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ALMOST RIGID HOPFIAN AND DUAL HOPFIAN ATOMIC BOOLEAN ALGEBRAS

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There are no nontrivial constraints on the number of atoms and the size of an almost rigid dual Hopfian atomic Boolean algebra with no more than c atoms; and no nontrivial constraints on the number of atoms of an almost rigid Hopfian, dual Hopfian atomic Boolean algebra of size c.

0. **Introduction.** In [3], one of the authors showed the (real) existence of a Hopfian (onto endomorphisms are 1–1), dual Hopfian ($1 - 1$ endomorphism are onto) atomic Boolean algebra. This algebra was of size C, had countably many atoms, and was almost rigid (each automorphism moves at most finitely many atoms). Thus it had only countably many automorphisms. Are more automorphisms or more atoms possible? In this paper we show the (consistent) answer is yes.

It was known that under MA, a Boolean algebra with infinitely many atoms and size less than c is neither Hopfian nor dual Hopfian, [3], and must have exactly c automorphisms, [4]. Van Douwen gave a consistent counterexample to this theorem in a model of not $- MA$ [1]. The techniques used to answer our first question show that his example is in fact dual Hopfian, and that there are no nontrivial restrictions on the size of the algebra or on the number of atoms of an atomic almost rigid dual Hopfian Boolean algebra of size $\leqq c$.

Since our constructions need only the presence of cofinally many Cohen reals (in a sense to be made precise in §3), they can be carried out in many models. The most interesting examples of our techniques are the following two theorems.

THEOREM 1. *Let γ have uncountable cofinality and add γ many Cohen reals to a model of* CH. *Then in the new model the following holds:*

If $\omega \leqq \lambda \leqq k \leqq c$ and $\kappa \geqq \omega_1$, then

(a) *there is an almost rigid dual Hopfian atomic Boolean algebra of size κ with λ many atoms.*

(b) *there is a almost rigid Hopfian, dual Hopfian atomic Boolean algebra of size c with λ many atoms.*

THEOREM 2. *Assume* MA. *Then if $\omega \leqq \lambda \leqq c$, the conclusion* 1(b) *holds.*

It will turn out that c need not be a constraint on the size of our dual Hopfian algebras. However, the statement of this theorem is even more technical than Theorem 1, so we delay its statement to Corollary 3.5.

Some conventions: all algebras are infinite Boolean subalgebras of some $\mathscr{P}(\kappa)$ which contain all finite and cofinite subsets of κ. We use map to mean a Boolean endomorphism and denote $\{\alpha\}$ by $\bar{\alpha}$.

If E is an infinite set and ϕ a formula, we say "ϕ infinitely often on E" iff $\{\alpha \in E: \phi(\alpha)\}$ is infinite. "Infinitely often" means infinitely often on ω.

Finally, "model of set theory" always means "countable transitive model of ZFC".

1. Motivation. Fix $B \subset \mathscr{P}(\kappa)$, an algebra and a map $f: B \to B$. We want to extend B so that f has no homomorphic extension to the new algebra.

DEFINITION 1.1. Let D be a pairwise disjoint subset of B, $D^* \subset D$, and suppose for some d^*, $d^* \cap \bigcup D = \bigcup D^*$. Then b is a d^*/D split if $b \supset d^* \cap \bigcup D$ and $b \cap d = \varnothing$ for all $d \in D \sim D^*$.

LEMMA 1.2. *Let $f: B \to B$ be a map and C a pairwise disjoint subset of B, $D = \{f(c): c \in C\}$. If $C^* \subset C$, $c^* \cap \bigcup C = \bigcup C$, $c^* \in B$, and $D^* = \{f(c): c \in C^*\}$ and $d^* = \bigcup D^*$, then $f(C^*)$ is a d^*/D split.*

Lemma 1.2, whose proof follows immediately from the Boolean properties of f, is used in the following two ways.

