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A COMPARISON OF THE AUTOMORPHIC REPRESENTATIONS OF GL (3) AND ITS TWISTED FORMS

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Jacquet and Langlands have proved the existence of a deep relationship between the representation theory of the group GL (2) over a local or global field and of the group of invertible elements in a quaternion algebra over the same field. It is the purpose of this thesis to extend these results to the case of GL (3) and a central division algebra of rank 3^2.

0. Introduction. The theorems are deduced as consequences of the Arthur-Selberg trace formula. The proofs have been patterned after those used in [17] in comparing the representation theory of the groups GL (2) over two distinct fields.

The two main theorems of this thesis are as follows.

Let F be a nonarchimedean local field of characteristic zero, let $G = \mathrm{GL}(3, F)$, and let G' be the group of invertible elements in a central division algebra of rank 3^2 over F. Define admissible irreducible representations π of G and π' of G' to be related, and write $\pi \sim \pi'$, if $\Theta_\pi(g) = \Theta_{\pi'}(g')$ for all pairs of elements $g \in G$ and $g' \in G'$ which have the same irreducible characteristic polynomial, where Θ_π (resp., $\Theta_{\pi'}$) is the character of π (resp., π').

THEOREM 1. *The relation $\sim$ establishes a 1-1 correspondence between the set of isomorphism classes of admissible irreducible representations of G' and the set of isomorphism classes of admissible irreducible representations of G which are special or supercuspidal.*

Now let F be a number field, let A be the adele ring of F, let $G = \mathrm{GL}(3)$, and let G' be the group of invertible elements in a central division algebra D of rank 3^2 over F. Let S be the set of places v

of F at which D does not split. Define irreducible representations $\pi = \bigotimes_v \pi_v$ of $G(A)$ and $\pi' = \bigotimes_v \pi'_v$ of $G'(A)$ to be related, and write $\pi \sim \pi'$, if $\pi_v \simeq \pi'_v$ for almost all $v \notin S$.

THEOREM 2. *The relation $\sim$ establishes a 1–1 correspondence between the set of irreducible cuspidal automorphic representations π' of $G'(A)$ and the set of irreducible cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $G(A)$ for which π_v is special or supercuspidal for all $v \in S$.*

Moreover, if $\pi \sim \pi'$ for such π and π', then

i) $\pi_v \simeq \pi'_v$ *for all* $v \notin S$

and

ii) $\pi_v \sim \pi'_v$ *for all* $v \in S$.

Using the theory of L-series rather than that of the trace formula, Jacquet, Pyatetskii-Shapiro, and Shalika [14] have obtained related results.

It is a pleasure to extend my sincerest thanks to R. Langlands, who first suggested the topic of this research to me and then provided me with both encouragement and technical advice. Others who have been especially helpful to me while working on this thesis are my advisor J. Tate, D. Kazhdan, and J. Arthur.

1. Disconnected spaces. In this paper a topological space T will be said to be a disconnected space if it is Hausdorff, locally compact, and totally disconnected; it amounts to the same to say that T is Hausdorff and that every element of T has a fundamental system of compact neighborhoods which are open in T. A locally closed subspace of a disconnected space is a disconnected space.

Let T be a disconnected space. Define $C(T)$ to be the space of locally constant complex-valued functions on T with compact support. Define $D(T)$, the space of distributions on T, to be the space $\mathrm{Hom}_{\mathbf{C}}(C(T), \mathbf{C})$. An element D of $D(T)$ is said to be positive if $D(f) \geqq 0$ for each $f \in C(G)$ that assumes only nonnegative real values.

Let Y be an open subset of T and let X be a closed subset of T. The maps $i_Y: C(Y) \to C(T)$ and $r_X: C(T) \to C(X)$ are defined as follows:

$i_Y(f)$ is the extension of f by zero to T, and
$r_X(f)$ is the restriction of f to X.

PROPOSITION 1.1. *Let X be a closed subspace of T. Then the following is an exact sequence*:

$$0 \longrightarrow C(T-X) \xrightarrow{i_{T-X}} C(T) \xrightarrow{r_X} C(X) \longrightarrow 0\ .$$

Proof. Only the surjectivity of r_X is not obvious. Let $f \in C(X)$. It must be shown that f is in the image of r_X. In fact, it may be assumed that f takes only one value different from zero, say a. Let $Z = f^{-1}\{a\}$. Z is compact and open in X. It is immediate that Z may be written $Z = X \cap W$, where W is a compact open subset of T. Then $f = r_X(g)$, where g is a times the characteristic function of W.

