \frac{1}{2}\Big(\frac{4}{5}\Big)^{q_{n+1}-q_n}\cdot\frac{v(\psi(n+1))}{v(\psi(n))} > \Big(\frac{2}{5}c_0\Big)^{q_{n+1}-q_n}$$

i.e.,

$$\frac{\psi(n+1)}{\psi(n)} \geqq \Big[100\cdot\Big(\frac{2}{c_1}\Big)^{1/\varepsilon}\Big]^{q_{n+1}-q_n} > 100\Big(\frac{2}{c_1}\Big)^{1/\varepsilon}\,. \tag{32}$$

(23) and (32) imply

$$\frac{v(\psi(n+1))}{v(\psi(n))} \geqq c_1\Big(\frac{\psi(n+1)}{\psi(n)}\Big)^{\varepsilon} \geqq c_1\Big(100\Big(\frac{2}{c_1}\Big)^{1/\varepsilon}\Big)^{\varepsilon} \geqq c_1\cdot 100^{\varepsilon}\cdot\frac{2}{c_1} > 2\,. \tag{33}$$

From (28) and (30) it follows that

$$\frac{v(\psi(n))}{2\sqrt{\psi(n)}} : \frac{v(\psi(n+1))}{\sqrt{\psi(n+1)}} \leqq \frac{\gamma_{2^{c_0^{q_n}}}}{\gamma_{2^{c_0^{q_{n+1}}}}} \leqq \Big(\frac{6}{5}c_0\Big)^{q_{n+1}-q_n} \tag{34}$$

or

$$\Big\{\frac{v(\psi(n))}{\psi(n)^{1/2-\varepsilon}} : \frac{v(\psi(n+1))}{\psi(n+1)^{1/2-\varepsilon}}\Big\}\cdot\Big(\frac{\psi(n+1)}{\psi(n)}\Big)^{\varepsilon} \leqq 2\Big(\frac{6}{5}c_0\Big)^{q_{n+1}-q_n}\,. \tag{35}$$

From (35) using (23) we get

$$\frac{\psi(n+1)}{\psi(n)} \leqq \Big(\frac{2}{c_1}\Big)^{1/\varepsilon}\Big(\frac{6}{5}c_0\Big)^{1/\varepsilon(q_{n+1}-q_n)}$$

and thus

$$\begin{aligned}\frac{2^{c_0^{q_n}}}{\psi(n)} : \frac{2^{c_0^{q_{n+1}}}}{\psi(n+1)} &= \frac{2^{c_0^{q_n}}}{2^{c_0^{q_{n+1}}}}\cdot\frac{\psi(n+1)}{\psi(n)}\\ &\leqq \frac{2^{c_0^{q_n}}}{2^{c_0^{q_{n+1}}}}\cdot\Big(\frac{2}{c_1}\Big)^{1/\varepsilon}\Big(\frac{6}{5}c_0\Big)^{1/\varepsilon(q_{n+1}-q_n)}\,.\end{aligned} \tag{36}$$

Similarly (32) and (24) give

$$\begin{aligned}(37)\qquad &\frac{v(\psi(n+1))}{\psi(n+1)} : \frac{v(\psi(n))}{\psi(n)} = \left[\frac{v(\psi(n+1))}{\psi(n+1)^{1/2-\varepsilon}} : \frac{v(\psi(n))}{\psi(n)^{1/2-\varepsilon}}\right] \cdot \left(\frac{\psi(n)}{\psi(n+1)}\right)^{1/2+\varepsilon} \\ &\leqq \frac{1}{c_1}\left(\frac{\psi(n)}{\psi(n+1)}\right)^{1/2} \leqq \frac{1}{c_1}\frac{1}{10}\left(\frac{c_1}{2}\right)^{1/2\varepsilon} \leqq \frac{1}{10}\left(\frac{1}{2}\right)^{1/2\varepsilon} .\end{aligned}$$

