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AN INVESTIGATION OF REAL DIVISION ALGEBRAS USING DERIVATIONS

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In a recent paper, "The derivation algebra of a real division algebra", we showed that if $\operatorname{Der} A$ is the derivation algebra of a finite dimensional division algebra A over the reals, then

(i) $\dim A = 1$ or 2 implies $\operatorname{Der} A = 0$,

(ii) $\dim A = 4$ implies $\operatorname{Der} A$ is $su(2)$ or $\dim \operatorname{Der} A = 0$ or 1,

(iii) $\dim A = 8$ implies $\operatorname{Der} A$ is one of the following Lie algebras:

(1) compact G_2

(2) $su(3)$

(3) $su(2) \oplus su(2)$

(4) $su(2) \oplus N$ where N is an abelian ideal and $\dim N = 0$ or 1

(5) N where N is abelian and $\dim N = 0, 1$ or 2.

Moreover, any subalgebra of $\operatorname{Der} A$ is isomorphic to one of the Lie algebras listed above.

For each Lie algebra L appearing in (i), (ii), and (iii) we also exhibited a real division algebra having L as its derivation algebra without proving that the derivation algebra was as asserted. One of the goals of this present paper is to verify that these examples have the derivation algebra claimed, but our main purpose is broader than this. Using the representation theory of Lie algebras we investigate those real division algebras A having L as its derivation algebra for each of the nonzero Lie algebras L mentioned above. The larger that L is, the more detailed is the information concerning the structure of A. As one might expect, most of the classes of division algebras are natural generalizations of the quaternions and octonions. The principal exception is a family of division algebras which includes the pseudo-octonions introduced by Okubo in "Pseudo-quaternion and pseudo-octonion algebras."

1. A review of some basic results on representations. *Throughout this paper we will assume that all algebras and modules are finite dimensional.* Let A be an algebra over a field F of characteristic 0, and assume L is a semisimple subalgebra of the derivation algebra $\operatorname{Der} A$. Since A is an L-module, it decomposes into irreducible summands: $A = V_1 \oplus \cdots \oplus V_n$. Moreover, the product of $V_r \times V_s$ into A followed by the projection onto V_t induces an L-module homomorphism of $V_r \otimes V_s$ into V_t. Conversely, by taking a sum of irreducible L-modules $A = V_1 \oplus \cdots \oplus V_n$ and prescribing

L-module homomorphisms from $V_r \otimes V_s$ into V_t for all r, s, t, one achieves an algebra structure on A such that $L \subseteq \operatorname{Der} A$. In case F is algebraically closed the dimension of $\operatorname{Hom}_L(V_r \otimes V_s, V_t)$ can be determined using

PROPOSITION 1.1. *Let L be a semisimple Lie algebra over an algebraically closed field of characteristic* 0. *Assume U is an L-module and W is an irreducible L-module. If $U = U_1 \oplus \cdots \oplus U_m$ where the U_i are irreducible L-submodules, then* $\dim \operatorname{Hom}_L(U, W)$ *equals the number of U_i isomorphic to W.*

Since this is a standard result we give only a brief outline of the proof. Using the uniqueness of the decomposition of U and Schur's lemma, one can show that the homomorphisms π_i (projection of U onto U_i followed by an isomorphism onto W) form a basis for $\operatorname{Hom}_L(U, W)$.

In case U is an L-module over an arbitrary field F of characteristic 0, we can take the algebraic closure K of F and form the module $U_K = U \otimes_F K$ for $L_K = L \otimes_F K$, and then apply Proposition 1.1 to U_K. We examine the effect of this field extension on certain submodules of U.

Suppose $U = U_1 \oplus \cdots \oplus U_m$ is a decomposition of U into irreducible L-submodules. Let U_0 be the sum of all the trivial 1-dimensional summands and U_* be the sum of the others. Then $U = U_0 \oplus U_*$ and one readily verifies that:

$$U_0 = \{u \in U \mid lu = 0 \text{ for all } l \in L\}$$
$$U_* = LU .$$

The submodules U_0 and U_* behave nicely relative to field extensions as the next lemma indicates.

LEMMA 1.2. (i) $(U_0)_K = (U_K)_0$
(ii) $(U_*)_K = (U_K)_*$.

Proof. From our alternate characterizations above, it follows that $(U_0)_K \subseteq (U_K)_0 = \{x \in U_K \mid lx = 0 \text{ for all } l \in L_K\}$, and $(U_*)_K = (LU)_K \subseteq L_K U_K = (U_K)_*$. But since $U_K = (U_0)_K \oplus (U_*)_K \subseteq (U_K)_0 \oplus (U_K)_* = U_K$, equality must hold in each case. $\square$

In view of the above remarks, an equivalent formulation of Lemma 1.2 (ii) is that the extension $(LU)_K$ equals the image of U_K under L_K, which is $L_K U_K$.

We now turn our attention to the case that A is a real division algebra. According to the result stated in the introduction, the only possible semisimple subalgebras of $\operatorname{Der} A$ are compact G_2, $su(3)$, $su(2) \oplus su(2)$, and $su(2)$. Each of these Lie algebras contains a copy of $su(2)$ so that if $\operatorname{Der} A$ contains a semisimple algebra, A decomposes into irreducible $su(2)$-modules. Irreducible $su(2)$-modules are most easily described by complexifying and regarding the resulting module as an $sl(2)$-module. Again the results we mention are quite well-known ([4] or [6]), but our aim is to develop the background needed for later sections.

Let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be the standard basis for $sl(2)$ over the complex numbers $\boldsymbol{C}$. Given any integer $m \geqq 0$, there is a unique irreducible $sl(2)$-module $V(m)$ having dimension $m+1$. We can choose a basis $Z_m, Z_{m-2}, \cdots, Z_{m-2m} = Z_{-m}$ for $V(m)$ so that:

$$\begin{aligned} hZ_r &= rZ_r \\ eZ_r &= \frac{m-r}{2} Z_{r+2} \text{ where } Z_{m+2} = 0 \\ fZ_r &= \frac{m+r}{2} Z_{r-2} \text{ where } Z_{-m-2} = 0 \,. \end{aligned} \tag{1.3}$$

Now $su(2) = \{x \in sl(2) \mid \bar{x}^t = -x\}$, and a basis for $su(2)$ can be obtained by taking $\partial_1 = ih$, $\partial_2 = e - f$, and $\partial_3 = ie + if$. The effect of these elements on the basis of Z's is given by:

$$\begin{aligned} \partial_1 Z_r &= irZ_r \\ \partial_2 Z_r &= \left(\frac{m-r}{2}\right) Z_{r+2} - \left(\frac{m+r}{2}\right) Z_{r-2} \\ \partial_3 Z_r &= i\left(\frac{m-r}{2}\right) Z_{r+2} + i\left(\frac{m+r}{2}\right) Z_{r-2} \,. \end{aligned} \tag{1.4}$$

Let us consider the case that $m = 2n$, and hence that $\dim V(m) = 2n+1$ is odd. In this situation we define:

$$\begin{aligned} U_q &= Z_{2q} + (-1)^q Z_{-2q} \quad \text{for } q = 0, \cdots, n \\ V_q &= iZ_{2q} - (-1)^q iZ_{-2q} \quad \text{for } q = 1, \cdots, n \\ V_0 &= V_{n+1} = U_{n+1} = 0 \,. \end{aligned}$$

The action of $su(2)$ on the U's and V's can be readily computed using (1.4) to show that for $q = 1, \cdots, n$:

$$\begin{aligned} \partial_1 U_q &= 2qV_q & \qquad \partial_1 U_0 &= 0 \\ \partial_1 V_q &= -2qU_q \end{aligned}$$

$$
\begin{aligned}
&\partial_2 U_q = (n-q)U_{q+1} - (n+q)U_{q-1} \quad \partial_2 U_0 = 2nU_1 \\
&\partial_2 V_q = (n-q)V_{q+1} - (n+q)V_{q-1} \\
&\partial_3 U_q = (n-q)V_{q+1} - (n+q)V_{q-1} \quad \partial_3 U_0 = 2nV_1 \\
&\partial_3 V_q = -(n-q)U_{q+1} - (n+q)U_{q-1}\,.
\end{aligned}
\tag{1.5}
$$

Thus, if we regard $V(m)$ where $m = 2n$, as a real $su(2)$-module, the U's and V's generate a $su(2)$-submodule of dimension $m+1$ over $\boldsymbol{R}$, call it $W(m)$. It is not difficult to verify that $W(m)$ is irreducible and that $V(m) = W(m) \oplus iW(m)$ as a real $su(2)$-module.

The situation when m is odd is completely different. Here $V(m)$ is an irreducible $su(2)$-module over $\boldsymbol{R}$.

Let us assume W is any irreducible $su(2)$-module. Then $W_C = W \otimes_R \boldsymbol{C}$ is an $sl(2)$-module and as such, it decomposes into irreducible submodules of the type $V(m)$. Now W_C as an $su(2)$-module is isomorphic to exactly two copies of W. Thus when we regard the $V(m)$ summands as real $su(2)$-modules we must have a total of two irreducible $su(2)$-summands each isomorphic to W. When W has dimension $2n+1$ this implies W is isomorphic to $W(2n)$ and $W_C \approx V(2n)$. If W has dimension $4n$, then $W_C \approx V(2n-1) \oplus V(2n-1)$ and $W \approx V(2n-1)$ when $V(2n-1)$ is regarded an $su(2)$-module. There can be no irreducible $su(2)$-module of dimension $2(2n+1)$, so in fact, the smallest nontrivial $su(2)$-module is $su(2)$ itself.

The Clebsch-Gordan formula provides the answer as to how the tensor product of two irreducible $sl(2)$-modules decomposes:

$$
V(m) \otimes V(n) = V(m+n) \oplus \cdots \oplus V(|m-n|)\,. \tag{1.6}
$$

Thus

$$
\dim \mathrm{Hom}_{sl(2)}(V(m) \otimes V(n),\ V(s)) = \begin{cases} 1 & \text{if } s = m+n,\ m+n-2,\ \cdots,\ |m-n| \\ 0 & \text{otherwise.} \end{cases}
$$

Since for any real Lie algebra L and for any three L-modules U, V, W, $\dim_R \mathrm{Hom}_L(U \otimes V, W) \leqq \dim_C \mathrm{Hom}_{L_C}(U_C \otimes V_C, W_C)$, the Clebsch-Gordan formula determines a bound for $\dim_R \mathrm{Hom}_{su(2)}(U \otimes V, W)$.

In addition to results on $su(2)$ and $sl(2)$-modules we require some facts concerning irreducible modules for $sl(2) \oplus sl(2)$, $sl(3)$, and G_2. These facts can be established using arguments in ([4], Chapter 6) or ([6], Chapters 7 and 8).

Given a semisimple Lie algebra L over an algebraically closed field of characteristic 0 with Cartan decomposition $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$, there are certain linear functionals $\lambda_1, \cdots, \lambda_l$ on H, (the so called fundamental weights) which span the dual H^* of H. The irreducible L-modules are in one-to-one correspondence with the elements in H^* of the form $\lambda = m_1\lambda_1 + \cdots + m_l\lambda_l$ where the m_i are non-negative integers. Following Humphreys we denote the irreducible module corresponding to λ as $V(\lambda)$. (In this notation the $sl(2)$-module $V(m)$ would be $V(m\lambda_1)$). The dimension of the module $V(\lambda)$ is given by Weyl's formula ([4], p. 140), and the tensor product of $V(\lambda)$ and $V(\lambda')$ can be resolved into irreducibles using either Steinberg's formula ([4], p. 141) or calculations involving weights and their multiplicities.

Real division algebras exist only in dimensions 1, 2, 4 and 8, and as the result in the introduction indicates, the only time that $su(2) \oplus su(2)$, $su(3)$, and compact G_2 occur in $\operatorname{Der} A$ is when $\dim A = 8$. Therefore when we decompose A_C into irreducible summands for $sl(2) \oplus sl(2)$, $sl(3)$, or G_2, the $V(\lambda)$ are constrained by $\dim V(\lambda) \leqq 8$, and the sum of the dimensions must total 8.

Since every $sl(2) \oplus sl(2)$ irreducible module is just the tensor product of two irreducible $sl(2)$-modules, one can handle these modules using the above considerations.

For the Lie algebra $sl(3) = A_2$, Weyl's dimension formula reads: $\dim V(m_1\lambda_1 + m_2\lambda_2) = 1/2(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$. Using this expression one easily computes that the only modules of dimension less than 8 are given by

(1.7)

	dimension	notation
$V(0)$	1	1
$V(\lambda_1)$	3	3
$V(\lambda_2)$	3	$\overline{3}$
$V(2\lambda_1)$	6	6
$V(2\lambda_2)$	6	$\overline{6}$
$V(\lambda_1 + \lambda_2)$	8	8 .

We adopt the convention of denoting a module by its dimension, and in the event of two or more of equal dimensions distinguish them by a bar or star or both.

Consider now the tensor products of these modules. For any irreducible $V(\lambda)$, $V(0) \otimes V(\lambda) = V(\lambda)$. Of the remaining products we list only those needed in the study of real division algebras.

$$\begin{aligned} 3 \otimes 3 &= 6 + \overline{3} \\ 3 \otimes \overline{3} &= 8 + 1 \\ \overline{3} \otimes \overline{3} &= \overline{6} + 3 \\ 6 \otimes 6 &= 15 + 15^* + \overline{6} \end{aligned} \tag{1.8}$$

$$\overline{6} \otimes \overline{6} = \overline{15} + \overline{15}^* + 6$$
$$8 \otimes 8 = 27 + 10 + \overline{10} + 8 + 8 + 1\,.$$

Here $15 = V(4\lambda_1)$, $15^* = V(2\lambda_1 + \lambda_2)$, $\overline{15} = V(4\lambda_2)$, $\overline{15}^* = V(\lambda_1 + 2\lambda_2)$, $27 = V(2\lambda_1 + 2\lambda_2)$, $10 = V(3\lambda_1)$ and $\overline{10} = V(3\lambda_2)$.

The case $8 \otimes 8$ is noteworthy because the two 8-dimensional summands imply there are two linearly independent $sl(3)$-homomorphisms of $8 \otimes 8 \to 8$. The 8-dimensional module is $sl(3)$ itself under the adjoint representation, and a basis of homomorphisms may be concretely given by: $x \otimes y \to [xy]$ and $x \otimes y \to xy + yx - 2/3\, tr(xy)I$ where $tr(xy)$ denotes the trace of xy.