For a closed subspace X of T, $D(X)$ may be viewed as a subspace of $D(T)$ via the map adjoint to r_X.

PROPOSITION 1.2. *Let $\{X_a \mid a \in A\}$ be a family of closed subspaces of T. Then $D(\bigcap_{a \in A} X_a) = \bigcap_{a \in A} D(X_a)$.*

Proof. It is clear that $D(\bigcap_{a \in A} X_a) \subset \bigcap_{a \in A} D(X_a)$. If $A = \{1, 2\}$, the opposite inclusion may be proved by chasing the exact commutative diagram below.

$$\begin{array}{ccccccc}
D(T - X_1) & \longleftarrow & D(T) & \longleftarrow & D(X_1) & \longleftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
D(X_2 - (X_1 \cap X_2)) & \longleftarrow & D(X_2) & \longleftarrow & D(X_1 \cap X_2) & \longleftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The case in which A is any finite set follows from this by induction.

Now let A be arbitrary, and let $J \in \bigcap_{a \in A} D(X_a)$. The exactness of

$$D(T - \bigcap_{a \in A} X_a) \longleftarrow D(T) \longleftarrow D(\bigcap_{a \in A} X_a) \longleftarrow 0$$

implies that in order to show that $J \in D(\bigcap_{a \in A} X_a)$ it suffices to show that $J(f) = 0$ for each $f \in C(T - \bigcap_{a \in A} X_a)$. Let $f \in C(T - \bigcap_{a \in A} X_a)$. Because the support of f is compact, there exists a finite subset B of A such that $f \in C(T - \bigcap_{b \in B} X_b)$. By the result above, $J \in D(\bigcap_{b \in B} X_b)$; hence $J(f) = 0$ as required.

This proposition justifies the

DEFINITION. Let $D \in D(T)$. The support of D, written $\operatorname{supp} D$, is the smallest closed subspace X of T such that $D \in D(X)$.

For a disconnected space T and complex vector space V, define $C(T, V)$ to be the space of locally constant functions from T to V with compact support. It is evident that $C(T, V) \simeq C(T) \otimes V$.

Let X and Y be disconnected spaces. Define the map

$S: C(X, C(Y)) \to C(X \times Y)$ by the formula $Sf(x, y) = f(x)(y)$.

PROPOSITION 1.3. *S is an isomorphism.*

Proof. Triviality.

A topological group G will be said to be a disconnected group if it is a disconnected space. Such a group is one which contains an open profinite subgroup. See [7], p. 118, and [10], p. 62. If H is a closed subset of d disconnected group G, then the homogeneous space $H\backslash G$ is a disconnected space.

Let G be a disconnected group. Define actions L and R of G on $C(G)$ and L and R of G on $D(G)$ as follows. For $s, g \in G$, $f \in C(G)$, and $D \in D(G)$, then

$$\begin{aligned} L(s)f(g) &= f(s^{-1}g) \qquad & L(s)D(f) &= D(L(s^{-1})f) \\ R(s)f(g) &= f(gs) \qquad & R(s)D(f) &= D(R(s^{-1})f)\ . \end{aligned}$$

An element D of $D(G)$ is said to be left invariant if $L(s)D = D$ for all $s \in G$. A Haar measure on G is a nonzero left invariant positive element of $D(G)$.

PROPOSITION 1.4. *The subspace of left invariant elements of $D(G)$ has dimension one and contains a Haar measure. There exists a continuous homomorphism Δ_G from G to the multiplicative group of positive real numbers such that $R(s)D = \Delta_G(s)^{-1}D$ for all $s \in G$ and all left invariant $D \in D(G)$.*

Proof. Let K be a compact open subgroup of G. The space $C(G)$ is spanned by functions of the form $L(s)X_N$, where X_N is the characteristic function of an open subgroup N of K and $s \in G$. Thus a left invariant element J of $D(G)$ is uniquely determined by its values on the X_N. But $J(X_N)$ is determined by $J(X_K)$ through the formula $J(X_N) = (1/(K\colon N))J(X_K)$.

Conversely, it is easy to see that by using this formula a left invariant distribution taking a prescribed value on X_K can be constructed.