The Hopfian motivation. Suppose f is an onto map of the algebra B which is not $1-1$. Since f is not $1-1$ there is some $\alpha \neq \beta$ and some γ_0 where $f(\bar{\beta}) = f(\bar{\alpha}) = \bar{\gamma}_0$. Let $d_{\gamma_0} = \{\beta, \alpha\} \sim \bar{\gamma}_0$. Continuing by induction, given distinct γ_i, $i \leqq k$, and d_{γ_i} so $\gamma_i \in d_{\gamma_i}$ and $f(d_{\gamma_i}) = \bar{\gamma}_i$, choose $\gamma_{k+1} \in d_{\gamma_k}$ and let $d_{\gamma_{k+1}} \supset \bar{\beta}$ for some β with $f(\bar{\beta}) = \bar{\gamma}_{k+1}$. In this way there is a countably infinite set of atoms, A, and a pairwise disjoint family $D = \{d_\alpha: \bar{\alpha} \in A\} \subset B$ where $f(d_\alpha) = \bar{\alpha}$ and $A \cap d_\alpha \sim \bar{\alpha} \neq \varnothing$. During the inductive construction of B, we want to have put a set J (of the form $\bigcup_{\alpha \in x} d_\alpha$ where $x \subset A$) into B in such a way that there is no x/A split in the new algebra. If in the final algebra B, there is still no x/A split, then by Lemma 1.2, f has no place to send J. x will be a Cohen subset of A.

The almost rigid, dual Hopfian motivation. If f is a $1-1$ map of the algebra B which is not onto, there is some α not in the range of f. Let $\bar{\alpha}_0$ be such, $d_0 = f(\bar{\alpha}_0)$. Pick $\alpha_1 \in d_0$ with $\alpha_1 \neq \alpha_2$ and let $d_1 = f(\bar{\alpha}_1)$. Continue. You have constructed a countably infinite set of atoms A and a pairwise disjoint family $D = \{d_\alpha : \bar{\alpha} \in A\} \subset B$ where $f(\bar{\alpha}) = d_\alpha$ and $A \cap d_\alpha \sim \bar{\alpha} \neq \emptyset$ infinitely often on A. Such an A, D can also be constructed if f is an automorphism of B moving infinitely many atoms: Let $A = D$ be an infinite set of atoms moved. Given such A, D, as we construct B, we want to have put $x \subset \kappa$ into B so that if $J = \bigcup_{\alpha \in x \cap A} d_\alpha$, there is no J/D split in the new algebra. As before, if the final algebra B still has no J/D split, there will be no image for x under f. In this case, x will be a Cohen subset of κ.

Doomed endomorphisms. Here we define the sorts of endomorphisms we have a chance of killing.

DEFINITION 1.4. Let A be countably infinite subset of κ. Then $D = \{d_\alpha : \alpha \in A\}$ is a candidate if D is pairwise disjoint and $A \cap d_\alpha \sim \bar{\alpha} \neq \emptyset$ infinitely often on A.

DEFINITION 1.5. (a) f is expanding if there is a candidate $D = \{d_\alpha : \alpha \in A\}$ where $f(\bar{\alpha}) = d_\alpha$ for $\alpha \in A$ and some $|d_\alpha \cap A| > 1$.

(b) f is contracting if there is a candidate $D = \{d_\alpha : \alpha \in A\}$ where $f(d_\alpha) = \bar{\alpha}$ and some $|d_\alpha \cap A| > 1$.

(c) f is kinetic if it is neither expanding nor contracting and for some candidate $D = \{d_\alpha : \alpha \in A\}$, $f(\bar{\alpha}) = d_\alpha$ for $\alpha \in A$. (In particular, a kinetic map moves infinitely many atoms to atoms.)

An algebra with no contracting maps is Hopfian. An algebra with neither expanding maps nor kinetic maps is almost rigid, dual Hopfian.

2. Combinatorial lemmas.

In this section we prove the combinatorial lemmas needed to construct our algebras. In 2.1, we work only in the case $B \subset \mathscr{P}(\omega)$; in §2.2 we give the lemmas allowing us to extend our results to $B \subset \mathscr{P}(\kappa)$ for arbitrary $\kappa \leqq \mathfrak{c}$.