The dimension formula for G_2 modules is

$$\dim V(\lambda)$$
$$= \frac{1}{5!}(m_1+1)(m_2+1)(m_1+m_2+2)(m_1+2m_2+3)(m_1+3m_2+4)(2m_1+3m_2+5)\,.$$

Thus, there are only two modules of dimension less than or equal to 8: the 1-dimensional module $V(0)$, and the 7-dimensional module $V(\lambda_1)$. The resolution of $7 \otimes 7$ into irreducibles is given by

$$7 \otimes 7 = V(2\lambda_1) \oplus V(\lambda_2) \oplus V(\lambda_1) \oplus V(0) \tag{1.9}$$

where these modules have dimensions 27, 14, 7, and 1 respectively.

2. The case Der A = compact G_2. We are now ready to consider individually the different possibilities for Der A, and to investivate for each one the division algebras A with that derivation algebra. We take the possible derivation algebras in the order in which they are listed at the beginning of this paper, starting with the case when Der A is a compact form of G_2. As we noted in § 1, there are only two irreducible G_2-modules of dimension 8 or less over the complex numbers—one of dimension 1 and one of dimension 7. Thus, if A is a real division algebra with Der A = compact G_2, the scalar extension $A_C = A \otimes_R C$ must be a sum of one 1-dimensional module and one 7-dimensional module. (Note A_C could not be a sum of eight 1-dimensional modules because Der A must act faithfully on A.) Since the decomposition of A_C into irreducible modules is necessarily a refinement of the decomposition of A, we see that either A is a direct sum of a 1-dimensional module and an irreducible 7-dimensional module, or else A is an irreducible 8-dimensional module. But the last possibility can be ruled out by Lemma 1.2. Hence $A = U + V$ where U is a 1-dimensional G_2-module and V is an irreducible 7-dimensional G_2-module.

As was observed at the beginning of § 1, the homomorphisms from $U \otimes U$, $U \otimes V$, $V \otimes U$, and $V \otimes V$ into U and V determine the possible products between the summands. Since for G_2-modules over C, $1 \otimes 1 \cong 1$, $1 \otimes 7 \cong 7 \cong 7 \otimes 1$, and $7 \otimes 7 \cong 27 + 14 + 7 + 1$, it follows from Proposition 1.1 that there is at most one homomorphism up to scalar multiple in each of the cases: $U \otimes U \to U$, $U \otimes V \to V$, $V \otimes U \to V$, $V \otimes V \to V$, and $V \otimes V \to U$, and only the zero homomorphism in the other cases. From this we deduce first that $U^2 \subseteqq U$. But since A is a division algebra, $U^2 \neq 0$, so it must be $U^2 = U$. Thus, there exists an idempotent $u \in U$. Now $u \otimes v \to v$ and $v \otimes u \to v$ define module homomorphisms from $U \otimes V$ and $V \otimes U$ onto V. Therefore, left (right) multiplication by u is just the identity transformation on V multiplied by the scalar η (ζ). To determine homomorphisms for $V \otimes V \to V$, $V \otimes V \to U$, we examine the best known example in the class we are describing—the octonion algebra $\boldsymbol{O}$. In $\boldsymbol{O}$ there is a basis $u, e_1, \cdots, e_7$ with multiplication given by table (2.1) below with $\beta = \eta = \zeta = 1$. Here u spans a 1-dimensional module and $e_1, \cdots, e_7$ a 7-dimensional module for $\operatorname{Der} \boldsymbol{O} =$ compact G_2. Since the modules being discussed are unique up to isomorphism, and since $\dim_R \operatorname{Hom}_{G_2}(V \otimes V, V) \leqq 1$ and $\dim_R \operatorname{Hom}_{G_2}(V \otimes V, U) \leqq 1$, the products in the general case are the same as in the octonions up to multiplication by a constant. After replacing the basis elements of V by a fixed scalar multiple of themselves, we may assume that the multiplication from $V \times V$ to V is identical to that of the octonions, but that the products from $V \times V$ to U involve the scalar β. To be specific, there is a basis $u, e_1, \cdots, e_7$ with multiplication given by

(2.1)

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	ηe_1	ηe_2	ηe_3	ηe_4	ηe_5	ηe_6	ηe_7
e_1	ζe_1	$-\beta u$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	ζe_2	$-e_4$	$-\beta u$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	ζe_3	$-e_7$	$-e_5$	$-\beta u$	e_6	e_2	$-e_4$	e_1
e_4	ζe_4	e_2	$-e_1$	$-e_6$	$-\beta u$	e_7	e_3	$-e_5$
e_5	ζe_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-\beta u$	e_1	e_4
e_6	ζe_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-\beta u$	e_2
e_7	ζe_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-\beta u$

.

The final thing that we wish to determine in this case is for which values of β, η, ζ the algebra with the above table is a division algebra. In particular, we shall establish

THEOREM 2.2. *A real algebra A is a division algebra with the* *compact form of G_2 as its derivation algebra if and only if A has a basis $u, e_1, \cdots, e_7$ with multiplication given by* (2.1) *for some real numbers β, η, ζ such that $\beta\eta\zeta > 0$.*

Proof. In view of our preceding discussion, it remains only to show that the algebra A whose multiplication table is given by (2.1) is a division algebra exactly when $\beta\eta\zeta > 0$. The condition for A to be a division algebra is that the relation

$$0 = \left(a_0 u + \sum_{i=1}^{7} a_i e_i\right)\left(b_0 u + \sum_{i=1}^{7} b_i e_i\right) \tag{2.3}$$

can hold for real a's and b's only if either all a's are zero or all b's are zero. As in the proof of ([1], Theorem 20), we multiply out the right side of (2.3) and set the coefficients equal to zero. If the b's are regarded as variables in the resulting equations, the coefficient matrix is

$$M_1 = \begin{pmatrix} a_0 & -\beta a_1 & -\beta a_2 & -\beta a_3 & -\beta a_4 & -\beta a_5 & -\beta a_6 & -\beta a_7 \\ \zeta a_1 & \eta a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 & a_3 \\ \zeta a_2 & a_4 & \eta a_0 & -a_5 & -a_1 & a_3 & -a_7 & a_6 \\ \zeta a_3 & a_7 & a_5 & \eta a_0 & -a_6 & -a_2 & a_4 & -a_1 \\ \zeta a_4 & -a_2 & a_1 & a_6 & \eta a_0 & -a_7 & -a_3 & a_5 \\ \zeta a_5 & a_6 & -a_3 & a_2 & a_7 & \eta a_0 & -a_1 & -a_4 \\ \zeta a_6 & -a_5 & a_7 & -a_4 & a_3 & a_1 & \eta a_0 & -a_2 \\ \zeta a_7 & -a_3 & -a_6 & a_1 & -a_5 & a_4 & a_2 & \eta a_0 \end{pmatrix}.$$

The statement that A is a division algebra is equivalent to the condition that the determinant of M_1 is nonzero unless all the a's are zero. Since η and ζ must be nonzero for A to be a division algebra, we can replace each a_0 with $\eta^{-1}a_0$ and then multiply each entry in the first column by ζ^{-1}. If the resulting matrix is called M_2, then $\det M_1 = 0$ if and only if $\det M_2 = 0$. Now let us suppose that the matrix M is the same matrix as M_2 only with $\eta^{-1} = \zeta^{-1} = \beta = 1$. Since M corresponds to the octonion algebra, $\det M \neq 0$ unless all a_i are 0. Thus $\det M_2 = 0$ if and only if $\det M_2M^t = 0$. But M_2M^t has no entries below the diagonal since the last 7 rows of M_2 and M are the same and are pairwise orthogonal. Hence, the determinant of M_2M^t is the product of the diagonal elements. The first diagonal entry is $\eta^{-1}\zeta^{-1}a_0^2 + \sum \beta a_i^2$ and the remaining entries are $a_0^2 + \sum_{i=1}^{7} a_i^2$. It is clear that if $\eta^{-1}\zeta^{-1}$ and β are both positive or both negative, then the determinant of M_2M^t is not 0 unless all the a's are, hence A is a division algebra. Conversely, if A is a division algebra, $\eta^{-1}\zeta^{-1}a_0^2 + \sum_{i=1}^7 \beta a_i^2$ is not zero unless all the a's vanish, which implies that $\eta^{-1}\zeta^{-1}$ and β have the same sign. Thus, A is a division algebra if and only if $\beta\eta\zeta > 0$. □

3. The case $\mathrm{Der}\, A = su(3)$ begun. In this section we investigate the case when A is an 8-dimensional irreducible $su(3)$-module. Then A is isomorphic to $su(3)$ when it is regarded as an $su(3)$-module under the adjoint representation. As we saw in § 1 there are two independent homomorphisms from $sl(3) \otimes sl(3)$ to $sl(3)$, and this is indeed true for $su(3)$ as well. One of the homomorphisms is obviously the Lie product, and to obtain the other we consider $su(3)$ as 3×3 complex skew-Hermitian matrices ($\bar{x}^t = -x$) of trace zero. For x and y in $su(3)$, $xy + yx - (2/3)tr(xy)I$ is a Hermitian matrix, so multiplying it by i gives a skew-Hermitian matrix which also has trace zero. Now for z also in $su(3)$,

$$\left[z,\, i\left\{xy + yx - \frac{2}{3}tr(xy)I\right\}\right] = i\{[zx]y + y[zx] + x[zy] + [zy]x\}$$
$$= i\left\{[zx]y + y[zx] - \frac{2}{3}tr([zx]y)I\right\}$$
$$+ i\left\{x[zy] + [zy]x - \frac{2}{3}tr(x[zy])I\right\}$$

since $tr([zx]y) = -tr([xz]y) = -tr(x[zy])$. This calculation demonstrates that the map $x \otimes y \to i\{xy + yx - (2/3)tr(xy)I\}$ is indeed an $su(3)$-homomorphism. Our argument shows that for any real 8-dimensional algebra A on which $su(3)$ acts irreducibly as derivations, the product in A is given by

$$x*y = \alpha[xy] + \beta i\left\{xy + yx - \frac{2}{3}tr(xy)I\right\}. \tag{3.1}$$

In fact, $su(3)$ is the entire derivation algebra whenever $\alpha \neq 0$. For if A^- denotes the algebra A under the product $x*y - y*x = 2\alpha[xy]$, then every derivation of A is also a derivation of A^-. But A^- is isomorphic to $su(3)$, which has only inner derivations (see [4] p. 23), so $\mathrm{Der}\, A = su(3)$ in this instance. Our investigations of this case will be complete, once we establish a criterion for such an algebra to be a division algebra. To this purpose we prove

THEOREM 3.2. *Let A be an 8-dimensional real algebra defined on the vector space $su(3)$ with multiplication given by* (3.1). *Then A is a division algebra if and only if $\alpha\beta \neq 0$. For such a division algebra,* $\mathrm{Der}\, A = su(3)$ *and A is an irreducible $su(3)$-module. Conversely any real division algebra on which $su(3)$ acts irreducibly* *as derivations is given by this construction.*

It is easy to see the necessity of the condition $\alpha\beta \neq 0$ to have a division algebra since any element squares to zero if $\beta = 0$, and since the product of $ie_{11} - ie_{22}$ with $e_{12} - e_{21}$ is zero if $\alpha = 0$. To establish the sufficiency of the condition we need the following results.

Let x be a skew-Hermitian complex matrix. Then there is a unitary matrix u such that $u^{-1}xu = v$ where v is diagonal (see for instance, Herstein [3] p. 302, Theorem $6.Z_2$). Since v is skew-Hermitian also, it follows that all the characteristic roots of v, hence of x, are purely imaginary.

LEMMA 3.3. *Let x and y belong to $su(3)$, and assume $\gamma, \delta \in \boldsymbol{C}$ are such that $\delta \neq \pm \gamma$. If $\gamma xy + \delta yx = \lambda I$ for some $\lambda \in \boldsymbol{C}$, then x or y is 0.*

Proof. Let u be a unitary matrix which diagonalizes x as above. Then $\gamma(u^{-1}xu)(u^{-1}yu) + \delta(u^{-1}xu)(u^{-1}yu) = \lambda I$. Hence we may assume without loss of generality that x is diagonal, say $x = \operatorname{diag}\{a_1, a_2, a_3\}$. If $y = (b_{ij})$, then the equation $\gamma xy + \delta yx = \lambda I$ gives for $i \neq j$:

$$(\gamma a_i + \delta a_j)b_{ij} = 0 \ .$$

Since y is skew-Hermitian, $b_{ji} = -\bar{b}_{ij}$. Thus, for each pair i, j with $i \neq j$, we obtain the system of equations:

$$\begin{aligned} (\gamma a_i + \delta a_j)b_{ij} &= 0 \ , \\ (\gamma a_j + \delta a_i)\bar{b}_{ij} &= 0 \ . \end{aligned} \tag{3.4}$$

If some $b_{ij} \neq 0$ for $i \neq j$, then since $\gamma^2 - \delta^2 \neq 0$, we have $a_i = a_j = 0$. However, x has trace 0, so it follows that $a_1 = a_2 = a_3 = 0$ in this case, and $x = 0$. We may assume then that y is diagonal, say $y = \operatorname{diag}\{b_1, b_2, b_3\}$. Equating entries in $\gamma xy + \delta yx = \lambda I$ gives

$$a_1b_1 = a_2b_2 = a_3b_3 = (\gamma + \delta)^{-1}\lambda \ . \tag{3.5}$$

Using the fact that x and y have trace 0, we obtain

$$a_1b_1 = a_2b_2 = (-a_1 - a_2)(-b_1 - b_2) \tag{3.6}$$

which simplifies to show:

$$a_1b_1 - a_2b_2 = 0, \quad a_1(b_1 + b_2) + a_2b_1 = 0 \ . \tag{3.7}$$

If not both a_1 and a_2 are zero then

$$(3.8) \qquad 0 = \begin{vmatrix} b_1 & -b_2 \\ b_1 + b_2 & b_1 \end{vmatrix} = b_1^2 + b_1 b_2 + b_2^2 \,.$$

If $b_2 = 0$, then $b_1 = 0$ and $y = 0$ as well. So we may assume $b_2 \neq 0$. Then it is apparent from (3.8) that $b_1 b_2^{-1}$ satisfies the equation $z^2 + z + 1$. Hence $b_1 b_2^{-1} = \omega$, a complex cube root of 1. But then (3.5) implies $a_2 = a_1 \omega$ and $a_3 = -a_1 - a_2 = -a_1(1 + \omega) = a_1 \omega^2$. If $a_1 = \alpha i$ for $\alpha \in \boldsymbol{R}$, then $a_2 = (\alpha/2)i \pm \sqrt{(3/2)}\alpha$ which contradicts the fact that all roots of x are purely imaginary unless $\alpha = 0$. But then $x = 0$ as desired. $\square$

Proof of Theorem 3.2. It remains to show that if $\alpha\beta \neq 0$ then A is a division algebra. Suppose x and y are complex skew-Hermitian 3×3 matrices of trace zero with the property that

$$\begin{aligned} 0 = x*y &= \alpha[xy] + \beta i\left\{xy + yx - \frac{2}{3} tr(xy) I\right\} \\ &= (\alpha + \beta i)xy + (-\alpha + \beta i)yx - \frac{2}{3}\beta i tr(xy) I \,. \end{aligned}$$

Letting $\gamma = \alpha + \beta i$ and $\delta = -\alpha + \beta i$, we have $\gamma + \delta = 2\beta i \neq 0$ and $\gamma - \delta = 2\alpha \neq 0$. Since the hypotheses of Lemma 3.3 are satisfied, we are forced to conclude that $x = 0$ or $y = 0$, and hence that A is a division algebra. $\square$

Those special cases in which $\beta = \pm\sqrt{3}\alpha$ have been studied recently by Okubo [8], and have been shown to have many interesting properties. For example these algebras have a quadratic form permitting composition. They are not composition algebras in the usual sense since they do not have an identity element.