The homomorphism Δ_G is certainly continuous, for K is contained in its kernel.

G is said to be a unimodular if Δ_G is identically one.

Let H be a closed subgroup of a disconnected group G. Define actions R of G on $C(H\backslash G)$ and of G on $D(H\backslash G)$ as follows. For $s \in G$, $\bar{g} \in H\backslash G$, $f \in C(H\backslash G)$, and $D \in D(H\backslash G)$, then

$$R(s)f(\bar{g}) = f(\bar{g}s) \qquad R(s)D(f) = D(R(s^{-1})f)\ .$$

An element D of $D(H\backslash G)$ is said to be G-invariant if $R(s)D = D$ for all $s \in G$.

PROPOSITION 1.5. *Assume that H and G are unimodular. Then the subspace of G-invariant elements of $D(H\backslash G)$ has dimension one and contains a nonzero positive element. If D is a nonzero G-invariant element of $D(H\backslash G)$, then the kernel of D is spanned by functions of the form $R(s)f - f$, where $s \in G$ and $f \in C(H\backslash G)$.*

Proof. The second assertion is a consequence of the first. The first follows from a study of the map P from $C(G)$ to $C(H\backslash G)$ defined by $Pf(g) = \int_H f(hg)\mu(h)$, where μ is a Haar measure on H. Specifically, one must show that P is surjective and that the kernel of P is contained in the kernel of a Haar measure on G. Details are left to the reader.

DEFINITION. Let $s \in G$, and let $f \in C(G)$. $\mathrm{Ad}(s)f \in C(G)$ is defined by the formula $\mathrm{Ad}(s)f(g) = f(s^{-1}gs)$. Define $I(G)$ to be the space of conjugation invariant distributions on G; that is, $I(G)$ is the set of $D \in D(G)$ such that $D(\mathrm{Ad}\,(s)f) = D(f)$ for all $s \in G$ and $f \in C(G)$. For each closed subset X of G, let $I(X) = D(X) \cap I(G)$.

2. Orbital integrals. Let F be a nonarchimedean local field of characteristic zero with valuation ring R. Let G be GL (3, F), let K be GL (3, R), and let Z be the center of G. For an element γ of G, write $\mathrm{cl}\,(\gamma)$ for the set of elements of G conjugate to γ, and write $G(\gamma)$ for the centralizer of γ in G. If X is a subset of G, write X^G for the set of elements in G which are conjugate to an element of X.

For a maximal torus T of G, write T' for the subset of its regular elements; that is, for the (open) subset of all its elements which have three distinct eigenvalues. These are precisely the elements whose centralizer in G is T. Write W_T or just W for the finite group N_T/T, where N_T is the normalizer of T in G. Some useful facts about a maximal torus T are assembled in the elementary

LEMMA 2.1.1. *If t is a regular element of T, then there exists an open closed conjugation invariant neighborhood of $\mathrm{cl}\,(t)$ in G which is contained in the open subset T'^G.*

2.1.2 *Let W act on $T' \times T\backslash G$ via $(t, \bar{g})^w = (w^{-1}tw, \overline{w^{-1}g})$. Then the map $\begin{matrix} T' \times T\backslash G \to T'^G \\ t \qquad \bar{g} \to g^{-1}tg \end{matrix}$ realizes T'^G as the quotient space of $T' \times T\backslash G$ by the action of W.*

2.1.3 *The map* char: $\begin{array}{c} T \to F \times F \times F^* \\ t \to c_1(t),\, c_2(t),\, c_3(t) \end{array}$ *where the characteristic polynomial of t is $\lambda^3 - c_1(t)\lambda^2 + c_2(t)\lambda - c_3(t)$, is a proper map.*

2.1.4 *If t is a regular element of T, then* $\mathrm{cl}\,(t)$ *is closed in G, and the map* $\begin{array}{c} T\backslash G \to \mathrm{cl}\,(t) \\ \bar{g} \to g^{-1}tg \end{array}$ *is a homeomorphism.*

Given a maximal torus T and a G-invariant measure ω on $T\backslash G$, define the map $F^\omega = F$ from $C(G)$ to functions on T' by the formula $F_f(t) = \int_{T\backslash G} f(g^{-1}tg)\omega(\bar{g})$. The integral converges because the restriction of f to $\mathrm{cl}\,(t)$ has compact support, $\mathrm{cl}\,(t)$ being closed in G.