2.1. Countably many atoms.

Fix M, a model of set theory. The Cohen partial order P_ω is the set of finite functions from ω into 2. We say $x \subset \omega$ is a Cohen subset of ω over M iff its characteristic function is an M-generic filter on P_ω (M-generic means it meets every dense set in M). Notice that if x is Cohen over $N \supset M$, then x is Cohen over M.

Canonical situation. In M, the following hold:

$D = \{d_n : n \in \omega\} \subset \mathscr{P}(\omega)$, D is a candidate, E is an infinite subset of ω. Let x be a Cohen subset of ω over M, $J = \bigcup_{n \in x} d_n$.

Here are three key lemmas, followed by their proofs and their interpretations. If $a \subset \omega$, we let a^0 denote $\{n \in \omega: a \cap d_n \sim \bar{n} \neq \emptyset\}$.

LEMMA 2.1.1. *Assume the canonical situation. If E^0 is infinite, then*

(a) $\bigcup D \cap E \cap x \cap J$ *is infinite,*
(b) $\bigcup D \cap E \cap J \sim X$ *is infinite,*
(c) $\bigcup D \cap E \cap x \sim J$ *is infinite, and*
(d) $\bigcup D \cap E \sim (x \cup J)$ *is infinite.*

LEMMA 2.1.2. *Assume the canonical situation. If $E \cap d_n \neq \emptyset$ for infinitely many n, then $E \cap J$ and $E \sim J$ are infinite.*

LEMMA 2.1.3. *Assume the canonical situation. Then $E \cap x$ and $E \sim x$ are infinite.*

Proofs. Lemma 2.1.3 is a standard fact about Cohen reals. The other two lemmas follow from the fact that if $S \in M$ is a collection of ordered pairs on ω containing an infinite $1-1$ relation, then there are infinitely many $(m, n) \in S$ with $m, n \in x$; infinitely many $(m, n) \in S$ with $m \in x$ and $n \notin x$; infinitely many $(m, n) \in S$ with $m \notin x$ and $n \in x$; and infinitely many $(m, n) \in S$ with $m \notin x$, $n \in x$.

Thus for 2.1.1, let S be the set of all pairs (m, n) where $m \in E \cap d_n$ and $m \neq n$. If $m, n \in x$, and $(m, n) \in S$, then $m \in \bigcup D \cap E \cap x \cap J$. If $m \in x$, $n \in x$, and $(m, n) \in S$, then $m \in \bigcup D \cap E \cap J \sim x$. If $m \in x$, $n \in x$, and $(m, n) \in S$, then $m \in \bigcup D \cap E \cap x \sim J$. By hypothesis, each of the above possibilities occurs infinitely often, so 2.1.1 is proved.

For 2.1.2, let S be the set of all pairs (m, n) where $m \neq n$, $d_n \cap E \neq \emptyset$ and $d_m \cap E \neq \emptyset$. Then for infinitely many pairs $(m, n) \in S$, $m \notin x$ and $n \in x$ we have $d_n \cap E \subset J$ and $d_m \cap E \subset E \sim J$, so we're done.

Interpretations. D represents a function f we're trying to kill—either $f(\bar{n}) = d_n$ for all n or $f(d_n) = \bar{n}$ for all n. Lemma 2.1.1 allows us to add J without adding x and x without adding J. Lemmas 2.1.2 and 2.1.3 will ensure that we don't add an earlier split which we want to avoid.

To apply these lemmas, we turn to the *Second canonical situation.* The following holds in M: B is a subalgebra of $\mathscr{P}(\omega)$; $\mathscr{D}$ is a collection of candidates, each a subset of B, where if $D = \{d_n: n \in \omega\} \in \mathscr{D}$ then for some infinite $b_D \in B$ there is no

$$\bigcup_{n \in b_D} d_n / D$$

split in B (for brevity this is called a b_D-split); and $D^* = \{d_n^*: n \in \omega\} \subset B$ is a candidate not in $\mathscr{D}$. Finally, x is a Cohen subset of ω over m, $J = \bigcup_{n \in x} d_n^*$.

(Interpretation: B is the algebra so far, $\mathscr{D}$ is the set of candidates we have killed so far, D^* is the candidate we are about to kill by adding either J or x.)