It turns out that the two algebras studied by Okubo are the only ones in the class defined by Theorem 3.2 which have a quadratic form permitting composition. However, we can show that every algebra A described by Theorem 3.2 is flexible. For this we take x, y skew-Hermitian matrices of trace 0, we let $\lambda_{x,y} = (2/3)tr(xy)$ and use (3.1) to calculate that

$$\begin{aligned} &(x*y)*x - x*(y*x) \\ &\quad = (\alpha[xy] + \beta i\{xy + yx - \lambda_{x,y} I\})*x - x*(\alpha[yx] + \beta i\{xy + yx - \lambda_{x,y} I\}) \\ &\quad = \alpha^2([[xy]x] - [x[yx]]) + \alpha\beta i([xy]x + x[xy] - \lambda_{[xy],x} I - x[yx] \\ &\qquad - [yx]x + \lambda_{x,[xy]} I + [xy + yx, x] - [x, xy + yx]) \end{aligned}$$

$$- \beta^2(xyx + yx^2 + x^2y + xyx - 2\lambda_{x,y}x - \lambda_{xy+yx,x}I - x^2y - 2xyx$$
$$- yx^2 + 2\lambda_{x,y}x + \lambda_{x,xy+yx}I)$$
$$= \alpha\beta i(2[xy]x + 2x[xy] + 2[xy + yx, x]) = 0 \, .$$

It is also clear from (3.1) that A is Lie admissible, since $A^- \cong su(3)$.

4. The case $\operatorname{Der} A = su(3)$ concluded. Having dealt with the situation when A is a single irreducible $su(3)$-module, we turn to the case when A is a sum of at least two irreducible $su(3)$-modules. The only irreducible $sl(3)$-modules of dimension less than 8 are the ones which in the notation of (1.7) are given by 1, 3, $\bar{3}$, 6 and $\bar{6}$. Thus A_C must be a sum of modules of these types which add up to give $\dim A_C = 8$. We consider the various possibilities.

First, if A_C consists of a sum of 1's and 3's, then the relation $3 \otimes 3 = 6 + \bar{3}$ in (1.8) shows that the product of any two elements in the 3-summand(s) must be zero. However, by Lemma 1.2 the complexification of the image $(su(3)A)_C$ equals $sl(3)A_C$ which is the sum of copies of 3. Thus, the product of any two elements in $su(3)A$ would be zero and would contradict the fact that A is a division algebra. This demonstrates that A_C cannot consist solely of 1's and 3's. Similarly we can rule out each case where in addition to 1's there is exactly one of the types $\bar{3}$, 6, or $\bar{6}$ occurring in A_C by using the relations $\bar{3} \otimes \bar{3} = \bar{6} + 3$, $6 \otimes 6 = 15 + 15^* + \bar{6}$, and $\bar{6} \otimes \bar{6} = \overline{15} + \overline{15}^* + 6$ from (1.8).

Thus, there must be at least two of the types 3, $\bar{3}$, 6, $\bar{6}$ present in A_C, and this implies $A_C = 1 + 1 + 3 + \bar{3}$. Looking again at $su(3)A$ and $sl(3)A_C$, we see that A is the sum of two 1-dimensional modules and either two nonisomorphic 3-dimensional irreducible $su(3)$-modules or one irreducible 6-dimensional module. In the former case let us suppose W, $\bar{W}$ are the two 3-dimensional modules such that $W_C = 3$ and $\bar{W}_C = \bar{3}$, and U and V are the 1-dimensional modules. Then the relations $3 \otimes 1 = 3$, $3 \otimes 3 = 6 + \bar{3}$, and $\bar{3} \otimes 3 = 8 + 1$ show that for each $w \in W$, $wA \subseteq \boldsymbol{R}w + U + V + \bar{W}$. Hence, left multiplication by w is not onto, and this case cannot happen if A is a division algebra. Thus A is the direct sum of two 1-dimensional modules and an irreducible 6-dimensional module Z. Moreover we have the following

THEOREM 4.1. *If A is a real division algebra such that $\operatorname{Der} A = su(3)$ and A is not an irreducible $su(3)$-module, then A has a basis $u, v, z_1, \cdots, z_6$ with multiplication table given by* (4.2). *Conversely an algebra A defined by* (4.2) *admits $su(3)$ as derivations.*

(4.2)

	u	v	z_1	z_2	z_3	z_4	z_5	z_6
u	$\eta_1 u+\theta_1 v$	$\eta_2 u+\theta_2 v$	$\sigma_1 z_1+\sigma_2 z_3$	$\sigma_1 z_2+\sigma_2 z_6$	$-\sigma_2 z_1+\sigma_1 z_3$	$\sigma_1 z_4+\sigma_2 z_5$	$-\sigma_2 z_4+\sigma_1 z_5$	$-\sigma_2 z_2+\sigma_1 z_6$
v	$\eta_3 u+\theta_3 v$	$\eta_4 u+\theta_4 v$	$\sigma_3 z_1+\sigma_4 z_3$	$\sigma_3 z_2+\sigma_4 z_6$	$-\sigma_4 z_1+\sigma_3 z_3$	$\sigma_3 z_4+\sigma_4 z_5$	$-\sigma_4 z_4+\sigma_3 z_5$	$-\sigma_4 z_2+\sigma_3 z_6$
z_1	$\tau_1 z_1+\tau_2 z_3$	$\tau_3 z_1+\tau_4 z_3$	$-u$	z_4	v	$-z_2$	z_6	$-z_5$
z_2	$\tau_1 z_2+\tau_2 z_6$	$\tau_3 z_2+\tau_4 z_6$	$-z_4$	$-u$	z_5	z_1	$-z_3$	v
z_3	$-\tau_2 z_1+\tau_1 z_3$	$-\tau_4 z_1+\tau_3 z_3$	$-v$	$-z_5$	$-u$	z_6	z_2	$-z_4$
z_4	$\tau_1 z_4+\tau_2 z_5$	$\tau_3 z_4+\tau_4 z_5$	z_2	$-z_1$	$-z_6$	$-u$	v	z_3
z_5	$-\tau_2 z_4+\tau_1 z_5$	$-\tau_4 z_4+\tau_3 z_5$	$-z_6$	z_3	$-z_2$	$-v$	$-u$	z_1
z_6	$-\tau_2 z_2+\tau_1 z_6$	$-\tau_4 z_2+\tau_3 z_6$	z_5	$-v$	z_4	$-z_3$	$-z_1$	$-u$

Proof. We have already determined that such an algebra is the sum of two 1-dimensional $su(3)$-modules and an irreducible 6-dimensional module. In order to deduce the various products between the summands let us first consider a well-known example in which this type of decomposition occurs—namely the octonions. Let $\boldsymbol{O}$ be an octonion algebra with basis $u, e_1, \cdots, e_7$ and multiplication given by (2.1) with $\beta = \zeta = \eta = 1$. Let $L = \{\partial \in \mathrm{Der}\, \boldsymbol{O} \mid \partial(e_7) = 0\}$. Then L is isomorphic to $su(3)$. (See for example, [2], [5], or [7].) One can actually verify this assertion directly in the following manner. Let us complexify $\boldsymbol{O}$ and obtain a basis for $\boldsymbol{O}_C$ by taking:

$$\begin{aligned} u_0 &= \frac{1}{2}(u + ie_7) & u_0^* &= \frac{1}{2}(u - ie_7) \\ u_1 &= \frac{1}{2}(e_1 + ie_3) & u_1^* &= \frac{1}{2}(e_1 - ie_3) \\ u_2 &= \frac{1}{2}(e_2 + ie_6) & u_2^* &= \frac{1}{2}(e_2 - ie_6) \\ u_3 &= \frac{1}{2}(e_4 + ie_5) & u_3^* &= \frac{1}{2}(e_4 - ie_5)\ . \end{aligned}$$

Products between these elements can be calculated using (2.1). We list the results below where we adopt the convention that $\varepsilon_{jkl} = 1$ if (jkl) is an even permutation of $\{1, 2, 3\}$, $\varepsilon_{jkl} = -1$ if the permutation is odd, and $\varepsilon_{jkl} = 0$ if (jkl) is not a permutation of $\{1, 2, 3\}$, and δ_{jk} is the Kronecker delta.

$$\begin{array}{llll} u_0 u_j = u_j & u_0^* u_j = 0 & u_j u_0 = 0 & u_j u_0^* = u_j \\ u_0 u_j^* = 0 & u_0^* u_j^* = u_j^* & u_j^* u_0 = u_j^* & u_j^* u_0^* = 0 \\ u_0^2 = u_0 & \multicolumn{2}{l}{u_0 u_0^* = 0 = u_0^* u_0} & (u_0^*)^2 = u_0^* \end{array}$$

$$\begin{aligned} u_j u_k &= \varepsilon_{jkl} u_l^* & u_j u_k^* &= -\delta_{jk} u_0 \\ u_j^* u_k^* &= \varepsilon_{jkl} u_l & u_j^* u_k &= -\delta_{jk} u_0^*\ . \end{aligned}$$

Now $\partial \in L$ implies $\partial(u_0) = 0 = \partial(u_0^*)$. Moreover if X denotes the span of the u's and Y the span of the u^*'s, then $X = \{x \in \boldsymbol{O}_C \mid u_0 x = x = x u_0^*\}$ and $Y = \{y \in \boldsymbol{O}_C \mid u_0^* y = y = y u_0\}$. It is easy to see from these characterizations that X and Y are L_C invariant. From applying ∂ to the relation $u_j u_k^* = -\delta_{jk} u_0$, it follows that for each $\partial \in L_C$ the matrix of ∂ on Y relative to the u_j^* is minus the transpose of the matrix of ∂ on X relative to the u_j. In addition the trace of ∂ on X and on Y must be 0. These are the only restrictions on the elements of L_C. Thus $L_C \cong sl(3)$, and X is the module which we have been denoting by 3 (it is 3×1 matrices on which $sl(3)$ acts by matrix multiplication), and Y is $\bar{3}$ (it is 1×3 matrices on which the action of $sl(3)$ is right multiplication by minus the matrix). From these observations it follows that if Z denotes the span of $e_1, \cdots, e_6$ in $\boldsymbol{O}$, then L leaves Z invariant, while our previous remarks show that Z must be an irreducible module for $su(3)$.

Let us consider the L-module homomorphisms of $Z \otimes Z$ into Z. Since $Z_C = X + Y$, and since $3 \otimes 3 = 6 + \bar{3}$, $\bar{3} \otimes \bar{3} = \bar{6} + 3$ and $3 \otimes \bar{3} = 8 + 1$ we see $\dim_C \mathrm{Hom}_{L_C}((X+Y) \otimes (X+Y),\ X+Y) = 2$. It is spanned by the homomorphisms φ_1, φ_2 where $\varphi_1(u_j \otimes u_k) = \varepsilon_{jkl} u_l^*$, $\varphi_2(u_j^* \otimes u_k^*) = \varepsilon_{jkl} u_l$, and φ_1 and φ_2 are 0 on all products of basis elements not of the specified type.

Given $\varphi \in \mathrm{Hom}_L(Z \otimes Z, Z)$, then φ lifts to an L_C-homomorphism of $(X+Y) \otimes (X+Y)$ into $X+Y$, and so $\varphi = a\varphi_1 + b\varphi_2$ where $a, b \in \boldsymbol{C}$. Therefore $\varphi((u_j + u_j^*) \otimes (u_k + u_k^*)) = \varepsilon_{jkl}(a u_l^* + b u_l)$. But since $u_j + u_j^*$ and $u_k + u_k^*$ lie in Z, so does $a u_l^* + b u_l$, and $a u_l^* + b u_l = \alpha(u_l + u_l^*) + \beta i(u_l^* - u_l)$ where $\alpha, \beta \in \boldsymbol{R}$. Thus $a = \alpha + \beta i$, $b = \alpha - \beta i$ and $b = \bar{a}$. It follows that

$$\begin{aligned}
&\varphi((u_j + u_j^*) \otimes (u_k + u_k^*)) = \varepsilon_{jkl}\{\alpha(u_l + u_l^*) + \beta i(u_l^* - u_l)\} \\
&\varphi((u_j + u_j^*) \otimes i(u_k^* - u_k)) = \varepsilon_{jkl}\{\beta(u_l + u_l^*) - \alpha i(u_l^* - u_l)\} \\
&\varphi(i(u_j^* - u_j) \otimes (u_k + u_k^*)) = \varepsilon_{jkl}\{\beta(u_l + u_l^*) - \alpha i(u_l^* - u_l)\} \\
&\varphi(i(u_j^* - u_j) \otimes i(u_k^* - u_k)) = \varepsilon_{jkl}\{-\alpha(u_l + u_l^*) - \beta i(u_l^* - u_l)\}\ .
\end{aligned}$$

These equations determine the effect of φ on the $e_r \otimes e_s$ basis of $Z \otimes Z$.

Since the modules involved are unique up to isomorphism, the general case of an irreducible 6-dimensional $su(3)$-module Z which becomes $3 + \bar{3}$ upon complexification is no different from the behavior just observed. There is a basis $e_1, \cdots, e_6$ of Z such that any $su(3)$-module homomorphism φ is given as above for some $\alpha, \beta \in \boldsymbol{R}$. If Z is a summand in an algebra A which admits $su(3)$ as derivations, then these homomorphisms determine the possible products from $Z \times Z$ to Z, and since the homomorphisms are all skew-symmetric, the products will be anticommutative.

Thus we may assume that the products from $Z \times Z$ to Z are given by (4.3) for some $\alpha, \beta \in \boldsymbol{R}$.