LEMMA 2.2. *For $f \in C(G)$, F_f is locally constant, has support which is relatively compact in T, and is invariant under conjugation by elements of W.*

Proof. To check that F_f is locally constant, it is enough, by 2.1.1, to consider the case in which the support of f is contained in T'^G, in which case the result follows from the properness of the map in 2.1.2 together with Proposition 1.3.

The support of F_f is contained in $\mathrm{char}^{-1}(\mathrm{char}\,(\mathrm{supp}\, f)) \cap T$, which is compact by 2.1.3.

The invariance of F_f under W is clear.

LEMMA 2.3. *The map $F\colon C(T'^G) \to C(T')^W$ is surjective, where $C(T')^W$ is the set of W-conjugation invariant elements of $C(T')$.*

Proof. That the map is defined, that is, that F_f has compact support for f in $C(T'^G)$, follows from the properness of the map in 2.1.2. For surjectivity, note that the map

$$\begin{array}{l} C(T' \times T\backslash G) \longrightarrow C(T') \\ \qquad\qquad f \longrightarrow \left(t \longrightarrow \int_{T\backslash G} f(t, \bar{g})\omega(\bar{g})\right) \end{array}$$

is onto. This is a W-map, and so its restriction to a function from $C(T'^G) = C(T' \times T\backslash G)^W$ to $C(T')^W$ is also onto.

If γ is any element of G, semisimple or not, $\mathrm{cl}\,(\gamma)$ is locally closed in G, $\mathrm{cl}\,(\gamma)$ is homeomorphic to $G(\gamma)\backslash G$, and $G(\gamma)$ is unimodular.

For γ in G and f in $C(G)$, let $D(\gamma, f)$ equal $\int_{G(\gamma)\backslash G} f(g^{-1}\gamma g)d\bar{g}$, where $d\bar{g}$ is the G-invariant measure on $G(\gamma)\backslash G$ which assigns measure 1 to $G(\gamma)\backslash G(\gamma)K$. That this integral converges even in the case in which $\mathrm{cl}\,(\gamma)$ is not closed may be seen from the expressions in Table 1.

Table 1

$$D_x^1(f) = D(\gamma_x^1, f) = \psi_f^1(x)$$

$$D_x^2(f) = D(\gamma_x^2, f) = \frac{1}{1-(1/q)^2}\psi_f^2(x, x)$$

$$D_x^3(f) = D(\gamma_x^3, f) = \frac{1}{(1-(1/q))^2}\psi_f^3(x, x)$$

$$x \neq y \quad D_{x,y}^2(f) = D(\gamma_{x,y}^2, f) = \frac{1}{|x-y|^2}\psi_f^2(x, y)$$

$$x \neq y \quad D_{x,y}^3(f) = D(\gamma_{x,y}^3, f) = \frac{1}{(1-(1/q))\,|x-y|^2}\psi_f^3(x, y)$$

where

$$\gamma_x^1 = x \cdot 1_G \qquad \gamma_x^2 = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix} \qquad \gamma_x^3 = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}$$

$$x \neq y \qquad \gamma_{x,y}^2 = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \qquad \gamma_{x,y}^3 = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}$$

and

$$\psi_f^1(x) = f(\gamma_x^1)$$

$$\psi_f^2(x, y) = \int f\left(k^{-1}\begin{pmatrix} x & 0 & A \\ 0 & x & B \\ 0 & 0 & y \end{pmatrix}k\right)dkdAdB$$

$$\psi_f^3(x, y) = \int f\left(k^{-1}\begin{pmatrix} x & A & B \\ 0 & x & C \\ 0 & 0 & y \end{pmatrix}k\right)dkdAdBdC$$

$$k \text{ varies over } K \quad \int_K dk = 1$$

and

$$A,\ B,\ C \text{ vary over } F \quad \int_R dA = \int_R dB = \int_R dC = 1$$

$$x,\ y \in F^* .$$

The unexplained symbols at the left of each line of the table are to be defined by the equations in which they appear. Note that $\psi_f^1 \in C(F^*)$ and that ψ_f^2, $\psi_f^3 \in C(F^* \times F^*)$.