LEMMA 2.1.4. *Assume the second canonical situation. If $b, c \in B$, then $x \neq (c \cap J) \cup (b \sim J)$.*

LEMMA 2.1.5. *Assume the second canonical situation. If $b, c \in B$, then $(c \cap x) \cup (b \sim x)$ is not a J/D^* split.*

LEMMA 2.1.6. *Assume the second canonical situation. If $D \in \mathscr{D}$, $b, c \in B$, then $(c \cap J) \cup (b \sim J)$ and $(c \cap x) \cup (b \sim x)$ are not b_D-splits.*

Respectively, these lemmas say that we can add J without adding x, that we can add x without adding J, and that we can add either x or J without bringing any dead D's in back to life via b_D-splits.

Proofs. First note that a set of the form $(c \cap x) \cup (b \sim x)$ is the disjoint union of $b \cap c$, $(c \sim b) \cap x$, and $(b \sim c) \sim x$.

Proof of 2.1.4. If $b \cap c$ is infinite, or if $\sim (b \cup c)$ is infinite, then by 2.1.3 $x \neq (c \cap J) \cup (b \sim J)$. So we may assume that $b \cap c$ is finite and $b \cup c$ is cofinite. Then for infinitely many n,

$$d_n^* \sim \bar{n} = (b \cup c) \cap d_n^* \sim \bar{n} \neq \varnothing \ .$$

So either $(c \sim b)^0$ or $(b \sim c)^0$ is infinite. If $(c \sim b)^0$ is infinite, then by Lemma 2.1.1(b),

$$(c \sim b) \cap J \cap x \neq (c \sim b) \cap J \ .$$

So we're done. On the other hand, if $(b \sim c)^0$ is infinite, then by 2.1.1(d),

$$[(b \sim c) \sim J] \cap x \neq (b \sim c) \sim J$$

and we're done.

Proof of 2.1.5. Let $S = \{n: d_n^* \subseteq b \cup c\}$. Define $T = \{d_n^* \sim (b \cup c): n \notin S$. If $\sim S$ is infinite, by 2.1.2 $T \cap J \neq \varnothing$ and bence no J/D^* split is contained in $b \cup c$. So we may assume S is cofinite. Then either

$(c \sim b)^0$ or $(b \sim c)^0$ is infinite. If the former, by 2.1.1(c) no set containing $(c \sim b) \cap x$ is a J/D^* split. If the latter, by 2.1.1(a) no set containing $(b \sim c) \cap x$ is a J/D^* split, and we're done.

Proof of 2.1.6. Let $D = \{d_n: n \in \omega\} \in \mathscr{D}$ and consider $\bigcup_{n \in b_D} d_n = d$. To show the first part by contradiction, suppose for some $b, c \in B$,

$$d = \cup D \cap [(c \cap J) \cup (b \sim J)] .$$

If $d \cap (c \sim b)$ meets infinitely many d_n^*, then by 2.1.2, $d \cap (c \sim b) \sim J$ is infinite, a contradiction. If $d \cap (b \sim c)$ meets infinitely many d_n^*, we get a similar contradiction. So we may assume $d \cap (c \sim b)$ and $d \cap (b \sim c)$ meet only finitely many d_n^*, say $d_0^*, \cdots, d_k^*$. Let $h = \bigcup_{ik} d_i^*$. Then if $(c \cap J) \cup (b \sim J)$ is a b_D-split, so is $(c \cap h) \cup (b \sim h)$ which is an element of B. Contradiction.

The second part of the lemma has a similar proof. Suppose

$$d = \cup D \cup [(c \cap x) \cup (b \sim x)] .$$

If $d \cap (c \sim b)$ is infinite, then by Lemma 2.1.3, $d \cap (c \sim b) \sim x$ is infinite, a contradiction. In case $d \cap (b \sim c)$ is infinite, the argument is similar. So we may assume $d \cap (c \triangle b)$ is finite. But then $[d \cap (C \triangle b)] \cup [b \cap c]$ belongs to B and is a b_D-split, a contradiction.