(4.3)

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	—	$\alpha e_4+\beta e_5$	—	$-\alpha e_2-\beta e_6$	$-\beta e_2+\alpha e_6$	$\beta e_4-\alpha e_5$
e_2	$-\alpha e_4-\beta e_5$	—	$-\beta e_4+\alpha e_5$	$\alpha e_1+\beta e_3$	$\beta e_1-\alpha e_3$	—
e_3	—	$\beta e_4-\alpha e_5$	—	$-\beta e_2+\alpha e_6$	$\alpha e_2+\beta e_6$	$-\alpha e_4-\beta e_5$
e_4	$\alpha e_2+\beta e_6$	$-\alpha e_1-\beta e_3$	$\beta e_2-\alpha e_6$	—	—	$-\beta e_1+\alpha e_3$
e_5	$\beta e_2-\alpha e_6$	$-\beta e_1+\alpha e_3$	$-\alpha e_2-\beta e_6$	—	—	$\alpha e_1+\beta e_3$
e_6	$-\beta e_4+\alpha e_5$	—	$\alpha e_4+\beta e_5$	$\beta e_1-\alpha e_3$	$-\alpha e_1-\beta e_3$	—

If u_j, u_j^* are as defined above using the e's, then $u_j u_k = \varepsilon_{jkl}\, a\, u_l^*$ and $u_j^* u_k^* = \varepsilon_{jkl}\bar{a} u_l$ where $a = \alpha + \beta i$. Let us suppose $v_j = a^{-2/3}\bar{a}^{1/3}u_j$ and $v_j^* = a^{-1/3}\bar{a}^{2/3}u_j^*$ so that $v_j v_k = \varepsilon_{jkl}v_l^*$ and $v_j^* v_k^* = \varepsilon_{jkl}v_l$. Now let $z_1 = v_1 + v_1^*$, $z_2 = v_2 + v_2^*$, $z_4 = v_3 + v_3^*$, $z_3 = i(v_1^* - v_1)$, $z_6 = i(v_2^* - v_2)$, $z_5 = i(v_3^* - v_3)$. Then the multiplication table for the z's is the same as (4.3) when $\alpha = 1$ and $\beta = 0$. Note $z_1 = \gamma e_1 + \zeta e_3$, $z_2 = \gamma e_2 + \zeta e_6$, $z_4 = \gamma e_4 + \zeta e_5$, $z_3 = -\zeta e_1 + \gamma e_3$, $z_6 = -\zeta e_2 + \gamma e_6$, $z_5 = -\zeta e_4 + \gamma e_5$ where $\gamma = 1/2(a^{-2/3}\bar{a}^{-1/3} + \bar{a}^{-2/3}a^{-1/3})$ and $\zeta = (1/2)i(a^{-2/3}\bar{a}^{-1/3} - \bar{a}^{-2/3}a^{-1/3})$. Since $\bar{\gamma} = \gamma$ and $\bar{\zeta} = \zeta$, the z's lie in Z, and they are the desired basis.

To calculate further entries in the (4.2) table let us recall that $3 \otimes \bar{3} = 8 + 1$. (This resolution can be concretely realized by the matrix multiplication of a 3×1 matrix with a 1×3 matrix followed by projection onto $sl(3)$ and $C\cdot I$). Thus, $v_j \otimes v_k^* \to \delta_{jk}w$ is an $sl(3)$-module homomorphism of $3 \otimes \bar{3}$ onto the 1-dimensional module spanned by w, and any other one is just a complex multiple of this homomorphism. From this it follows that any $su(3)$-module homomorphism $\psi_1 \colon Z \times Z \to \boldsymbol{R}w$ when lifted to $(3 + \bar{3}) \otimes (3 + \bar{3}) \to \boldsymbol{C}w$ is given by $\psi_1(v_j \otimes v_k^*) = c\delta_{jk}w$, $\psi_1(v_k^* \otimes v_j) = d\delta_{jk}w$ for $c, d \in \boldsymbol{C}$ and the condition $\psi_1(Z \otimes Z) \subseteq \boldsymbol{R}w$ forces $d = \bar{c}$. Thus if $c = \alpha_1 + \beta_1 i$

$$\begin{aligned}
\psi_1((v_j + v_j^*) \otimes (v_k + v_k^*)) &= 2\delta_{jk}\alpha_1 w \\
\psi_1((v_j + v_j^*) \otimes i(v_k^* - v_k)) &= 2\delta_{jk}\beta_1 w \\
\psi_1(i(v_j^* - v_j) \otimes (v_k^* + v_k)) &= -2\delta_{jk}\beta_1 w \\
\psi_1(i(v_j^* - v_j) \otimes i(v_k^* - v_k)) &= 2\delta_{jk}\alpha_1 w \,.
\end{aligned}$$

Similarly if $\boldsymbol{R}x$ is the other 1-dimensional summand any homomorphism ψ_2 is prescribed by scalars α_2, β_2. Thus any product of $Z \times Z$ into the two 1-dimensional summands is determined by four scalars $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \boldsymbol{R}$. This number can be reduced by making the change of basis $u = -2\alpha_1 w - 2\alpha_2 x$, $v = 2\beta_1 w + 2\beta_2 x$. For then $z_j^2 = -u$ for all j and $z_1 z_3 = z_2 z_6 = z_4 z_5 = v = -z_3 z_1 = -z_6 z_2 = -z_5 z_4$ as in the table. The elements u, v will seem less mysterious if one keeps the octonion example in mind, for there v corresponds to e_7 and u to the identity element.

The relations $1 \otimes 3 = 3$, $1 \otimes \bar{3} = \bar{3}$ similarly imply the existence of scalars $\sigma_1, \sigma_2 \in \boldsymbol{R}$ such that

$$\begin{aligned}
u(v_j + v_j^*) &= \sigma_1(v_j + v_j^*) + \sigma_2 i(v_j^* - v_j) \\
ui(v_j^* - v_j) &= -\sigma_2(v_j + v_j^*) + \sigma_1 i(v_j^* - v_j) \,.
\end{aligned}$$

In this fashion one obtains the entries in the table involving the σ's and τ's. Since $\boldsymbol{R}u + \boldsymbol{R}v$ is a subalgebra, the products u^2, uv, vu, v^2 are of the form indicated by (4.2), and the determination of the table is complete. This table has been constructed by using $su(3)$-modules and $su(3)$-module homomorphisms at each stage, so that any algebra having (4.2) as its table for σ's, τ's, η's, θ's in $\boldsymbol{R}$ will admit $su(3)$ as derivations. $\square$

If A is a division algebra with multiplication given by (4.2) and if $\operatorname{Der} A$ is larger than $su(3)$, then $\operatorname{Der} A$ is a compact G_2, since this is the only Lie algebra in our classification of derivation algebras of real division algebras which can properly contain $su(3)$. The criterion for when $\operatorname{Der} A$ is a compact G_2 is given in

THEOREM 4.4. *If A is a real division algebra with multiplication given by* (4.2), *then* $\operatorname{Der} A$ *is a compact form of G_2 if and only if the following relations hold*:

$$(4.5)\qquad \begin{aligned} &\eta_2 = 0 = \eta_3,\ \theta_1 = 0 = \theta_4,\ \theta_2 = \sigma_1,\ \theta_3 = \tau_1\ , \\ &\eta_4 = -1 = \tau_4,\ \sigma_2 = 0 = \tau_2,\ \sigma_3 = 0 = \tau_3,\ \sigma_4 = 1\ . \end{aligned}$$

Otherwise $\operatorname{Der} A = su(3)$.

Proof. If the relations (4.5) hold, then it is immediate that A is isomorphic to the algebra defined by (2.1) with $\zeta = \eta_1^{-1}\tau_1$, $\eta = \eta_1^{-1}\sigma_1$, and $\beta = \eta_1$ under the correspondence $u \leftrightarrow \eta_1^{-1}u$, $e_7 \leftrightarrow v$, and $e_j \leftrightarrow z_j$ for $j = 1, \cdots, 6$. Hence $\operatorname{Der} A$ is a compact G_2 in this case.

Conversely suppose $\operatorname{Der} A$ is a compact form of G_2. Then A decomposes relative to $\operatorname{Der} A$ into a 1-dimensional module U and a 7-dimensional irreducible module V as in § 2. Since Z is the image of A under $su(3) \subsetneqq \operatorname{Der} A$, Z must be contained in V, the image of A under $\operatorname{Der} A$. Every element of V is known to square to an element in U (see Table 2.1), but every element of Z squares to an multiple of u. Thus U is the span of u, and V is the span of the z's and $v + \lambda u$ for some $\lambda \in \boldsymbol{R}$. It follows that right or left multiplication by u on V, and also on U, must be a scalar multiple of the identity transformation, and this implies that $\sigma_2 = 0_1 = \tau_2$, $\theta_1 = 0$, $\theta_2 = \sigma_1$, and $\theta_3 = \tau_1$. Since A is a division algebra, left or right multiplication by a nonzero linear combination of u and v on z_1 must be nonzero, and this forces $\sigma_4 \neq 0$ and $\tau_4 \neq 0$.

We deduce further relations by recalling that products from $V \times V \rightarrow V$ are the same as in the octonions. Thus, they share the property that if x, y and yx are in V, then $(yx)x \in \boldsymbol{R}y$ and $y(yx) \in \boldsymbol{R}x$, since these properties follow from the alternativity of the octonions. (See for example, Schafer [9].) Such elements are $v + \lambda u$, z_1 and $(v + \lambda u)z_1 = (\sigma_3 + \lambda\sigma_1)z_1 + \sigma_4 z_3$, so that

$$(4.6)\qquad ((v + \lambda u)z_1)z_1 = -(\sigma_3 + \lambda\sigma_1)u - \sigma_4 v \in \boldsymbol{R}(v + \lambda u)\ .$$

$$(v+\lambda u)((v+\lambda u)z_1) = ((\sigma_3+\lambda\sigma_1)(\sigma_3+\lambda\sigma_1)-\sigma_4^2)z_1 + 2\sigma_4(\sigma_3+\lambda\sigma_1)z_4 \in \boldsymbol{R}z_1\,. \tag{4.7}$$

Since $\sigma_4 \neq 0$, equation (4.7) implies that $\sigma_3 + \lambda\sigma_1 = 0$, and this together with (4.6) says $-\sigma_4 v \in \boldsymbol{R}(v+\lambda u)$. Hence $\lambda = 0$, $\sigma_3 = 0$, and an analogous argument with $z_1(z_1 v)$ determines that $\tau_3 = 0$. Further consequences of the result that $v \in V$ are $\eta_2 = 0 = \eta_3$ and $\theta_4 = 0$, because $x^2 \in U$ for each $x \in V$.

An additional property of V inherited from the octonions is that if $y \in V$ and if for some $x \in V$, $xy \in V$ and $(xy)y = -\rho x$, for $\rho \in \boldsymbol{R}$, then for any $w \in V$ with $wy \in V$, $(wy)y = -\rho w$. Therefore, $(z_2 z_1)z_1 = -z_2$ and $(vz_1)z_1 = -\sigma_4 v$ imply $\sigma_4 = 1$, while by symmetry $z_1(z_1 z_2) = -z_2$ and $z_1(z_1 v) = \tau_4 v$ give $\tau_4 = -1$.

All that is left to be shown is that $\eta_4 = -1$. However if x, y, and xy are in V, and $(xy)y = -x$ and $x(xy) = -y$ then $x^2 = y^2$, since the corresponding result holds for the octonions. But then $(vz_1)z_1 = -v$ and $v(vz_1) = -z_1$ imply $-u = z_1^2 = v^2 = \eta_4 u$. From this we deduce that $\eta_4 = -1$, so that all the conditions in (4.5) do indeed hold when $\operatorname{Der} A =$ compact G_2. □

The question of when a real algebra with multiplication given by (4.2) is a division algebra is formidable because of the large number of scalars in the multiplication table. However, we can exhibit division algebras of this type which have $su(3)$ as their full derivation algebra. The easiest example is obtained by taking the values of the constants prescribed in (4.5) with the sole exception that η_4 is some negative number besides -1. This algebra was shown to be a division algebra in ([1], Theorem 20), and it has $su(3)$ as its derivation algebra according to Theorem 4.4.

5. The case $\operatorname{Der} A = su(2) \oplus su(2)$. An irreducible $sl(2) \oplus sl(2)$-module over $\boldsymbol{C}$ is just the tensor product of two irreducible $sl(2)$-modules (one for each summand of $sl(2) \oplus sl(2)$). If V_1 is an irreducible module for the first copy of $sl(2)$, and V_2 for the second copy, and if $\dim V_1 = m$ and $\dim V_2 = n$, then $V_1 \otimes V_2$ is an irreducible module for $sl(2) \oplus sl(2)$ of dimension mn, and we denote this module by $m \mathbin{\tilde{\otimes}} n$.

Suppose now $su(2) \oplus su(2) \subseteq \operatorname{Der} A$ where A is a real division algebra, and for convenience write S_1 and S_2 for the two copies of $su(2)$. As we explained in § 1, we have the S_1-module decomposition $A = A_0 \oplus A_*$ where A_0 is the space of elements annihilated by S_1 and A_* is the image of A under S_1. Since S_1 and S_2 commute, it is easy to see that A_0 and A_* are invariant under S_2. The S_2-action on A_0 and A_* affords the decompositions, $A_0 = A_{00} \oplus A_{0*}$ and $A_* = A_{*0} \oplus A_{**}$. Thus, relative to $S_1 \oplus S_2$

$$A = A_{00} \oplus A_{0*} \oplus A_{*0} \oplus A_{**} .$$

Since the smallest nontrivial $su(2)$-module has dimension 3, we see $A_{0*} = 0$ or $\dim A_{0*} \geqq 3$, and the same is true of A_{*0}. Because $(A_{**})_C$ is just the sum of all irreducible $sl(2) \oplus sl(2)$-modules not annihilated by either summand, it follows that either $A_{**} = 0$ or else $\dim A_{**} \geqq 4$. We consider the various possibilities.

Of course, not all A_{*0}, A_{0*}, A_{**} can be zero, since $S_1 \oplus S_2$ acts nontrivially on A. If $A_{0*} \neq 0 \neq A_{*0}$, then a simple dimension count shows $A_{**} = 0$. Since $(m \mathbin{\tilde{\otimes}} 1) \otimes (1 \mathbin{\tilde{\otimes}} n) = m \mathbin{\tilde{\otimes}} n$ for $sl(2) \oplus sl(2)$-modules, $A_{*0}A_{0*} \subseteqq (A_{*0})_C(A_{0*})_C \subseteqq (A_{**})_C = 0$. This contradiction enables us to conclude either $A_{*0} = 0$ or $A_{0*} = 0$. Without loss of generality we suppose that $A_{0*} = 0$, and hence $A = A_{00} \oplus A_{*0} \oplus A_{**}$. In this decomposition $A_{**} \neq 0$, since otherwise S_2 would act trivially on A. Now $(A_{**})_C$ is comprised of a sum of modules of the following types: $2 \mathbin{\tilde{\otimes}} 2$, $2 \mathbin{\tilde{\otimes}} 3$, $3 \mathbin{\tilde{\otimes}} 2$, $2 \mathbin{\tilde{\otimes}} 4$, $4 \mathbin{\tilde{\otimes}} 2$. In any event, $(A_{**})_C$ is the direct sum of copies of modules of dimension 2 when it is decomposed relative to one of the copies of $sl(2)$. Since $2 \otimes 2 = 3 + 1$ for $sl(2)$-modules it must be that $(A_{**})_C^2 \subseteqq (A_{00})_C + (A_{*0})_C$, and hence $A_{**}^2 \subseteqq A_{00} + A_{*0}$. For any $x \neq 0$ in A_{**}, $xA_{**} \subseteqq A_{00} + A_{*0}$, and because left multiplication by x is nonsingular, $\dim A_{**} \leqq \dim(A_{00} + A_{*0})$. Thus, there is only one possibility for $(A_{**})_C$, namely $(A_{**})_C = 2 \mathbin{\tilde{\otimes}} 2$.