We may as well assume that

$$2^{c_0^{q_n}} > 6\psi(n) .$$

Really, let $N_1 = \{n; 2^{c_0^{q_n}} \leqq 6\psi(n)\}$. Then

$$\begin{aligned}\sum_{n \in N_1} c_0^{q_n} \cdot \frac{v(\psi(n))}{\sqrt{\psi(n)}} &\leqq \sqrt{6} \sum_{n \in N_1} c_0^{q_n} \frac{v(6\psi(n))}{\sqrt{6\psi(n)}} \\ &\leqq \sqrt{6} \sum_{n \in N_1} c_0^{q_n} \cdot c_1 \frac{v(2^{c_0^{q_n}})}{\sqrt{2^{c_0^{q_n}}}} \leqq c \sum c_0^{q_n} \frac{v(2^{c_0^{q_n}})}{(2^{c_0^{q_n}})^{1/2-\varepsilon}} \cdot \frac{1}{(2^{c_0^{q_n}})^{\varepsilon}} \\ &\leqq c \sum_{n \in N_1} c_0^{q_n} (2^{c_0^{q_n}})^{-\varepsilon} < \infty ,\end{aligned}$$

all the more (see (30))

$$\sum_{n \in N_1} c_0^{q_n} \gamma_{2^{c_0^{q_n}}} < \infty .$$

This with (26) imply that the set N_1 may be neglected.

Hence we have constructed the sequences $\{q_n\}$ and $\{\psi(n)\}$ which satisfy the following conditions:

(1) The series (26) is divergent

(2) The sequence $\{\psi(n)\}$ satisfies the relations (32), (33), (34), (35).

(3) The inequality

$$2^{c_0^{q_n}} > 6\psi(n) , \qquad n = 1, 2, \cdots$$

holds.

Set $m_n = 2^{c_0^{q_n}}$, $k_n = \psi(n)$ and $a_n = v(k_n)$. Since $m_n > 6k_n > 2(2k_n + 1)$ the function

$$f_0(x) = \sum_{n=1}^{\infty} a_n f_{m_n, k_n}(x) \equiv \sum_{n=1}^{\infty} a_n f_n(x) \tag{38}$$

may be considered. We shall show that this function is just that one which satisfies the conditions of Theorem 2.

At first we shall prove that f_0 is continuous. Recally, according to the choice of a_n and to the inequality (18) we have

$$\| a_n f_n \|_C \leqq v(k_n) \cdot \frac{1}{2k_n + 1} \leqq \frac{v(k_n)}{\sqrt{k_n}} \cdot \frac{1}{\sqrt{k_n}} \leqq \frac{c}{\sqrt{k_n}} = \frac{c}{\sqrt{\psi(n)}} .$$

From this estimation by (32) it follows that the series (38) is convergent uniformly on [0, 1], thus $f_0 \in C(0, 1)$ and as f_n is periodic

f_0 is also periodic.

According to Lemma 8 and Lemma 10 for $2k_{n-1}+1<\nu\leqq 2k_n+1$ we have

$$v(\nu, f_0)=\sum_{j=1}^{n-1}a_jv(\nu, f_j)+\sum_{j=n}^{\infty}a_jv(\nu, f_j)$$
$$\leqq c\left\{\sum_{j=1}^{n-1}a_j+\sum_{j=1}^{\infty}\frac{\nu}{k_j}a_j\right\}\leqq c\left\{\sum_{j=1}^{n-1}v(\psi(j))+\nu\sum_{j=n}^{\infty}\frac{v(\psi(j))}{\psi(j)}\right\}.$$

Now if we use the estimations (33) in the first sum and the estimations (37) in the second one we shall get

$$v(\nu, f_0)\leqq c\left\{v(k_{n-1})+\nu\frac{v(k_n)}{k_n}\right\}\leqq cv(\nu)$$

i.e., $f_0\in V[v]$.

Now we shall show that $f_0\in H^{\omega}$.