2.2. More atoms. Fix M a model of set theory, $A \in M$, $A \subseteq ON$. The Cohen partial order $\boldsymbol{P}_A$ is the set of finite functions from A into 2. We say $x \subset A$ is a Cohen subset of A over M iff its characteristic function is an M-generic filter on $\boldsymbol{P}_A$.

The salient fact we use is the following well-known lemma.

LEMMA 2.2.0. *Let x be a Cohen subset of κ over M, $A \in M$, A an infinite subset of κ. Then $x \cap A$ is a Cohen subset of A over M.*

For the rest of this section we parallel 2.1. Proofs are omitted, since by using Lemma 2.2.0, they are nearly exactly the same as in 2.1.

Third canonical situation. In M the following hold:

$$D = \{d_\alpha: \alpha \in A\} \subset \mathscr{P}(\kappa) ,$$

D is a candidate, E is an infinite subset of A, a countable subset of κ. Let x be a Cohen subset of κ over M, and $J = \bigcup_{\alpha \in x \cap A} d_\alpha$.

LEMMA 2.2.1. *Assume the third canonical situation. If*

$$E^0 = \{\alpha \in E \colon E \cap d_\alpha \sim \bar{\alpha} \neq \varnothing\}$$

if infinite, then

(a) $A \cap \bigcup D \cap E \cap x \cap J$ *is infinite,*

(b) $A \cap \bigcup D \cap E \cap x \sim J$ *is infinite,*

(c) $A \cap \bigcup D \cap E \cap J \sim x$ *is infiite.*

LEMMA 2.2.2. *Assume the third canonical situation. If $E \cap d_\alpha \neq \varnothing$ infinitely often on A, then $E \cap J$ and $E \sim J$ are infinite.*

LEMMA 2.2.3. *Assume the third canonical situation. Then $E \cap x$ and $E \sim x$ are infinite.*

The interpretation is as before: D represents a function f we're trying to kill. Lemma 2.2.2 allows us to add J without adding an x/A split, and to add x without adding a J/D split, as desired. Again, 2.2.2 and 2.2.3 ensure we don't add an earlier split which we want to avoid.

That these lemmas will suffice follows by noting that if any E has a hope of adding either J or x, then it must meet the hypotheses of 2.2.1 and 2.2.2.

Fourth canonical situation. The following holds in M: B is a subalgebra of $\mathscr{P}(\kappa)$; $\mathscr{D}$ is a collection of candidates, each a subset of B, where if $D = \{d_\alpha \colon \alpha \in A\} \in \mathscr{D}$ then for some $b_D \in B$ there is no $\bigcup_{\alpha \in b_D} d_\alpha / D$ split in B (for brevity this is called a b_D-split); and $D^* = \{d_\alpha \colon \alpha \in A\} \subset B$ is a candidate not in $\mathscr{D}$. Finally, x is a Cohen subset of κ over M, $J = \bigcup_{\alpha \in x \cap A} d^*_\alpha$.

LEMMA 2.2.4. *Assume the fourth canonical situation. If $b, c \in B$ then $(c \cap J) \cup (b \sim J)$ is not an x/A split, that is*

$$[(c \cap J) \cup (b \sim J)] \cap A \neq x\ .$$

LEMMA 2.2.5. *Assume the fourth canonical situation. If $b, c \in B$ then $(c \cap x) \cup (b \sim x)$ is not a J/D^* split.*

LEMMA 2.2.6. *Assume the fourth canonical situation. If $D \in \mathscr{D}$, $b, c \in B$, then $(c \cap J) \cup (b \sim J)$ and $(c \cap x) \cup (b \sim x)$ are not b_D-splits.*

3. Final constructions.

Main hypothesis. The following holds in our universe V: $\omega \leqq \lambda$, γ has uncountable cofinality; $\{M_\alpha \colon \alpha < \gamma\}$ is an increasing γ-sequence of models, each has the same cardinals as V; in each $M_{\alpha+1}$ there is a Cohen subset x_α of λ over M_α, each $(M_\alpha)^\omega \subset \bigcup_{\alpha<\gamma} M_\alpha$ and $M_0 \models \kappa \leqq \alpha^\lambda$.