If $\dim A_{00} = 0$, then $\dim A_{*0} = \dim A_{**} = 4$ and $(A_C)_{*0} = 2 + 2$ relative to $(S_1)_C = sl(2)$. But then A_C is just the sum of 2-dimensional modules for $(S_1)_C$, and as above $2 \otimes 2 = 3 + 1$ shows that all products are zero. Thus, it is impossible for A_{00} to be zero.

Consider now the possibility $A_{*0} = 0$. In this instance $\dim A_{00} = \dim A_{**} = 4$, and every derivation of A in $S_1 \oplus S_2$ has rank $\leqq 4$. If this is the case, then any space of commuting derivations has dimension not more than one according to ([1], Corollary 16). However, $S_1 \oplus S_2$ has a 2-dimensional space of commuting derivations, so we arrive at a contradiction. Therefore $A_{*0} \neq 0$, and since $A_{00} \neq 0$ and $\dim A_{**} = 4$, it must be that $\dim A_{*0} = 3$. This is the first part of the principal result of this section which we are now ready to state and prove.

THEOREM 5.1. *Let A be a real division algebra such that $su(2) \oplus su(2) \subseteqq \mathrm{Der}\, A$. Then as an $su(2) \oplus su(2)$-module, A is a direct sum of a 1-dimensional module U annihilated by both copies of $su(2)$, a 3-dimensional module X irreducible under one copy of $su(2)$ and annihilated by the other, and a 4-dimensional module Y irreducible under both copies of $su(2)$. There exists a basis u, x_1, x_2, x_3, y_1, y_2, y_3, y_4 such that the multiplication in A is given by (5.2) for some $\beta, \gamma, \delta, \varepsilon, \eta, \zeta, \theta, \rho, \sigma \in \boldsymbol{R}$. Furthermore,* Der A *is either $su(2) \oplus su(2)$ or a compact*

G_2; the latter occurring exactly when the following relations hold: $\varepsilon = 1 = \eta$, $\beta\gamma = \delta$, $\zeta = \rho$, $\theta = \sigma$, $\gamma > 0$, and $\beta\rho\sigma < 0$.

(5.2)

	u	x_1	x_2	x_3	y_1	y_2	y_3	y_4
u	u	ζx_1	ζx_2	ζx_3	ρy_1	ρy_2	ρy_3	ρy_4
x_1	θx_1	βu	x_3	$-x_2$	εy_4	εy_3	$-\varepsilon y_2$	$-\varepsilon y_1$
x_2	θx_2	$-x_3$	βu	x_1	εy_2	$-\varepsilon y_1$	εy_4	$-\varepsilon y_3$
x_3	θx_3	x_2	$-x_1$	βu	$-\varepsilon y_3$	εy_4	εy_1	$-\varepsilon y_2$
y_1	σy_1	$-\eta y_4$	$-\eta y_2$	ηy_3	δu	γx_2	$-\gamma x_3$	γx_1
y_2	σy_2	$-\eta y_3$	ηy_1	$-\eta y_4$	$-\gamma x_2$	δu	γx_1	γx_3
y_3	σy_3	ηy_2	$-\eta y_4$	ηy_1	γx_3	$-\gamma x_1$	δu	γx_2
y_4	σy_4	ηy_1	ηy_3	ηy_2	$-\gamma x_1$	$-\gamma x_3$	$-\gamma x_2$	δu

Proof. Recall from §1 that $\partial_1 = ih$, $\partial_2 = e - f$ $\partial_3 = i(e + f)$ give a basis of $su(2)$ such that $[\partial_j, \partial_{j+1}] = 2\partial_{j+2}$ where the indices are interpreted modulo 3. Let $\partial_1, \partial_2, \partial_3$ denote such a basis for the copy of $su(2)$ which acts irreducibly on X, and let $\partial_1', \partial_2', \partial_3'$ be the corresponding basis for the other copy. Since the module X is just the adjoint representation of $su(2)$, there is a basis x_1, x_2, x_3 of X such that the action of $su(2) \oplus su(2)$ on X is given by

(5.3) $\partial_j(x_{j+1}) = 2x_{j+2}$ where the subscripts are read modulo 3
$\partial_j'(x_k) = 0$ for all j and k.

Now $Y_C = 2 \tilde{\otimes} 2$, or in the notation of §1, $Y_C = V(1) \otimes V(1)$. Recall $V(1) \otimes V(1)$ has as basis $\{v_j \otimes v_k\}$ where $j, k = \pm 1$. Let

$$\begin{aligned}
y_1 &= v_1 \otimes iv_1 + iv_1 \otimes v_1 + v_{-1} \otimes iv_{-1} + iv_{-1} \oplus v_{-1} \\
y_2 &= v_{-1} \otimes iv_1 + iv_{-1} \otimes v_1 - v_1 \otimes iv_{-1} - iv_1 \otimes v_{-1} \\
y_3 &= iv_{-1} \otimes iv_1 - v_{-1} \otimes v_1 + iv_1 \otimes iv_{-1} - v_1 \otimes v_{-1} \\
y_4 &= v_1 \otimes v_1 - iv_1 \otimes iv_1 - v_{-1} \otimes v_{-1} + iv_{-1} \otimes iv_{-1} .
\end{aligned}$$

Then y_1, y_1, y_3, y_4 span an $su(2) \oplus su(2)$-module as the table below indicates.

(5.4)

	y_1	y_2	y_3	y_4
∂_1	$-y_4$	$-y_3$	y_2	y_1
∂_2	$-y_2$	y_1	$-y_4$	y_3
∂_3	y_3	$-y_4$	$-y_1$	y_2
∂_1'	$-y_4$	y_3	$-y_2$	y_1
∂_2'	y_2	$-y_1$	$-y_4$	y_3
∂_3'	y_3	y_4	$-y_1$	$-y_2$

Thus, Y must be isomorphic to the span of the y's.

In order to deduce the products X^2, UX, and XU, we recall that for $sl(2)$-modules $3 \otimes 3 = 5 + 3 + 1$, and $3 \otimes 1 = 3$ (see (1.6)). Thus, $X^2 \subseteqq X + U$, and $XU + UX \subseteqq X$, and up to scalar multiple there is just one possible product in each case. The product from $X \times X$ to X is just the Lie product on $su(2)$, from $X \times X$ to U the product is simply the inner product (as seen from the quaternions on which $su(2)$ acts as derivations), and from $X \times U$ to X or $U \times X$ to X the product is just multiplication by a scalar. After replacing each x_i by an appropriate scalar multiple of itself, we obtain the portion of the multiplication table (5.2) pertaining to products on $X + U$.

Now for products involving Y, we have $(3 \tilde{\otimes} 1) \otimes (2 \tilde{\otimes} 2) = (3 \otimes 2) \tilde{\otimes} (1 \otimes 2) = (4 + 2) \tilde{\otimes} 2 = 4 \tilde{\otimes} 2 + 2 \tilde{\otimes} 2$. Thus, $XY + YX \subseteqq Y$, $\dim_C \mathrm{Hom}_{sl(2)\oplus sl(2)}(X_C \otimes Y_C, Y_C) = 1$, and consequently

$$\dim_R \mathrm{Hom}_{su(2)\oplus su(2)}(X \otimes Y, Y) \leqq 1 .$$

Moreover, $(2 \tilde{\otimes} 2) \otimes (2 \tilde{\otimes} 2) = (3 + 1) \tilde{\otimes} (3 + 1)$ demonstrates that $Y^2 \subseteqq X + U$, $\dim_R \mathrm{Hom}_{su(2)\oplus su(2)}(Y \otimes Y, X) \leqq 1$, and $\dim_R \mathrm{Hom}_{su(2)\oplus su(2)}(Y \otimes Y, U) \leqq 1$. Finally $(1 \tilde{\otimes} 1) \otimes (2 \tilde{\otimes} 2) = 2 \tilde{\otimes} 2$ shows that constants $\sigma, \rho \in \boldsymbol{R}$ exist so that $uy_j = \rho y_j$ and $y_j u = \sigma y_j$ for all j.

In order to find the products XY, YX, and Y^2 we again turn to the octonions for guidance. It is known that the transformations

$$D_{v,w} = -ad_{[v,w]} + 3[L_v, R_w]$$

are derivations of the octonions for any two elements v, w in the octonion algebra, where $L_v(t) = vt$, $R_w(t) = tw$, and $ad_{[v,w]}(t) = [[vw]t]$. (See [7, page 2].) Using the fact that $[\partial, D_{v,w}] = D_{\partial(v),w} + D_{v,\partial(w)}$, one can verify readily that

$$\partial_1 = -\frac{1}{2} D_{e_2,e_4},\ \partial_2 = -\frac{1}{2} D_{e_1,e_1},\ \partial_3 = -\frac{1}{2} D_{e_1,e_2}$$

$$\partial_1' = \frac{1}{2}(D_{e_3,e_7} - D_{e_5,e_6}),\ \partial_2' = \frac{1}{2}(D_{e_6,e_7} - D_{e_3,e_5}),\ \partial_3' = \frac{1}{2}(D_{e_6,e_3} - D_{e_5,e_7})$$

span a $su(2) \oplus su(2)$ subalgebra of the derivation algebra of the octonions with multiplication as above. Moreover, if one makes the following identifications $u \leftrightarrow 1$, $x_1 \leftrightarrow e_1$, $x_2 \leftrightarrow e_2$, $x_3 \leftrightarrow e_4$, $y_1 \leftrightarrow e_3$, $y_2 \leftrightarrow e_5$, $y_3 \leftrightarrow e_6$, and $y_4 \leftrightarrow e_7$, the action of $su(2) \oplus su(2)$ on the x's and y's is exactly that given by (5.3) and (5.4). Therefore, since there is at most one $su(2) \oplus su(2)$-homomorphism up to scalar multiple in each of the cases: $X \otimes Y \to Y$, $Y \otimes X \to Y$, $Y \otimes Y \to X$, and $Y \otimes Y \to U$, the homomorphism can be computed easily from the corresponding products in the octonions. This calculation gives the remaining entries in (5.2).

If $\operatorname{Der} A$ properly contains $su(2) \oplus su(2)$ for a division algebra A with multiplication given by (5.2), then $\operatorname{Der} A$ must be a compact G_2, since this is the only Lie algebra in our classification of derivation algebras of real division algebras which can properly contain $su(2) \oplus su(2)$. The proof of Theorem 5.1 will be complete if we can show that $\operatorname{Der} A$ is a compact G_2 if and only if

$$(5.5) \qquad \varepsilon = 1 = \eta,\ \beta\gamma = \delta,\ \zeta = \rho,\ \theta = \sigma,\ \gamma > 0, \text{ and } \beta\rho\sigma < 0\ .$$

If A satisfies the relations (5.5), then (5.2) reduces to the multiplication given in (2.1) under the correspondence given by $x_1 \leftrightarrow e_1$, $x_2 \leftrightarrow e_2$, $x_3 \leftrightarrow e_4$, $y_1 \leftrightarrow \sqrt{\gamma} e_3$, $y_2 \leftrightarrow \sqrt{\gamma} e_5$, $y_3 \leftrightarrow \sqrt{\gamma} e_6$, $y_4 \leftrightarrow \sqrt{\gamma} e_7$, and so $\operatorname{Der} A =$ compact G_2 when the relations (5.5) hold.

Conversely, suppose that $\operatorname{Der} A$ is a compact G_2 for a certain choice of the constants in (5.2). Then A is isomorphic to one of the algebras of the form (2.1), and this isomorphism φ must take $X+Y$ onto $V = \langle e_1, e_2, \cdots, e_7\rangle$. Now V inherits from the alternativity of the octonions the property that if $v_1, v_2 \in V$ and if $v_1v_2 \in V$ then $v_1(v_1v_2) \in \langle v_2\rangle$. The same property must also hold for $\varphi^{-1}(V) = X+Y$, so that using (5.2) we obtain

$$\begin{aligned}(x_2 + y_1)((x_2 + y_1)x_1) = (x_2 + y_1)(-x_3 - \eta y_4) &= -(1 + \gamma\eta)x_1 \\ &\quad + (\varepsilon\eta - \eta)y_3 \in \langle x_1\rangle\ .\end{aligned}$$

Thus, $\varepsilon\eta - \eta = 0$, and $\varepsilon = 1$ because $\eta \neq 0$ in a division algebra. Since V is anticommutative, we also have $\eta = \varepsilon$, and so $\eta = 1$. Then,

$$\begin{aligned}(x_2 + y_1)((x_2 + y_1)y_2) = (x_2 + y_1)(-y_1 + \gamma x_2) &= -(1 + \gamma)y_2 \\ &\quad + (\beta\gamma - \delta)u \in \langle y_2\rangle\ ,\end{aligned}$$

giving $\beta\gamma = \delta$. Since left multiplication by u is just a multiple of the identity on V, it follows the $\zeta = \rho$, and similarly $\theta = \sigma$. If $\gamma < 0$, then $\sqrt{-\gamma}$ is a real number and

$$\begin{aligned}(\sqrt{-\gamma}u + \rho y_1))\sqrt{-\gamma}x_3 - y_3) &= -\gamma\rho x_3 - \sqrt{-\gamma}\rho y_3 \\ &\quad + \sqrt{-\gamma}\rho y_3 + \gamma\rho x_3 = 0\end{aligned}$$

using (5.2). Hence, $\gamma > 0$ when A is a division algebra. Finally, for any $c \in \boldsymbol{R}$,

$$\begin{aligned}(\sigma u + cx_1)(\sigma u - cx_1) &= \rho\sigma u - \rho\sigma c x_1 + \rho\sigma c x_1 - \beta c^2 u \\ &= (\rho\sigma - \beta c^2)u\ ,\end{aligned}$$

using $\zeta = \rho$ and $\theta = \sigma$. If $\beta\rho\sigma > 0$, we can set $c = \sqrt{\beta^{-1}\rho\sigma}$ in the last calculation and obtain zero divisors. Thus, $\beta\rho\sigma < 0$ in a division algebra, and we have verified all the relations of (5.5). $\square$

Although we shall not attempt to derive necessary and sufficient conditions on the constants for the algebra A given by (5.2) to be a division algebra, we note that there do exist division algebras of this form with $\operatorname{Der} A = su(2) \oplus su(2)$. For example, if we choose $\varepsilon = 1 = \eta = \gamma$, $\beta < 0$, $\beta \neq \delta < 0$, $\zeta = \rho = 1 = \theta = \sigma$, then A is isomorphic to the division algebra of ([1], Theorem 20) using the map $u \leftrightarrow u$, $x_1 \leftrightarrow e_1$, $x_2 \leftrightarrow e_2$, $x_3 \leftrightarrow e_4$, $y_1 \leftrightarrow e_3$, $y_2 \leftrightarrow e_5$, $y_3 \leftrightarrow e_6$, $y_4 \leftrightarrow e_7$.