Let $m_{n-1}<\nu\leqq m_n$. Then using Lemma 9 we get

$$\omega\left(\frac{1}{\nu}, f_0\right)\leqq\sum_{j=1}^{n-1}a_j\omega\left(\frac{1}{\nu}, f_j\right)+\sum_{j=n}^{\infty}a_j\omega\left(\frac{1}{\nu}, f_j\right)$$
$$\leqq c\left\{\sum_{j=1}^{n-1}a_j\frac{m_j}{k_j}\cdot\frac{1}{\nu}+\sum_{j=n}^{\infty}\frac{a_j}{k_j}\right\}$$
$$=c\left\{\frac{1}{\nu}\cdot\frac{a_{n-1}m_{n-1}}{k_{n-1}}\sum_{j=1}^{n-1}\frac{a_jm_j}{k_j}\cdot\frac{k_{n-1}}{a_{n-1}m_{n-1}}+\sum_{j=n}^{\infty}\frac{a_j}{k_j}\right\}.$$

With a view to estimate the first sum we shall use the relation (36), to estimate the second one the relation (37) and then to estimate the both sums the relation (29), we obtain

$$\omega\left(\frac{1}{\nu}, f_0\right)\leqq c\left\{\frac{m_{n-1}}{\nu}\frac{a_{n-1}}{k_{n-1}}+\frac{a_n}{k_n}\right\}$$
$$\leqq c\left\{\frac{m_{n-1}}{\nu}\frac{v(\psi(n-1))}{\psi(n)}+\frac{v(\psi(n))}{\psi(n)}\right\}\tag{39}$$
$$\leqq c\left\{\frac{m_{n-1}}{\nu}\omega\left(\frac{1}{m_{n-1}}\right)+\omega\left(\frac{1}{m_n}\right\}\right).$$

But as $\omega(\delta)$ is nondecreasing and for $\omega(\delta)/\delta$ Lemma 6 is valid (39) implies

$$\omega\left(\frac{1}{\nu}, f_0\right)\leqq c\omega\left(\frac{1}{\nu}\right)$$

i.e., $f_0\in H^{\omega}$.

It remains to prove that $f\,\overline{\in}\,A$.

Let $m_{n-1}/2<\nu\leqq m_n/2$, then using Lemma 11 we get

$$|c_\nu(f_0)| = \left|\int_0^1 e^{2\pi i\nu x}f(x)dx\right| \geqq \left|\int_0^1 e^{2\pi i\nu x}a_n f_n(x)dx\right| - \sum_{j=1}^{n-1} a_j \left|\int_0^1 f_j(x)e^{2\pi i\nu x}dx\right|$$

$$(40) \qquad - \sum_{j=n+1}^{\infty} a_j \left|\int_0^1 f_j(x)e^{2\pi i\nu x}dx\right| \geqq \frac{2}{\pi^2}\cdot\frac{1}{\nu}\frac{a_n}{\sqrt{2k_n+1}}\left|\sin \pi\nu\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)\right|$$

$$- \frac{1}{\pi^2}\sum_{j=1}^{n-1}\frac{a_j m_j}{\nu^2\sqrt{2k_j+1}} - \frac{1}{\pi}\cdot\frac{1}{\nu}\sum_{j=n+1}^{\infty}\frac{a_j}{\sqrt{2k_j+1}}\,.$$

By the choice of a_j, m_j, k_j and by the inequality (34)

$$\frac{1}{\pi^2\nu^2}\sum_{j=1}^{n-1}\frac{a_j m_j}{\sqrt{2k_j+1}} \leqq \frac{c}{\nu^2}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}}\sum_{j=1}^{n-1}\frac{a_j m_j}{\sqrt{k_j}}\cdot\frac{\sqrt{k_{n-1}}}{a_{n-1}m_{n-1}}$$

$$(41) \qquad = \frac{c}{\nu^2}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}}\sum_{j=1}^{n-1}\left[\frac{v(\psi(j))}{\sqrt{\psi(j)}} : \frac{v(\psi(n-1))}{\sqrt{\psi(n-1)}}\right]\cdot\frac{2^{c_0^{q_j}}}{2^{c_0^{q_{n-1}}}}$$

$$\leqq \frac{c}{\nu^2}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}}\sum_{j=1}^{n-1}2^{n-1-j}\left(\frac{6}{5}c_0\right)^{q_{n-1}-q_j}2^{-(c_0^{q_{n-1}}-c_0^{q_j})} \leqq \frac{c}{\nu^2}\frac{a_{n-1}m_{n-1}}{k_{n-1}}\,.$$