THEOREM 3.1. *Suppose the main hypothesis holds. Then there is an atomic Boolean algebra $B \subset \mathscr{P}(\lambda)$ where $|B| = \gamma \cdot \kappa$ and B has no expanding or kinetic maps.*

Proof. Let $B_0 \in M_0$ be an algebra $\subset \mathscr{P}(\lambda)$ of size κ. For $\beta < \gamma$, let B_β be the algebra generated by $B_0 \cup \{x_\alpha : \alpha < \beta\}$; let $B = \cup B$. Suppose $f: B \to B$ is expanding or kinetic. Then there is a candidate $D = \{d_\alpha : \alpha \in A\}$, $D \subset B$, and $f(\bar{\alpha}) = d_\alpha$ for all $\alpha \in A$. Since γ has uncountable cofinality, $D \subset M_\alpha$ for some α; and since $(M_\alpha)^\omega \subset \bigcup_{\alpha<\gamma} M_\alpha$, $D \in M_\beta$ for some β. Let $J = \bigcup_{\alpha \in x_\beta \cap A} d_\alpha$. Then by Lemma 2.2.5, there is no J/D split in $B_{\alpha+1}$, and by Lemma 2.2.6, there remains no J/D split in B.

Modified main hypothesis. The following holds in our universe V: $\omega \leqq \lambda \leqq \gamma$; γ is regular and uncountable; $\{M_\alpha : \alpha < \gamma\}$ is an increasing sequence of models; each M_α has the same ordinals as V; if A is a countable subset of λ then $A \in M_\alpha$ for some α and there is a Cohen subset $x_{\alpha,A}$ of A over M_α with $x_{\alpha,A} \in M_\beta$ for some β; each $(M_\alpha)^\omega \subset \bigcup_{\alpha<\gamma} M_\alpha$ and $\gamma^\omega = \gamma$.

THEOREM 3.2. *Assume the modified main hypothesis. Then there is an atomic Boolean algebra $B \subset \mathscr{P}(\lambda)$ where $|B| = \gamma$ and B has no expanding, contracting or kinetic maps.*

Proof. Again let $B_0 \in M_0$ be an algebra $\subset \mathscr{P}(\lambda)$ of size κ. We will construct by induction an increasing sequence of algebras $\{B_\beta : \beta < \gamma\}$ as follows (where the final algebra $B = \bigcup_{\beta<\gamma} B_\beta$).

If β is even, B_β is the algebra generated by $\bigcup_{\alpha<\beta} B_\alpha \cup \{x_\beta\}$ where x_β is defined below.

If β is odd, B_β is the algebra generated by $\bigcup_{\alpha<\beta} B_\alpha \cup \{J_\beta\}$ where J_β is defined below.

Let X be the set of names of element of B, $|X| = \gamma \cdot \kappa$. Let $\mathscr{D}$ be the set of all $D = \{d_\alpha : \alpha \in A\}$ where A is a countable subset of λ and $d_\alpha \in X$. Let $\{D_\alpha : \alpha < \gamma\}$ enumerate $\mathscr{D}$. Let $B' = \bigcup_{\alpha<\beta} B_\alpha$ for all β.

We define x_β: Suppose β is even, each $\beta_\alpha, \alpha < \beta$, has been defined, and for $\alpha < \beta, \alpha$ even, have defined $\delta(\alpha)$ so that $D_{\delta(\alpha)} = \{d_\gamma^{\delta(\alpha)} : \sigma \in A_{\delta(\alpha)}\}$ and x_α is a Cohen subset of $A_{\delta(\alpha)}$. Let

$$G_\beta = \{\tau : \tau \notin \{\delta(\alpha) : \alpha < \beta, \alpha \text{ even}\} \quad \text{and} \quad D_\tau \subset B'_\beta\}$$

and let $\delta(\beta) = \inf G_\beta$. Let ρ be the least so that $B'_\beta \in M_\rho$ and $D_{\delta(\beta)} =$

$\{d_\sigma^{\delta(\beta)}: \sigma \in A_{\delta(\beta)}\} \in M_\rho$. Let x_β be a Cohen subset of $A_{\delta(\beta)}$ over M_ρ.