6. The case $\operatorname{Der} A = su(2)$ and $\operatorname{Der} A = su(2) + N$. Suppose now A is a real division algebra and that $su(2) \subseteqq \operatorname{Der} A$. Using the convention explained in §1 of denoting an irreducible $su(2)$-module by its dimension, we can state

PROPOSITION 6.1. *If A is a real division algebra such that $su(2) \subseteqq \operatorname{Der} A$, then the decomposition of A into irreducible $su(2)$-modules has one of the following forms*: $1 + 3$, $1 + 7$, $3 + 5$, $1 + 1 + 3 + 3$, $1 + 3 + 4$, $1 + 1 + 1 + 1 + 4$.

Proof. We suppose first that A is a direct sum of odd-dimensional irreducible modules. At least one irreducible module of dimension $\geqq 3$ must be present, since $su(2)$ cannot act trivially on all of A. Then the only possibility when $\dim A = 4$ is $1 + 3$. For $\dim A = 8$, we note that the elements of A annihilated by all of $su(2)$ form a subalgebra which has dimension 0, 1, 2, or 4. With this restriction on the number of 1's in the decomposition, it is immediate that the only possible decompositions are $1 + 7$, $3 + 5$, and $1 + 1 + 3 + 3$.

Suppose then that A has an even-dimensional irreducible module. Since by (1.6) the product of even-dimensional irreducible modules in A_C must lie in the sum of the odd-dimensional irreducible modules, the same is true in A. Thus A must also have odd-dimensional irreducible modules. In fact, the dimension of the sum of the odd-dimensional irreducible modules must be the same as the dimension of the sum of the even-dimensional modules, since right multiplication by any nonzero element of an even-dimensional irreducible module will map each of these two spaces into the other. As the smallest even-dimensional irreducible $su(2)$-module has dimension 4, it follows that $\dim A = 8$ and that A is the sum of a single 4-dimensional irreducible module and some odd-dimensional irreducible modules. The only possibilities are $1 + 3 + 4$ and $1 + 1 + 1 + 1 + 4$. □

We discuss in turn each of the cases that arise in Proposition 6.1 beginning with the case $1 + 3$. This case is very similar to the case when Der $A =$ compact G_2, since we see that there is exactly one product from 3×3 to 3 and one from 3×3 to 1. Then A is just like the quaternions except that there are several constants in the table. Specifically the multiplication table for A is given by

$$\begin{array}{c|cccc|} & u & e_1 & e_2 & e_4 \\ \hline u & u & \eta e_1 & \eta e_2 & \eta e_4 \\ e_1 & \zeta e_1 & -\beta u & e_4 & -e_2 \\ e_2 & \zeta e_2 & -e_4 & -\beta u & e_1 \\ e_4 & \zeta e_4 & e_2 & -e_1 & -\beta u \\ \hline \end{array} \tag{6.2}$$

where we have normalized e_1, e_2, e_4 to make the scalar involved in the product from 3×3 to 3 become 1, and we have normalized u so that $u^2 = u$. Since this algebra is a subalgebra of the algebra given by (2.1), it is a division algebra if $\beta\eta\zeta > 0$ by Theorem 2.2. Conversely, if the algebra given by (6.2) is a division algebra, then the equation

$$0 = (a_0 u + a_1 e_1 + a_2 e_2 + a_4 e_4)(b_0 u + b_1 e_1 + b_2 e_2 + b_4 e_4)$$

can hold only if either all the a's or all the b's are zero. An argument identical to the proof of Theorem 2.2 shows that this condition implies $\beta\eta\zeta > 0$. We have proved

THEOREM 6.3. *A 4-dimensional real algebra is a division algebra with* $su(2)$ *as its derivation algebra if and only if* A *has a basis* u, e_1, e_2, e_4 *with multiplication given by* (6.2) *for some real numbers* β, η, ζ *such that* $\beta\eta\zeta > 0$.

The best known algebra belonging to the class defined by (6.2) is of course the algebra of quaternions, which arises by taking $\beta = \eta = \zeta = 1$. If we take $\beta = 1$ and $\eta = -1 = \zeta$, we obtain the pseudo-quaternions of Okubo [8].

We consider next the case when A has the decomposition $1+7$. Here we can establish

THEOREM 6.4. *If* A *is a real division algebra with* $su(2) \subseteq \mathrm{Der}\, A$, *and if* A *breaks up as an* $su(2)$*-module into a sum of a* 1*-dimensional module and an irreducible* 7*-dimensional module, then* Der A *is a compact* G_2. *Hence the structure of* A *is described by Theorem* 2.2.

Proof. Let A be an algebra satisfying the hypotheses of Theorem 6.4, let U be the 1-dimensional module, and let E be the irreducible 7-dimensional module. Then $U \otimes U \cong U$, and so U is a subalgebra spanned by an idempotent u. Also, $U \otimes E \cong E$, and right multiplication by u acts on E as a scalar multiple of the identity transformation. Similarly, left multiplication by u acts on E as a scalar multiple of the identity. By the Clebsch-Gordan formula, there is up to a scalar multiple exactly one homomorphism from $E \otimes E$ to E, and exactly one from $E \otimes E$ to U. If we can show that these are the same two homomorphisms which come out of the algebras defined by (2.1), we will have shown that the present algebra A belongs to the class of algebras defined by (2.1). In order to demonstrate that these homomorphisms are the same, it is sufficient to exhibit an algebra which satisfies the hypotheses of Theorem 6.4 and which also has the form (2.1), since the modules involved are unique up to isomorphism. Thus, it suffices to establish that the octonions $\boldsymbol{O}$ satisfy the hypotheses of Theorem 6.4.

Letting $\boldsymbol{O}$ be spanned by $u, e_1, \cdots, e_7$ where multiplication is given by (2.1) with $\beta = \eta = \zeta = 1$, we show that there exists a subalgebra of $\operatorname{Der} \boldsymbol{O}$ isomorphic to $su(2)$ which acts irreducibly on $E = \langle e_1, \cdots, e_7 \rangle$. As we noted in § 5, the maps

$$D_{ij} = -ad_{[e_i, e_j]} + 3[L_{e_i}, \boldsymbol{R}_{e_j}]$$

are known to be derivations of $\boldsymbol{O}$. Then the linear transformations

$$\partial_1 = \frac{1}{3} D_{2,6} - \frac{4}{3} D_{4,5}$$

$$\partial_2 = -\frac{1}{2}\sqrt{6}\, D_{3,7} + \frac{1}{6}\sqrt{10}(D_{1,2} - D_{6,3})$$

$$\partial_3 = \frac{1}{2}\sqrt{6}\, D_{7,1} + \frac{1}{6}\sqrt{10}(D_{6,1} - D_{2,3})$$

are also derivations of $\boldsymbol{O}$, and one can verify that the action of the ∂_j's on E is given by

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
∂_1	$-2e_3$	$-4e_6$	$2e_1$	$6e_5$	$-6e_4$	$4e_2$	0
∂_2	$\sqrt{10}e_2$	$\sqrt{10}\,e_1 - \sqrt{6}e_4$	$-\sqrt{10}e_6 + 2\sqrt{6}e_7$	$\sqrt{6}\,e_2$	$-\sqrt{6}\,e_6$	$\sqrt{6}e_5 + \sqrt{10}e_3$	$-2\sqrt{6}e_3$
∂_3	$\sqrt{10}e_6 + 2\sqrt{6}e_7$	$\sqrt{10}\,e_3 - \sqrt{6}e_5$	$-\sqrt{10}e_2$	$\sqrt{6}\,e_6$	$\sqrt{6}\,e_2$	$-\sqrt{6}e_4 - \sqrt{10}e_1$	$-2\sqrt{6}e_1$

It is straightforward to check using this table that $[\partial_j, \partial_{j+1}] = 2\partial_{j+2}$ where the subscripts are interpreted modulo 3. Thus, $\partial_1, \partial_2, \partial_3$ span a subalgebra of $\operatorname{Der} \boldsymbol{O}$ which is isomorphic to $su(2)$.

It remains to show that E is irreducible under this copy S of $su(2)$. We show first that each basis element e_j generates all of E under the action of S. Let $M(e_j)$ denote the S-submodule of E

generated by e_j. From the action of ∂_1, we see that $M(e_1) = M(e_3)$, $M(e_2)=M(e_6)$, and $M(e_4)=M(e_5)$. Since $\partial_2(e_4)=\sqrt{6}e_2$ and $\partial_2(e_2)=\sqrt{10}e_1-\sqrt{6}e_4$, we have $e_1, \cdots, e_6 \in M(e_4)$. Also, $\partial_2(e_2) = \sqrt{10}e_1 - \sqrt{6}e_4$ and $\partial_3(e_6) = -\sqrt{6}e_4 - \sqrt{10}e_1$ imply that $e_1, \cdots, e_6 \in M(e_2)=M(e_6)$. Similarly, we obtain $e_1, \cdots, e_6 \in M(e_1) = M(e_3)$. Since any submodule containing $e_1, \cdots, e_6$ contains e_7 using $\partial_3(e_1)=\sqrt{10}e_6+2\sqrt{6}e_7$, we see that $M(e_1)= M(e_2) = \cdots = M(e_6) = E$. Then $M(e_7) = E$ also, because $e_1 \in M(e_7)$ follows from $\partial_3(e_7) = 2\sqrt{6}e_1$.

If E is not an irreducible S-module, there exists an element which generates a nonzero proper submodule, and among all such elements we pick one, $w = \lambda_1e_1 + \cdots + \lambda_7e_7$, of shortest length (i.e., with as many λ's zero as possible). It is easy to see that the element

$$\begin{aligned} w_1 = \alpha^2 w + \partial_1^2 w &= (\alpha^2 - 4)\lambda_1e_1 + (\alpha^2 - 16)\lambda_2e_2 + (\alpha^2 - 4)\lambda_3e_3 \\ &\quad + (\alpha^2 - 36)\lambda_4e_4 + (\alpha^2 - 36)\lambda_5e_5 + (\alpha^2 - 16)\lambda_6e_6 + \alpha^2\lambda_7e_7 \end{aligned}$$

will have shorter length than w for some $\alpha \in \{0, 2, 4, 6\}$ and that $M(w_1) \subseteq M(w) \neq E$. Since w has the shortest length among all nonzero elements, we obtain $w_1 = 0$ for some α, which implies that w has one of the forms

$$\lambda_1e_1 + \lambda_3e_3,\ \lambda_2e_2 + \lambda_6e_6,\ \lambda_4e_4 + \lambda_5e_5,\ \lambda_7e_7 .$$

The case when $w = \lambda_7e_7$ has already been eliminated. If $w = \lambda_1e_1 + \lambda_3e_3$, then $2\lambda_1w + \lambda_3\partial_1w = (2\lambda_1^2 + 2\lambda_3^2)e_1$, showing that e_1 is in the submodule generated by w. But we have shown that e_1 generates all of E, so w could not be of the form $\lambda_1e_1 + \lambda_3e_3$. An identical argument rules out the cases when $w = \lambda_2e_2 + \lambda_6e_6$ and $w=\lambda_4e_4+\lambda_5e_5$. Thus E is an irreducible S-module. □

We turn now to the case when A is a direct sum of an irreducible 3-dimensional $su(2)$-module and an irreducible 5-dimensional $su(2)$-module. Since each of 3×3, 3×5, 5×3, 5×5 has one multiplication into each of 3 and 5, there will be eight constants in the multiplication table of A. One can construct A by thinking of A as the 3×3 skew-Hermitian complex matrices of trace zero, where both $su(2)$ and the 3-dimensional submodule of A are identified with the subspace of matrices which are skew (as well as skew-Hermitian), and where the 5-dimensional module is those matrices which are symmetric (and skew-Hermitian). The action of $su(2)$ on the two modules is the Lie product, and the different multiplications between the two modules in A are obtained by resolving into the 3 and 5-components the two products on this set of matrices given in (3.1).

As is obvious from the construction of A, the algebras occurring here include the class of algebras studied in § 3. On the other hand, when A has no 1-dimensional submodule for $su(2)$, it cannot have a 1-dimensional submodule for all of $\operatorname{Der} A$, which rules out the cases that $\operatorname{Der} A$ is either a compact G_2 or $su(2) \oplus su(2)$, and the case when $\operatorname{Der} A = su(3)$ and A is not an irreducible $su(3)$-module. We have established most of

THEOREM 6.5. *If A is a real division algebra with $su(2) \subsetneqq \operatorname{Der} A$, and if A is the sum of an irreducible 3-dimensional $su(2)$-module and an irreducible 5-dimensional $su(2)$-module, then either $\operatorname{Der} A = su(2)$, or else $\operatorname{Der} A = su(3)$ and A is an irreducible $su(3)$-module.*

Proof. In view of our classification of the derivation algebras of division algebras and of the remarks in the paragraph before the statement of the theorem, it is only necessary to rule out the case that $\operatorname{Der} A = su(2) \oplus N$ where N is a 1-dimensional Lie algebra. Employing the representation of A explained above, we let $\partial_1, \partial_2, \partial_3$ be the basis of $su(2)$ and x_1, x_2, x_3 the basis for the 3-dimensional module X defined by

$$\partial_1 = e_{12} - e_{21} = x_1,\ \partial_2 = e_{23} - e_{32} = x_2,\ \partial_3 = e_{13} - e_{31} = x_3\ , \tag{6.6}$$

where the e_{ij}'s are 3×3 matrix units. We let

$$\begin{gathered} y_1 = i(e_{12} + e_{21}),\ y_2 = i(e_{23} + e_{32}),\ y_3 = i(e_{13} + e_{31})\ , \\ y_4 = i(e_{11} - e_{22}),\ y_5 = i(e_{22} - e_{33}) \end{gathered} \tag{6.7}$$

be the basis of the 5-dimensional module Y.

