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POINTWISE DOMINATION OF MATRICES AND COMPARISON OF $\mathscr{I}_p$ NORMS

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Let p be a real number in $[1, \infty)$ which is not an even integer. Let $N = 2[p/2] + 5$. We give examples of $N \times N$ matrices A and B, so that $|a_{ij}| \leqq b_{ij}$ but $\mathrm{Tr}([A^*A]^{p/2}) > \mathrm{Tr}([B^*B]^{p/2})$.

Let A and B be $N \times N$ matrices with

$$|a_{ij}| \leqq b_{ij} . \tag{1}$$

If we define the p norm of a matrix by

$$\|A\|_p = \mathrm{Tr}([A^*A]^{p/2})^{1/p} \tag{2}$$

then it is trivial that, if p is an even integer, then

$$\|A\|_p \leqq \|B\|_p \tag{3}$$

when (1) holds. For one need only write out the trace explicitly in terms of matrix elements. In a more general context, we conjectured in [5] that (1) implies (3) whenever $p \geqq 2$. The attractiveness of this conjecture is shown by the fact that I know of at least five people other than myself who have worked on proving it.

It was thus quite surprising that Peller [3] announced that (3) fails for some infinite matrices whenever p is not an even integer. In correspondence, Peller described his counterexample which relies on his beautiful but elaborate theory of $\mathscr{I}_p$ Hankel operators (4) and on a paper of Boas (2). It follows from Peller's example that (3) must fail for some finite N but it is not clear for which N. Our purpose here is to give explicit N and to avoid the complications of Peller's $\mathscr{I}_p$-Hankel theory.

The idea of the construction is very simple. Boas [2] constructed polynomials $f(z)$, $g(z)$ with $\int |f(e^{i\theta})|^p d\theta > \int |g(e^{i\theta})|^p d\theta$ even though the coefficients, a_n, of f and coefficients, b_n, of g obey $|a_n| \leqq b_n$. a and b should be thought of as Fourier coefficients of $f(e^{i\theta})$ and $g(e^{i\theta})$. It is obvious that for sufficiently large N, $\sum_{j=0}^{N-1} |f(e^{ij\theta_N})|^p \geqq \sum_{j=0}^{N-1} |g(e^{ij\theta_N})|$ where $\theta_N = 2\pi/N$. Again f and g should be viewed as functions on Z_N and the coefficients of the polynomial (if N is larger than the degrees) as Z_N-Fourier components. But the functions on Z_N are naturally imbedded in $N \times N$ matrices in such a way $\|A\|_p^p$ is just $\sum |f(e^{ij\theta_N})|^p$ and so that the order (1) is equivalent to the order on Fourier coefficients.

To be explicit, given N and $c_0, \cdots, c_{N-1}$ let A be the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & \cdots c_{N-3} & c_{N-2} \\ c_{N-2} & c_{N-1} & \cdots c_{N-4} & c_{N-3} \\ \vdots & & & \vdots \\ c_1 & c_2 & & c_0 \end{pmatrix}$$

$z_N = \exp(i\theta_N)$ and let φ_j be the vector with components $(1, z_N^j, z_N^{2j}, \cdots, z_N^{(N-1)j})$; $j = 0, \cdots, N-1$ and observe that

$$A\varphi_j = f(j)\varphi_j \tag{4}$$

where

$$f(j) = \sum_{i=0}^{N-1} c_\ell \chi_\ell(j) \tag{5}$$

with

$$\chi_\ell(j) = z_N^{\ell j} . \tag{6}$$

We use (6) to define χ_ℓ for any integer ℓ although, of course, χ_ℓ is periodic in ℓ with period N.

Of course, we have just exploited the fact that if σ is the matrix which cyclicity permutes the coordinates by one component, then $A\sigma = \sigma A$ (indeed $A = \sum c_p \sigma^k$) and since $= \sigma^N = 1$, σ is naturally diagonalized in terms of the group Z_N. The χ's are just the characters of Z_N. (In Physicist's language, since A has periodic boundary conditions, one diagonalizes it in momentum space.)

Since the φ_j are orthogonal vectors, A is a normal operator. For such an operator $\|A\|_p^p$ is just the sum of the pth powers of the eigenvalues, i.e.,

$$\|A\|_p^p = \sum_{j=0}^{N} |f(j)|^p . \tag{7}$$

We take

$$k = \left[\frac{1}{2}p\right] + 2 \tag{8a}$$

$$N = 2k + 1 = 2\left[\frac{1}{2}p\right] + 5 . \tag{8b}$$

Motivated by Boas' example, we choose

$$c_0 = 1 ; \quad c_1 = r ; \quad c_k = \lambda r_k ; \quad c_\ell = 0 , \quad \text{if} \quad \ell \neq 0, 1, k \tag{9}$$

where r is sufficiently small and

$$\lambda = \left(\frac{1}{2}p - 1\right)\left(\frac{1}{2}p - 2\right) \cdots \left(\frac{1}{2}p - k + 1\right)\Big/ k! \,. \tag{10}$$

Notice that *since p is not an even integer* and since $p/2 + 1 < k < p/2 + 2$, we have that $\lambda < 0$. Let $d_j = |c_j|$ and let B the corresponding matrix so (1) certainly holds.