Using (31) we obtain

$$\frac{1}{\pi}\cdot\frac{1}{\nu}\sum_{j=n+1}^{\infty}\frac{a_j}{\sqrt{2k_j+1}} \leqq \frac{1}{\sqrt{2}\,\pi}\cdot\frac{1}{\nu}\frac{a_n}{\sqrt{k_n}}\sum_{j=n+1}^{\infty}\frac{v(\psi(j))}{\sqrt{\psi(j)}} : \frac{v(\psi(n))}{\sqrt{\psi(n)}}$$

$$\leqq \frac{1}{\sqrt{2}\,\pi}\cdot\frac{1}{\nu}\frac{a_n}{\sqrt{k_n}}\sum_{j=n+1}^{\infty}2^{j-n}\left(\frac{5}{4c_0}\right)^{q_j-q_n},$$

but as

$$(42) \qquad 2^{j-n}\left(\frac{5}{4c_0}\right)^{q_j-q_n} \leqq \left(\frac{5}{2c_0}\right)^{j-n} \leqq \left(\frac{1}{10}\left(\frac{c_1}{2}\right)^{1/2\varepsilon}\right)^{j-n} < \left(\frac{1}{20}\right)^{j-n}$$

we have

$$(43) \qquad \begin{aligned}\frac{1}{\pi}\cdot\frac{1}{\nu}\sum_{j=n+1}^{\infty}\frac{a_j}{\sqrt{2k_j+1}} &\leqq \frac{1}{\sqrt{2}\,\pi}\cdot\frac{1}{\nu}\cdot\frac{a_n}{\sqrt{k_n}}\sum_{j=1}^{\infty}\left(\frac{1}{20}\right)^j \\ &= \frac{1}{19\sqrt{2}\,\pi}\cdot\frac{1}{\nu}\cdot\frac{a_n}{\sqrt{k_n}}\,.\end{aligned}$$

From (40), (41) and (43) it follows that

$$(44) \qquad \begin{aligned}|c_\nu(f_0)| \geqq{}& \frac{2}{\pi^2}\cdot\frac{1}{\nu}\frac{a_n}{\sqrt{2k_n+1}}\left|\sin \pi\nu\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)\right| \\ &- c\cdot\frac{1}{\nu^2}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}} - \frac{1}{19\sqrt{2}\,\pi}\cdot\frac{1}{\nu}\frac{a_n}{\sqrt{k_n}}\,.\end{aligned}$$

It is easy to show that according to (31) and (42)

$$(45) \qquad \sum_{n=1}^{\infty}\sum_{\nu=(m_{n-1}/2)+1}^{m_n/2}\frac{1}{\nu^2}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}} \leqq c\sum_{n=1}^{\infty}\frac{a_{n-1}m_{n-1}}{\sqrt{k_{n-1}}}\cdot\frac{1}{m_{n-1}} < \infty\,.$$

Now we shall estimate the sum

$$\begin{aligned}\sigma = \frac{2}{\pi^2}\frac{a_n}{\sqrt{2k_n+1}}\sum_{\nu=(m_{n-1}/2)+1}^{m_n/2}\frac{1}{\nu}\left|\sin \pi\nu\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)\right| \\ -\frac{1}{19\sqrt{2}\,\pi}\cdot\frac{a_n}{\sqrt{k_n}}\sum_{\nu=(m_{n-1}/2)+1}^{m_n/2}\frac{1}{\nu}\,.\end{aligned} \tag{46}$$

Let

$$\left\{\begin{aligned} l_0 &= \left[\left(\frac{m_{n-1}}{2}+1\right)\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)-\frac{1}{6}\right]+1 \\ l_1 &= \left[\frac{m_n}{2}\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)-\frac{5}{6}\right],\end{aligned}\right. \tag{47}$$

and for $l = l_0,\ l_0+1, \cdots, l_1$

$$\left\{\begin{aligned} \mu_e' &= \left[\frac{l+\frac{1}{6}}{\frac{1}{2k_n+1}-\frac{1}{m_n}}\right]+1 \\ \mu_e'' &= \left[\frac{l+\frac{5}{6}}{\frac{1}{2k_n+1}-\frac{1}{m_n}}\right].\end{aligned}\right. \tag{48}$$