If $\operatorname{Der} A = su(2) \oplus N$, then there exists a nonzero derivation ∂ commuting with $\partial_1, \partial_2, \partial_3$. By ([1], Lemma 15), the rank of any derivation on an 8-dimensional real division algebra is 0, 4, or 6. But $\partial(A)$ is an $su(2)$-submodule and so must have dimension 0, 3, 5, or 8. Hence, $\partial(A) = 0$, and $\partial = 0$. This rules out the case $\operatorname{Der} A = su(2) + N$ here. □

REMARK. The question of whether real division algebras satisfying the hypotheses of Theorem 6.5 and having $\operatorname{Der} A = su(2)$ actually exist has not been settled, to the best of our knowledge.

Consider next the case when the decomposition of A as $su(2)$-modules is $1 + 1 + 3 + 3$. By determining all possible homomorphisms from the tensor product of two summands into a third summand, one can obtain a general multiplication table with 40 different scalars, but the number of constants can be decreased by making a judicious choice of basis. This class of algebras clearly contains those division algebras with $\operatorname{Der} A = su(3)$ where A is not an irreducible $su(3)$-module, and hence also the division algebras where $\operatorname{Der} A =$ compact G_2. It also contains the algebras with $\operatorname{Der} A = su(2) \oplus su(2)$, since in the notation of (5.3) and (5.4), the elements $\partial_1 + \partial_1'$, $\partial_2 + \partial_2'$, $\partial_3 + \partial_3'$ form a subalgebra of $\operatorname{Der} A$ isomorphic to $su(2)$ under which A has the decomposition $1 + 1 + 3 + 3$. We don't know whether the case when $\operatorname{Der} A = su(3)$ and A is an irreducible $su(3)$-module is included in the present case, or whether there exist real division algebras with the decomposition $1 + 1 + 3 + 3$ where $\operatorname{Der} A$ is either just $su(2)$ or $su(2) + N$.

We turn briefly to the case where A has the $su(2)$-module decomposition $1 + 3 + 4$. The general multiplication table here can be written out using 21 constants. It is clear that those division algebras where $\operatorname{Der} A = su(2) \oplus su(2)$ or $\operatorname{Der} A =$ compact G_2 are included in this class. The division algebras with $\operatorname{Der} A = su(3)$ and A not an irreducible $su(3)$-module are clearly not included in the $1 + 3 + 4$ case, but it is less clear whether the case when $\operatorname{Der} A = su(3)$ and A is an irreducible $su(3)$-module is included. We have not attempted to settle whether there are division algebras of this type with $\operatorname{Der} A = su(2)$ or $\operatorname{Der} A = su(2) \oplus N$ for the case $1 + 3 + 4$.

Our final case is when the $su(2)$-module decomposition is $1+1+1+1+4$. Again those division algebras where $\operatorname{Der} A = su(2) \oplus su(2)$ or $\operatorname{Der} A =$ compact G_2 are included in this class. We don't know whether either type of division algebra with $\operatorname{Der} A = su(3)$ occurs here. For this case we will prove that there are division algebras with $\operatorname{Der} A = su(2)$ and also with $\operatorname{Der} A = su(2) \oplus N$.

Let A be an algebra with basis $u, e_1, e_2, \cdots, e_7$ and multiplication as in the octonions except that the squares of the e_i's are not all equal. Specifically, products in A are given by

$$u^2 = u,\ ue_i = e_i = e_i u,\ e_i^2 = -\beta_i u, \qquad \text{for } i = 1, \cdots, 7$$

$$e_i e_{i+1} = e_{i+3} = -e_{i+1} e_i,\ e_{i+1} e_{i+3} = e_i = -e_{i+1} e_{i+3}\ , \tag{6.8}$$

$$e_{i+3} e_i = e_{i+1} = -e_i e_{i+3}, \text{ where the subscripts are taken modulo 7,}$$

and where $\beta_1, \beta_2, \cdots, \beta_7$ are positive real numbers. We have shown [1, Theorem 20] that this algebra is a real division algebra, and we want to calculate its derivations for appropriate conditions on the β's. In particular, we shall establish

THEOREM 6.9. *Let A be the real division algebra defined by (6.8) and let $\beta_3 = \beta_5 = \beta_6 = \beta_7$. If $\beta_1, \beta_2, \beta_3, \beta_4$ are distinct, then* $\operatorname{Der} A = su(2)$. *If $\beta_1, \beta_2, \beta_3$ are distinct and $\beta_2 = \beta_4$, then* $\operatorname{Der} A = su(2) \oplus N$ *where N is a 1-dimensional Lie algebra.*

Proof. Suppose that $\beta_3 = \beta_5 = \beta_6 = \beta_7$, and define the linear transformations $\partial'_1, \partial'_2, \partial'_3$, on A by

$$
(6.10)\qquad
\begin{array}{c|cccc|cccc|}
 & u & e_1 & e_2 & e_4 & e_3 & e_5 & e_6 & e_7 \\
\hline
\partial'_1 & 0 & 0 & 0 & 0 & -e_7 & e_6 & -e_5 & e_3 \\
\partial'_2 & 0 & 0 & 0 & 0 & e_5 & -e_3 & -e_7 & e_6 \\
\partial'_3 & 0 & 0 & 0 & 0 & e_6 & e_7 & -e_3 & -e_5 \\
\hline
\end{array}\ .
$$

Comparing with (5.4), we see that (6.10) defines an irreducible module action of $su(2)$ spanned by $\partial'_1, \partial'_2, \partial'_3$ on the subspace $E_1 = \langle e_3, e_5, e_6, e_7\rangle$. By definition $su(2)$ acts trivially on the subalgebra $E_0 = \langle u, e_1, e_2, e_4\rangle$, and we need to verify that $\partial'_1, \partial'_2, \partial'_3$ act like derivations on products of the form E_0E_1, E_1E_0, and E_1E_1.

Since we showed in §5 that $\partial'_1, \partial'_2, \partial'_3$ are derivations of the octonions and since the present algebra is the same as the octonions except for the squares of the e'_is, the maps $\partial'_1, \partial'_2, \partial'_3$ will act as derivations on any product of basis vectors where the verification does not depend on calculating the square of an e_i. In particular, the ∂'_is act as derivations on all products of the form E_0E_1 or E_1E_0. For the remaining products—those of the type E_1E_1, one can verify directly using (6.10) and the fact that $\beta_3 = \beta_5 = \beta_6 = \beta_7$ that each ∂'_i behaves as a derivation. Thus, $\partial'_1, \partial'_2, \partial'_3$ are derivations and span a copy of $su(2)$ in Der A.

In order to find out which other derivations of A exist, we need

LEMMA 6.11. *If ∂ is a derivation of the algebra A defined by equations (6.8), then $\partial(u) = 0$ and there exist real numbers a_{ij} for $1 \leqq i, j \leqq 7$ such that $a_{ij} = -a_{ji}$ and $\partial(e_i) = \sum_{j=1}^{7} a_{ij}e_j$ for $1 \leqq i \leqq 7$. Futhermore, if $\beta_i \neq \beta_j$, then $a_{ij} = 0$.*

Proof. Since u is the identity element of A, $\partial(u) = 0$. If $\partial(e_i) = a_{i0}u + \sum_{j=1}^{7} a_{ij}e_j$ for $a_{i0}, a_{ij} \in \boldsymbol{R}$, we see from

$$0 = \partial(e_i^2) = \partial(e_i)e_i + e_i\partial(e_i) = 2a_{i0}e_i - 2\beta_i a_{ii}u$$

that $a_{i0} = 0 = a_{ii}$ for $1 \leqq i \leqq 7$. For fixed $i \neq j$, there exists k such that either $e_i = e_je_k$ or $e_i = e_ke_j$. In the former case, the e_j-component of

$$\partial(e_i) = \partial(e_je_k) = \partial(e_j)e_k + e_j\partial(e_k) = \sum_l a_{jl}e_le_k + \sum_l a_{kl}e_je_l$$

is

$$(6.12) \qquad a_{ij} = -a_{ji} ,$$

since $e_j e_l$ is never a multiple of e_j and since $e_l e_k$ is a multiple of e_j exactly when $l = i$, in which case $e_i e_k = -e_j$. If $e_i = e_k e_j$, then (6.12) also holds by the identical argument with left and right interchanged. If $\beta_i \neq \beta_j$, then

$$(6.13) \qquad 0 = \partial(e_i e_j + e_j e_i) = \partial(e_i)e_j + e_i\partial(e_j) + \partial(e_j)e_i + e_j\partial(e_i) ,$$

and the u-component of this is

$$0 = 2a_{ij}e_j^2 + 2a_{ji}e_i^2 = 2a_{ij}(e_j^2 - e_i^2) = 2a_{ij}(\beta_i - \beta_j)u ,$$

which implies that $a_{ij} = 0$. □

Returning to the proof of Theorem 6.9, we suppose first that $\beta_1, \beta_2, \beta_3, \beta_4$ are distinct. Then, for any $\partial \in \operatorname{Der} A$, we see from Lemma 6.11 that $a_{ij}=0$ unless i and j are both in the set $\{3, 5, 6, 7\}$, giving $\partial(E_0) = 0$ and $\partial(E_1) \subseteqq E_1$. Hence $\partial(e_3) = b_5e_5 + b_6e_6 + b_7e_7$ for some $b_5, b_6, b_7 \in \boldsymbol{R}$, and $\partial' = \partial - b_5\partial_2' - b_6\partial_3' + b_7\partial_1'$ has the property that $\partial'(e_3) = 0$. It follows that

$$\begin{aligned}
\partial'(e_5) &= \partial'(e_2e_3) = \partial'(e_2)e_3 + e_2\partial'(e_3) = 0 ,\\
\partial'(e_6) &= \partial'(e_3e_4) = \partial'(e_3)e_4 + e_3\partial'(e_4) = 0 ,\\
\partial'(e_7) &= \partial'(e_1e_3) = \partial'(e_1)e_3 + e_1\partial'(e_3) = 0 ,
\end{aligned}$$

giving $\partial' = 0$. Thus, $\partial = b_5\partial_2' + b_6\partial_3' - b_7\partial_1' \in su(2)$, and $\operatorname{Der} A = su(2)$.

Finally, suppose that $\beta_1, \beta_2, \beta_3$ are distinct and that $\beta_2 = \beta_4$. If $\partial \in \operatorname{Der} A$, then Lemma 6.11 implies that

$$\partial(e_1) = 0,\ \partial(e_2) = a_{24}e_4,\ \partial(e_4) = -a_{24}e_2,\ \partial(E_1) \subseteqq E_1 .$$

If $a_{24} = 0$, the argument of the last paragraph shows that $\partial \in su(2)$, implying that $\dim \operatorname{Der} A \leqq \dim su(2) + 1 = 4$. Thus, in order to prove that $\operatorname{Der} A = su(2) + N$, it is sufficient to show that $\operatorname{Der} A$ contains a nonzero derivation ∂_1 which commutes with $\partial_1', \partial_2', \partial_3'$. We claim that if ∂_1 is defined by

$$\begin{aligned}
&\partial_1(u) = 0,\ \partial_1(e_1) = 0,\ \partial_1(e_2) = 2e_4,\ \partial_1(e_4) = -2e_2 ,\\
&\partial_1(e_3) = -e_7,\ \partial_1(e_5) = -e_6,\ \partial_1(e_6) = e_5,\ \partial_1(e_7) = e_3 ,
\end{aligned}$$

then ∂_1 is a derivation of A commuting with $\partial_1', \partial_2', \partial_3'$. We saw in § 5 that ∂_1 is a derivation of the octonions commuting with $\partial_1', \partial_2', \partial_3'$ (see (5.4)). Thus ∂_1 must also commute here with $\partial_1', \partial_2', \partial_3'$, and ∂_1 must act as a derivation on any product of basis vectors, since in those cases where the calculation involves squaring on e_i, the two

β's involved will be equal. Therefore, ∂_1 is a derivation here also. $\square$

7. The case when Der A is abelian. We consider next the case when Der A is abelian of dimension 2. By [1, Corollary 16], $\dim A = 8$ and there is a basis ∂_1', ∂_2' of Der A such that ∂_1' and ∂_2' are diagonal relative to a suitable choice of basis of A_C and are of the form

$$\begin{aligned} \partial_1' &\longleftrightarrow \operatorname{diag}\{0, 0, \alpha i, -\alpha i, \beta i, -\beta i, (\alpha+\beta)i, -(\alpha+\beta)i\} \\ \partial_2' &\longleftrightarrow \operatorname{diag}\{0, 0, 0, 0, \gamma i, -\gamma i, \gamma i, -\gamma i\} \end{aligned}$$

for nonzero real numbers α, β, γ. Then $\partial_1 = (1/\alpha)\partial_1' - (\beta/\alpha\gamma)\partial_2'$ and $\partial_2 = (1/\gamma)\partial_2'$ are also a basis for Der A and

$$\begin{aligned} \partial_1 &\longleftrightarrow \operatorname{diag}\{0, 0, i, -i, 0, 0, i, -i\} \\ \partial_2 &\longleftrightarrow \operatorname{diag}\{0, 0, 0, 0, i, -i, i, -i\} . \end{aligned} \tag{7.1}$$

Hence there must exist a basis $u_1, u_2, x_1, x_2, y_1, y_2, z_1, z_2$ of A such that ∂_1 and ∂_2 are given by

$$\begin{aligned} &\partial_1(u_1) = 0 = \partial_1(u_2),\ \partial_1(x_1) = x_2,\ \partial_1(x_2) = -x_1,\ \partial_1(y_1) = 0 = \partial_1(y_2) , \\ &\partial_1(z_1) = z_2,\ \partial_1(z_2) = -z_1,\ \partial_2(u_1) = 0 = \partial_2(u_2),\ \partial_2(x_1) = 0 = \partial_2(x_2) , \\ &\partial_2(y_1) = y_2,\ \partial_2(y_2) = -y_1,\ \partial_2(z_1) = z_2,\ \partial_2(z_2) = -z_1 . \end{aligned} \tag{7.2}$$

Defining the subspaces U, X, Y, Z of A by

$$U = \langle u_1, u_2\rangle,\ X = \langle x_1, x_2\rangle,\ Y = \langle y_1, y_2\rangle,\ Z = \langle z_1, z_2\rangle ,$$

we will show first that the product of any two of these spaces is contained in one of the subspaces. Specifically, we will prove

LEMMA 7.3. *The products of the spaces U, X, Y, Z are given by the table*

(7.4)

	U	X	Y	Z
U	U	X	Y	Z
X	X	U	Z	Y
Y	Y	Z	U	X
Z	Z	Y	X	U

Proof. Since U is the kernel of Der A, we have $U^2 \subseteq U$. If

$u \in U$ and $x \in X$, then $u, x \in \ker \partial_2$, so $ux \in \ker \partial_2 = U + X$. On the other hand, x is the image of some element $x' \in X$ under ∂_1, and so $ux = u\partial_1(x') = \partial_1(ux')$ is also in the image of ∂_1 which is $X + Z$. Thus $ux \in (U + X) \cap (X + Z) = X$. Similarly, for $u \in U$ and $y \in Y$, we have

$$uy \in (\ker \partial_1) \cap (\operatorname{Im} \partial_2) = (U + Y) \cap (Y + Z) = Y .$$