We compute $\|A\|_p^p$ using (7) and the binomial theorem which is certainly legitimate if r is sufficient small

$$\begin{aligned} |f(j)|^{p/2} &= \sum_{\ell=0}^{\infty} \binom{p/2}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} r^{\ell + m(k-1)} \lambda^m \chi_{\ell + m(k-1)}(j) \\ &= f_1(j) + f_2(j) + f_3(j) + 0(r^{2k+1}) \end{aligned}$$

where

$$f_1 = \sum_{\ell=0}^{k-1} \binom{p/2}{\ell} r^\ell \chi_\ell$$

$$f_2 = \sum_{\ell=k}^{2k-1} \left[\binom{p/2}{\ell} + \lambda \binom{p/2}{\ell - k + 1}\binom{\ell - k + 1}{1}\right] r^\ell \chi_\ell$$

$$f_3 = r^{2k}\chi_{2k}\left[\binom{p/2}{2k} + \lambda(k+1)\binom{p/2}{k+1} + \lambda^2\binom{p/2}{2}\right] .$$

Because $N = 2k + 1$, the characters $\chi_0, \cdots, \chi_{2k}$ are orthogonal so squaring and summing:

$$\|A\|_p^p = \sum_{1=0}^{k-1} \binom{p/2}{j}^2 r^{2j} + r^{2k}\left[\binom{p/2}{k} + \lambda\binom{p/2}{1}\right]^2 + 0(r^{2k+1}) .$$

The formula for $\|B\|_p^p$ is identical, except λ is replaced by $|\lambda| = -\lambda$. But λ is exactly chosen so that

$$\binom{p/2}{k} - \lambda\binom{p/2}{1} = 0 .$$

Thus, for r small, $\|A\|_p > \|B\|_p$.

It was necessary to take $N = 2k + 1$ rather than just $k + 1$ to avoid cross terms between the r_0 and r^ℓ ($\ell \leqq 2k$) factors which have the wrong sign and only vanish because χ_0 and χ_ℓ are orthogonal for $\ell \leqq 2k$.

We close this paper with a series of remarks:

(1) Peller constructs infinite matrices A, B which are matrices of compact operators on ℓ_2 with (1) holding, $B \in \mathscr{I}_p$ and $A \notin \mathscr{I}_p$. It is easy to get such operators from our examples as follows: normalize A, B so that $\|A\|_p > 1 > \|B\|_p \geqq \|B\| \geqq \|A\|$. Let us view ℓ_2 as the tensor algebra over $\boldsymbol{C}^N$, i.e., as $\boldsymbol{C} \oplus \boldsymbol{C}^N \oplus \boldsymbol{C}^{N^2} \oplus \cdots$ and let $\Gamma(A) = 1 \oplus A \oplus (A \otimes A) \oplus \cdots$. Then $|\Gamma(A)_{ij}| \leqq |\Gamma(B)_{ij}|$ and $\Gamma(A)$, $\Gamma(B)$ are compact, $\Gamma(B) \in \mathscr{I}_p$ but $\Gamma(A) \notin \mathscr{I}_p$.

(2) Given any measure space, (M, μ) with $L^2(M, \mu)$ infinite dimensional, we cannot have that $\|A\|_p \leqq c\|B\|_p$ for some fixed c and all A, B with $|(Af)(m)| \leqq (B|f|)(m)$. For one can always imbed C^N into $L^2(M, \mu)$ in a way preserving $\|A\|_p$ norms and order (map $(a_1, \cdots, a_n)$ into $\sum a_i f_i(m)$ with f_i multiples of characteristic functions of disjoint sets). If $\|A\|_p \leqq c\|B\|_p$ held for $L^2(M)$ it would hold for any C^N. But by taking tensor products of our example one can arrange that $\|A\|_p/\|B\|_p$ is arbitrarily large. [It is interesting that this tensor product/operator theory version of Katznelson's remark (quoted in Bachelis [1]) is more natural than the function theoretic construction.]

(3) Let $N(p)$ be the smallest N for which there exist matrices for which (1) holds but (3) fails. Clearly we have shown

$$N(p) \leqq 2\left[\frac{1}{2}p\right] + 5$$

but equality is most unlikely for *any* p. Indeed for $1 \leqq p < 2$, we have $N(p) = 2$ since if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then $\|B\|_p^p = 2^p$, $\|A\|_p^p = 2(\sqrt{2})^p > \|B\|_p^p$ if $p < 2$. Moreover, we owe to S. Friedland the following simple argument showing that $N(p) \geqq 3$ if $p > 2$. If C, D are *positive* matrices with

$$|c_{ij}| \leqq d_{ij} \tag{11}$$

then with $\mu_j(\cdot) =$ singular values, we trivially have

$$\mu_1(C) \leqq \mu_1(D)\ ; \qquad \mu_1(C) + \mu_2(C) \leqq \mu_1(D) + \mu_2(D)$$

(since for 2×2 positive matrices $\mu_1(C) + \mu_2(C) = \operatorname{Tr}(C)$). By general rearrangement inequalities [5]

$$\operatorname{Tr}(C^p) \leqq \operatorname{Tr}(D^p)$$

for any $1 \leqq p \leqq \infty$. Given A, B obeying (1) and applying this remark to $C = A^*A$, $D = B^*B$, we see that (3) holds for any $p \geqq 2$ if $N = 2$. It would be interesting to know the precise value of $N(p)$. Two natural guesses are $[p/2] + 1$ and $2[p/2]$.

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