We define J_β for odd β: Suppose each B_α, $\alpha < \beta$, has been defined, and for $\alpha < \beta$, α odd, we have defined some $D_{\delta(\alpha)} = \{d_\sigma^{\delta(\alpha)}: \sigma \in A_{\delta(\alpha)}\}$ and for some x, a Cohen subset of $A_{\delta(\alpha)}$, $J_\alpha = \bigcup_{\sigma \in x} d_\sigma^{\delta(\alpha)}$. Let

$$H_\beta = \{\tau: \tau \notin \{\delta(\alpha): \alpha < \beta,\ \alpha \text{ odd}\} \text{ and } D_\tau \subset B'_\beta\},$$

and let $\delta(\beta) = \inf H_\beta$. Let ρ be the first so that $B'_\beta \in M_\rho$ and $D_{\delta(\beta)} = \{d_\sigma^{\delta(\beta)}: \sigma \in A_{\delta(\beta)}\} \in M_\rho$. Let x be Cohen subset of $A_{\delta(\beta)}$ over M_ρ, and let $J_\beta = \bigcup_{\sigma \in x} d_\sigma^{\delta(\beta)}$.

By 2.2.5, expanding and kinetic maps are destroyed at even stages; contracting maps are destroyed at odd stages. That every expanding, contracting or kinetic map is destroyed follows from the regularity of γ. Lemma 2.2.6 ensures that dead maps are not resurrected.

Theorem 1 is now restated as

COROLLARY 3.3. *Let γ have uncountable cofinality, and add γ many Cohen reals to a model of* CH. *Then in the new model the following holds*:

If $\omega \leqq \lambda \leqq \kappa \leqq c$ and $\kappa \geqq \omega$, then

(a) *there is a dual Hopfian almost rigid atomic Boolean algebra of size κ with λ many atoms.*

(b) *there is a Hopfian, dual Hopfian almost rigid atomic Boolean algebra of size c with λ many atoms.*

Proof. In such an extension $c = \gamma$ in the final model, and it is well-known (see e.g., [2]) that the hypothesis of 3.2 holds for $\kappa \leqq \gamma$; thus 3.3.(b) is proved.

By homogeneity we can rearrange the extension by first adding γ many Cohen reals and then adding ω_1 many Cohen reals. So we can arrange that $M_0 \models \gamma = c$. Again it is well-known that the hypothesis of 3.1 holds; thus 3.3(a) is proved.

Theorem 2 is restated as

COROLLARY 3.4. *Assume* MA. *Then if $\omega \leqq \lambda \leqq c$, the conclusion of* 3.3(b) *holds.*

Proof. A reader unfamiliar with logic may choose to parallel the arguments throughout the paper to show that small collections of dead functions stay dead, but it is less exhausting to show that every model of MA satisfies the main hypothesis.

Assume MA. Let $\{A_\alpha: \alpha < c\}$ enumerate λ^ω for $\lambda \leqq c$ (by MA, $\lambda^\omega = c$). Let M_β be the Skolem closure of the ordinals together with

$\{A_\alpha: \alpha < \beta\}$. If A is a countably infinite subset of λ and $A \in M_\beta$, then by MA, there is some x, a Cohen subset of A over M_β. Since $x \subset A$, x is in some M_δ.

Finally, we remove c as as a constraint on size.

COROLLARY 3.5. *Let M be a model of set theory, κ a cardinal in M, $M \models \lambda \leqq \kappa \leqq 2^\lambda$. Let N be the extension of M by ω_1 many Cohen subsets of λ. Then in N there is an almost rigid dual Hopfian Boolean algebra of size κ with λ many atoms. (Note that $N \models c = \sup\{c^M, (\lambda^\omega)^M\}$.*

Proof. Let $\{x_\alpha: \alpha < \omega_1\}$ be the Cohen subsets of λ added by N. Let $M_0 = M$, and for $0 < \alpha < \omega_1$ let M_α be the Skolem closure of $\bigcup_{\beta<\alpha} M_\beta \cup \{x_\beta: \beta < \alpha\}$. Then the main hypothesis is satisfied, so we may apply 3.1.

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