For $u \in U$ and $z \in Z$, we define $\partial_3 = \partial_1 - \partial_2$ and note that

$$uz \in (\ker \partial_3) \cap (\operatorname{Im} \partial_1) = (U + Z) \cap (X + Z) = Z .$$

The same calculations show that $XU \subseteq X$, $YU \subseteq Y$, and $ZU \subseteq Z$. Also, if $x \in X, y \in Y, z \in Z$, we obtain

$$\begin{aligned} xy, yx &\in (\operatorname{Im} \partial_1) \cap (\operatorname{Im} \partial_2) = (X + Z) \cap (Y + Z) = Z , \\ xz, zx &\in (\operatorname{Im} \partial_2) \cap (\operatorname{Im} \partial_3) = (Y + Z) \cap (X + Y) = Y , \\ yz, zy &\in (\operatorname{Im} \partial_1) \cap (\operatorname{Im} \partial_3) = (X + Z) \cap (X + Y) = X . \end{aligned}$$

Finally, if $x, x' \in X$, then we calculate that

$$\begin{aligned} \partial_1(xx') = \partial_1(x)x' + x\partial_1(x'),\ \partial_1^2(xx') = -xx' + 2\partial_1(x)\partial_1(x') - xx' , \\ \partial_1^3(xx') = -2\partial_1(x)x' - 2x\partial_1(x') - 2x\partial_1(x') - 2\partial_1(x)x' . \end{aligned}$$

But ∂_1^2 acting on $\operatorname{Im} \partial_1$ has the effect of multiplying by -1, and so

$$\begin{aligned} 0 &= \partial_1(xx') + \partial_1^3(xx') = \partial_1(x)x' + x\partial_1(x') - 4\partial_1(x)x' - 4x\partial_1(x') \\ &= -3\partial_1(xx') , \end{aligned}$$

showing that $xx' \in \ker \partial_1$. Since $x, x' \in \ker \partial_2$, so is xx', and

$$xx' \in (\ker \partial_1) \cap (\ker \partial_2) = (U + Y) \cap (U + X) = U .$$

By an identical argument, we obtain $Y^2 \subseteq U$ and $Z^2 \subseteq U$. $\square$

The existence of the two commuting derivations ∂_1, ∂_2 not only gives the block multiplication of Lemma 7.3 but also imposes some conditions on how the elements of these different blocks multiply. In particular, we have

THEOREM 7.5. *If A is a real division algebra which has two linearly independent commuting derivations, then A has a basis $u_1, u_2, x_1, x_2, y_1, y_2, z_1, z_2$ for which the multiplication table* (7.6) *holds.*

(7.6)

	u_1	u_2	x_1	x_2	y_1	y_2	z_1	z_2
u_1	$\rho_1 u_1+\rho_2 u_2$	$\rho_3 u_1+\rho_4 u_2$	$\alpha_1 x_1+\alpha_2 x_2$	$-\alpha_2 x_1+\alpha_1 x_2$	$\alpha_5 y_1+\alpha_6 y_2$	$-\alpha_6 y_1+\alpha_5 y_2$	$\alpha_9 z_1+\alpha_{10} z_2$	$-\alpha_{10} z_1+\alpha_9 z_2$
u_2	$\rho_5 u_1+\rho_6 u_2$	$\rho_7 u_1+\rho_8 u_2$	$\alpha_3 x_1+\alpha_4 x_2$	$-\alpha_4 x_1+\alpha_3 x_2$	$\alpha_7 y_1+\alpha_8 y_2$	$-\alpha_8 y_1+\alpha_7 y_2$	$\alpha_{11} z_1+\alpha_{12} z_2$	$-\alpha_{12} z_1+\alpha_{11} z_2$
x_1	$\eta_1 x_1+\eta_2 x_2$	$\eta_3 x_1+\eta_4 x_2$	$\beta_1 u_1+\beta_2 u_2$	$\delta_1 u_1+\delta_2 u_2$	$\gamma_1 z_1+\gamma_2 z_2$	$-\gamma_2 z_1+\gamma_1 z_2$	$\gamma_3 y_1+\gamma_4 y_2$	$-\gamma_4 y_1+\gamma_3 y_2$
x_2	$-\eta_2 x_1+\eta_1 x_2$	$-\eta_4 x_1+\eta_3 x_2$	$-\delta_1 u_1-\delta_2 u_2$	$\beta_1 u_1+\beta_2 u_2$	$-\gamma_2 z_1+\gamma_1 z_2$	$-\gamma_1 z_1-\gamma_2 z_2$	$\gamma_4 y_1-\gamma_3 y_2$	$\gamma_3 y_1+\gamma_4 y_2$
y_1	$\eta_5 y_1+\eta_6 y_2$	$\eta_7 y_1+\eta_8 y_2$	$\varepsilon_1 z_1+\varepsilon_2 z_2$	$-\varepsilon_2 z_1+\varepsilon_1 z_2$	$\beta_3 u_1+\beta_4 u_2$	$\delta_3 u_1+\delta_4 u_2$	$\gamma_5 x_1+\gamma_6 x_2$	$-\gamma_6 x_1+\gamma_5 x_2$
y_2	$-\eta_6 y_1+\eta_5 y_2$	$-\eta_8 y_1+\eta_7 y_2$	$-\varepsilon_2 z_1+\varepsilon_1 z_2$	$-\varepsilon_1 z_1-\varepsilon_2 z_2$	$-\delta_3 u_1-\delta_4 u_2$	$\beta_3 u_1+\beta_4 u_2$	$\gamma_6 x_1-\gamma_5 x_2$	$\gamma_5 x_1+\gamma_6 x_2$
z_1	$\eta_9 z_1+\eta_{10} z_2$	$\eta_{11} z_1+\eta_{12} z_2$	$\varepsilon_3 y_1+\varepsilon_4 y_2$	$\varepsilon_4 y_1-\varepsilon_3 y_2$	$\varepsilon_5 x_1+\varepsilon_6 x_2$	$\varepsilon_6 x_1-\varepsilon_5 x_2$	$\beta_5 u_1+\beta_6 u_2$	$\delta_5 u_1+\delta_6 u_2$
z_2	$-\eta_{10} z_1+\eta_9 z_2$	$-\eta_{12} z_1+\eta_{11} z_2$	$-\varepsilon_4 y_1+\varepsilon_3 y_2$	$\varepsilon_3 y_1+\varepsilon_4 y_2$	$-\varepsilon_6 x_1+\varepsilon_5 x_2$	$\varepsilon_5 x_1+\varepsilon_6 x_2$	$-\delta_5 u_1+\delta_6 u_2$	$\beta_5 u_1+\beta_6 u_2$

Proof. From (7.4) we know that $x_1y_1 \in Z$, say $x_1y_1 = \gamma_1z_1 + \gamma_2z_2$. Applying ∂_1 and ∂_2 respectively to this relation and using (7.2), we obtain

$$x_2y_1 = \partial_1(x_1)y_1 = \partial_1(x_1y_1) = \partial_1(\gamma_1z_1 + \gamma_2z_2) = \gamma_1z_2 - \gamma_2z_1 ,$$
$$x_1y_2 = x_1\partial_2(y_1) = \partial_2(x_1y_1) = \gamma_1z_2 - \gamma_2z_1 .$$

And applying ∂_1 to the last relation gives $x_2y_2 = \partial_1(x_1y_2) = -\gamma_1z_1 - \gamma_2z_2$. Similarly, there exist $\gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \boldsymbol{R}$ such that $x_1z_1 = \gamma_3y_1 + \gamma_4y_2$ and $y_1z_1 = \gamma_5x_1 + \gamma_6x_2$, and the application of $\partial_1, \partial_2, \partial_3 = \partial_1 - \partial_2$ to these equations gives the remaining products of the form XZ and YZ. The products of the form YX, ZX, and ZY follow by left-right symmetry.

Next, choosing $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \boldsymbol{R}$ such that $u_1x_1 = \alpha_1x_1 + \alpha_2x_2$ and $u_2x_1 = \alpha_3x_1 + \alpha_4x_2$, we get

$$u_1x_2 = u_1\partial_1(x_1) = \partial_1(u_1x_1) = \alpha_1x_2 - \alpha_2x_1,\ u_2x_2 = \partial_1(u_2x_1) = \alpha_3x_2 - \alpha_4x_1 .$$

By identical arguments, we obtain all the entries in (7.4) of the forms UX, UY, UZ, XU, YU, and ZU. Choosing $\beta_1, \beta_2, \delta_1, \delta_2 \in \boldsymbol{R}$ with $x_1^2 = \beta_1u_1 + \beta_2u_2$ and $x_1x_2 = \delta_1u_1 + \delta_2u_2$, we have the relations

$$0 = \partial_1(x_1^2) = \partial_1(x_1)x_1 + x_1\partial_1(x_2) = x_2x_1 + x_1x_2 ,$$
$$0 = \partial_1(x_1x_2) = \partial_1(x_1)x_2 + x_1\partial_1(x_2) = x_2^2 - x_1^2$$

which give us x_2^2 and x_2x_1. The entries in (7.4) of the form Y^2 and Z^2 are found in the same way. Finally, the derivations ∂_1 and ∂_2 impose no restrictions at all on the subspace U, so the constants have to be all different here. □

As our final result, we establish

THEOREM 7.7. *Let A be the modified octonion algebra defined by* (6.8). *Then*

(i) *if $\beta_1, \cdots, \beta_7$ are all distinct,* $\operatorname{Der} A = 0$.

(ii) *if $\beta_1, \beta_2, \beta_3, \beta_4, \beta_6$ are distinct, $\beta_3 = \beta_5$, and $\beta_6 = \beta_7$, then* $\dim \operatorname{Der} A = 1$.

(iii) *if $\beta_1, \beta_2, \beta_3, \beta_6$ are distinct, $\beta_1 = \beta_4$, $\beta_3 = \beta_5$ and $\beta_6 = \beta_7$, then* $\dim \operatorname{Der} A = 2$ *and* $\operatorname{Der} A$ *is abelian.*

Proof. If $\beta_1, \cdots, \beta_7$ are all distinct, it is immediate from Lemma 6.11 that A cannot have any nonzero derivations, giving part (i). If the hypotheses of part (ii) of Theorem 7.7 hold and if $\partial \in \operatorname{Der} A$, then Lemma 6.11 implies that

$$\begin{aligned}\partial(e_3) = a_{35}e_5,\ \partial(e_5) = -a_{35}e_3,\ \partial(e_6) = a_{67}e_7,\ \partial(e_7) = -a_{67}e_6\ ,\\ \partial(u) = 0 = \partial(e_1),\ \partial(e_2) = 0 = \partial(e_4)\ ,\end{aligned} \tag{7.8}$$

for some $a_{35}, a_{67} \in \boldsymbol{R}$. From

$$a_{35}e_5 = \partial(e_3) = \partial(e_7e_1) = \partial(e_7)e_1 = -a_{67}e_6e_1 = -a_{67}e_5$$

we get $a_{35} = -a_{67}$. Thus, $\operatorname{Der} A$ is at most 1-dimensional. To show that $\dim \operatorname{Der} A = 1$, it is sufficient to verify that the special case of (7.8) with $a_{35} = 1$ and $a_{67} = -1$ is a derivation of A. But this linear transformation was shown to be a derivation of the octonions in § 5 (under the correspondence $e_3 \leftrightarrow y_1$, $e_5 \leftrightarrow y_2$, $e_6 \leftrightarrow y_3$, $e_7 \leftrightarrow y_4$, ∂ corresponds to ∂_2' in (5.4)), and so ∂ will act as a derivation on any product of basis vectors where the verification does not depend on calculating the square of an e_i. Since $\beta_3 = \beta_5$ and $\beta_6 = \beta_7$ in the case we are considering, it is clear from (7.8) that ∂ will act as a derivation even in those cases where the verification depends on calculating the square of an e_i.

Finally, suppose that the hypotheses of part (iii) of Theorem 7.7 hold. Then Lemma 6.11 shows that any $\partial \in \operatorname{Der} A$ has the form

$$\begin{aligned}\partial(u) = 0 = \partial(e_2),\ \partial(e_1) = a_{14}e_4,\ \partial(e_4) = -a_{14}e_1\ ,\\ \partial(e_3) = a_{35}e_5,\ \partial(e_5) = -a_{35}e_3,\ \partial(e_6) = a_{67}e_7,\ \partial(e_7) = -a_{67}a_6\ ,\end{aligned} \tag{7.9}$$

for some $a_{14}, a_{35}, a_{67} \in \boldsymbol{R}$. Since

$$\begin{aligned} a_{67}e_7 = \partial(e_6) = \partial(e_3e_4) &= \partial(e_3)e_4 + e_3\partial(e_4) = a_{35}e_5e_4 - a_{14}e_3e_1 \\ &= (a_{35} + a_{14})e_7\ ,\end{aligned}$$

we see that $\dim \operatorname{Der} A \leqq 2$. It suffices to show that the special cases of (7.9) defined by the table

	u	e_1	e_2	e_4	e_3	e_5	e_6	e_7
∂_2	0	$-2e_4$	0	$2e_1$	$-e_5$	e_3	$-e_7$	e_6
∂_2'	0	0	0	0	e_5	$-e_3$	$-e_7$	e_6

are both derivations of A. Again ∂_2 and ∂_2' were shown in § 5 to be derivations of the octonions (see (5.4)), and as we argued in the last case, ∂_2 and ∂_2' must be derivations here because $\beta_1 = \beta_4$, $\beta_3 = \beta_5$, and $\beta_6 = \beta_7$. □

REMARK. If A is a finite-dimensional real algebra with $L = \operatorname{Der} A$ as its derivation algebra, then the connected Lie group G corresponding to the Lie algebra L acts as a group of automorphisms on A. Furthermore, G necessarily has finite index in $\operatorname{Aut} A$, the group of all automorphisms of A. One might ask in the case of a real division algebra whether G can be properly contained in $\operatorname{Aut} A$, and we shall give an example to show that this can happen. In the algebra A defined by (6.8) with all β's distinct, we have shown that $\operatorname{Der} A = 0$ and hence $G = 1$. On the other hand, this algebra has 8 automorphisms, as one sees by noting that for any choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$ the map

$$a_0 u + \sum a_i e_i \longrightarrow a_0 u + a_1\varepsilon_1 e_1 + a_2\varepsilon_2 e_2 + a_3\varepsilon_3 e_3 + a_4\varepsilon_1\varepsilon_2 e_4 + a_5\varepsilon_2\varepsilon_3 e_5 + a_6\varepsilon_1\varepsilon_2\varepsilon_3 e_6 + a_7\varepsilon_1\varepsilon_3 e_7$$

is an automorphism of A.

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