Then for $\mu_e' \leqq \nu \leqq \mu_e''$, $l = l_0, \cdots, l_1$

$$\left|\sin \pi\nu\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)\right| \geqq \frac{1}{2}$$

therefore from (46) using the inequalities

$$\frac{1}{x} \leqq \ln\left(1+\frac{1}{x-1}\right) \leqq \frac{1}{x-1}\,, \qquad x \in (1, \infty) \tag{49}$$

we get

$$\begin{aligned}\sigma &\geqq \frac{2}{\pi^2}\frac{a_n}{\sqrt{2k_n+1}}\sum_{l=l_0}^{l_1}\sum_{\nu=\mu_e'}^{\mu_e''}\frac{1}{\nu}\left|\sin \pi\nu\left(\frac{1}{2k_n+1}-\frac{1}{m_n}\right)\right| \\ &\quad -\frac{1}{19\sqrt{2}\,\pi}\frac{a_n}{\sqrt{k_n}}\sum_{\nu=(m_{n-1}/2)+1}^{m_n/2}\ln\left(1+\frac{1}{\nu-1}\right) \\ &\geqq \frac{2}{\pi^2}\frac{a_n}{\sqrt{2k_n+1}}\cdot\frac{1}{2}\sum_{l=l_0}^{l_1}\sum_{\nu=\mu_e'}^{\mu_e''}\ln\left(1+\frac{1}{\nu}\right) \\ &\quad -\frac{1}{19\sqrt{2}\,\pi}\cdot\frac{a_n}{\sqrt{k_n}}\ln\left(\prod_{\nu=(m_{n-1}/2)+1}^{m_n/2}\left(1+\frac{1}{\nu-1}\right)\right) \\ &\geqq \frac{1}{\pi^2}\frac{a_n}{\sqrt{2k_n+1}}\sum_{l=l_0}^{l_1}\ln\frac{\mu_e''+1}{\mu_e'}-\frac{1}{19\sqrt{2}\,\pi}\frac{a_n}{\sqrt{k_n}}\ln\frac{m_n}{m_{n-1}}\,.\end{aligned} \tag{50}$$

We have

$$\frac{\mu_e'' + 1}{\mu_e'} \geqq \frac{l + \frac{5}{6}}{l + \frac{1}{6}} = 1 + \frac{\frac{2}{3}}{l + \frac{1}{6}},$$

this with (49) implies

$$\ln \frac{\mu_e'' + 1}{\mu_e'} \geqq \ln\left(1 + \frac{1}{\frac{3}{2}l + \frac{1}{4}}\right) \geqq \frac{1}{\frac{3}{2}l + \frac{5}{4}} \geqq \frac{\frac{2}{3}}{l+1} \geqq \frac{2}{3} \ln \frac{l+2}{l+1}.$$

Using this estimation we obtain

$$\begin{aligned}
\sum_{l=l_0}^{l_1} \ln \frac{\mu_e'' + 1}{\mu_e'} &\geqq \frac{2}{3} \sum_{l=l_0}^{l_1} \ln\left(1 + \frac{1}{l+1}\right) = \frac{2}{3} \ln \frac{l_1 + 2}{l_0 + 1} \\
&\geqq \frac{2}{3} \ln \frac{\frac{m_n}{2}\left(\frac{1}{2k_n + 1} - \frac{1}{m_n}\right) - \frac{5}{6} + 1}{\left(\frac{m_{n-1}}{2} + 1\right)\left(\frac{1}{2k_n + 1} - \frac{1}{m_n}\right) - \frac{1}{6} + 2} \\
&\geqq \frac{2}{3}\left\{\ln \frac{m_n}{m_{n-1}} + \ln \frac{1 + \frac{1}{3} \cdot \frac{1}{m_n}\left(\frac{1}{2k_n + 1} - \frac{1}{m_n}\right)^{-1}}{1 + \frac{11}{3} \frac{1}{m_{n-1}}\left(\frac{1}{2k_n + 1} - \frac{1}{m_n}\right)^{-1} + \frac{2}{m_{n-1}}}\right\} \\
&\geqq \frac{2}{3} \ln \frac{m_n}{m_{n-1}} - c.
\end{aligned}$$

Applying the last relation by (50) we get

$$\begin{aligned}
\sigma &\geqq \frac{1}{\pi^2} \frac{a_n}{\sqrt{2k_n + 1}} \cdot \frac{2}{3} \ln \frac{m_n}{m_{n-1}} - c\frac{a_n}{\sqrt{2k_n + 1}} \\
&\quad - \frac{1}{19\sqrt{2}\,\pi} \frac{a_n}{\sqrt{k_n}} \ln \frac{m_n}{m_{n-1}} = \frac{a_n}{\sqrt{2k_n + 1}} \ln \frac{m_n}{m_{n-1}}\left(\frac{2}{3\pi^2} - \frac{\sqrt{6}}{38\pi}\right) - c\frac{a_n}{\sqrt{k_n}}.
\end{aligned} \tag{51}$$

But as

$$\frac{2}{3\pi^2} - \frac{\sqrt{6}}{38\pi} > 0$$

(44), (51), (45) and (26) give

$$\begin{aligned}
\sum_{\nu=1}^{\infty} |c_\nu(f_0)| &= \sum_{n=1}^{\infty} \sum_{\nu=(m_{n-1})/2+1}^{m_n/2} |c_\nu(f_0)| \geqq c \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{2k_n + 1}} \ln \frac{m_n}{m_{n-1}} - c \\
&\geqq c \sum_{n=1}^{\infty} \frac{v(\psi(n))}{\sqrt{\psi(n)}} (c_0^{q_n} - c_0^{q_{n-1}}) \geqq c \sum_{n=1}^{\infty} c_0^{q_n} \gamma_{2c_0^{q_n}} = \infty
\end{aligned}$$

i.e., $f_0 \overline{\in} A$.

This completes the proof of Theorem 2.

Theorems 1 and 2 imply

THEOREM 3. *For all Fourier series of class $H^\omega \cap V[n^\alpha]$, $0 < \alpha < 1/2$, to be absolutely convergent it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \frac{1}{n}\left\{\omega\left(\frac{1}{n}\right)\right\}^{1-2\alpha/2(1-\alpha)} < \infty . \tag{52}$$

Proof. The sufficieny is contained in Corollary 2. We shall prove the necessity of the condition (52). Assume the contrary. We shall verify that for $\alpha \in (0, 1/2)$ the modulus of variation $v(n) = n^\alpha$ satisfies the conditions of Theorem 2.

In fact, if

$$0 < \varepsilon < \min\left\{\alpha, \frac{1}{2} - \alpha\right\}$$

then $n^{-\varepsilon}v(n)$ is increasing and $n^{\varepsilon-1/2}v(n)$ is decreasing.

Furthermore

$$\varphi(n) = \max\left\{m; \frac{m^\alpha}{m} \geqq \omega\left(\frac{1}{n}\right)\right\} = \left[\left(\omega\left(\frac{1}{n}\right)\right)^{-1/1-\alpha}\right],$$

and so

$$\begin{aligned}\left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{v^2(k)}{k^2}\right)^{1/2} &= \left(\sum_{k=\varphi(n)}^{\varphi(n)+n} \frac{1}{k^{2-2\alpha}}\right)^{1/2} \geqq c\frac{1}{\varphi(n)^{1/2-\alpha}} \\ &\geqq c\left\{\omega\left(\frac{1}{n}\right)\right\}^{(1-2\alpha)/2(1-\alpha)} .\end{aligned} \tag{53}$$

Since we have assumed that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left\{\omega\left(\frac{1}{n}\right)\right\}^{1-2\alpha/2(1-\alpha)} = \infty ,$$

from (53) it follow that the series (17) is also divergent and therefore all the conditions of Theorem 2 are satisfied. Then according to this theorem there exists a function in the class $H^\omega \cap V[n^\alpha]$ such that its Fourier series is not absolutely convergent. Hence we have got the contradiction.

This completes the proof.

Remark. In the case $\alpha \geqq 1/2$, i.e., if we have the class $H^\omega \cap V[n^\alpha]$, $\alpha \geqq 1/2$, it is easy to verify that Theorem 1 is the same as the theorem of S. N. Bernstein.

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