The derivation of the formulas in Table 1 is quite similar to that of the analogous formulas for orbital integrals on lie algebras appearing in [18]. More precisely, for each element γ of G, a parabolic subgroup $P(\gamma)$ containing $G(\gamma)$ may be selected such that $G = P(\gamma)K$, and an algebraic subgroup $H(\gamma)$ of $P(\gamma)$ may be selected such that

$G(\gamma) \cap H(\gamma) = \{1\}$ and $G(\gamma)H(\gamma)$ is open and dense in $P(\gamma)$. For instance, one can take $P(\gamma^2_{x,y}) = \{(a_{ij}) \mid a_{31} = a_{32} = 0\}$ and $H(\gamma^2_{x,y}) = \{(a_{ij}) \mid a_{ii} = 1,\ a_{12} = 0, \text{ and } a_{ij} = 0 \text{ if } i > j\}$. By standard lemmas, if $f \in C(G(\gamma)\backslash G)$, then

$$\int_{G(\gamma)\backslash G} f(\bar{g})d\bar{g} = \int_{H(\gamma)\times K} f(hk)(\Delta_{H(\gamma)}/\Delta_{P(\gamma)})(h)dhdk\ ,$$

where dh and dk are suitably normalized Haar measures on $H(\gamma)$ and K. Writing this integral formula explicitly in each case and keeping careful track of the Haar measures which must be used leads directly to the formulas in Table 1.

DEFINITION. For $x \in F^*$, let $\mathscr{U}_x$ be the set of elements in G with characteristic polynomial $(\lambda - x)^3$. For $(x, y) \in F^* \times F^* - \Delta$, where Δ is the diagonal set in $F^* \times F^*$, let $\mathscr{U}_{x,y}$ be the set of elements in G with characteristic polynomial $(\lambda - x)^2(\lambda - y)$.

Note that if X is a compact subset of G such that $\mathscr{U}_x \cap X$ (respectively, $\mathscr{U}_{x,y} \cap X$) is empty, then $G - X$ contains an open closed invariant neighborhood of $\mathscr{U}_x$ (respectively, $\mathscr{U}_{x,y}$).

LEMMA 2.4. *Let $x \in F^*$ and let $f \in C(G)$ be such that $D^1_x(f) = D^2_x(f) = D^3_x(f) = 0$. (Respectively, let $x, y \in F^* \times F^* - \Delta$ and let $f \in C(G)$ be such that $D^2_{x,y}(f) = D^3_{x,y}(f) = 0$.)*

Then there exist functions $\varphi_i \in C(G)$ and elements $g_i \in G$ such that $f - \sum(\mathrm{Ad}(g_i)\varphi_i - \varphi_i)$ vanishes on an invariant closed neighborhood V of $\mathscr{U}_x$ (respectively of $\mathscr{U}_{x,y}$).

Proof. Suppose $f \in C(G)$ is such that $D^1_x(f) = D^2_x(f) = D^3_x(f) = 0$. Because f is zero on $\mathrm{cl}(\gamma^1_x)$, the restriction $\tilde{f}$ of f to $\mathrm{cl}(\gamma^2_x)$ has compact support. Proposition 1.5 applies to produce functions $\tilde{\varphi}_i \in C(\mathrm{cl}(\gamma^2_x))$ and elements $g_i \in G$ such that $\tilde{f} - \sum(\mathrm{Ad}(g_i)\tilde{\varphi}_i - \tilde{\varphi}_i)$ is identically zero. After extending the $\tilde{\varphi}_i$ by zero to the set $\mathrm{cl}(\gamma^1_x) \cup \mathrm{cl}(\gamma^2_x)$ which is closed in G, then further extending to elements $\varphi_i \in C(G)$ in any way, which is possible by Proposition 1.1, and then replacing f by $f - \sum(\mathrm{Ad}(g_i)\varphi_i - \varphi_i)$, it may be assumed that f is zero on $\mathrm{cl}(\gamma^1_x) \cup \mathrm{cl}(\gamma^2_x)$. The argument just used on $\mathrm{cl}(\gamma^2_x)$ now applies to $\mathrm{cl}(\gamma^3_x)$, which proves the desired result.

The proof in the respective case of $\mathscr{U}_{x,y}$ is similar. The chain $\mathrm{cl}(\gamma^1_x) \subset \mathrm{cl}(\gamma^1_x) \cup \mathrm{cl}(\gamma^2_x) \subset \mathscr{U}_x$ of closed subsets of G must be replaced by $\mathrm{cl}(\gamma^2_{x,y}) \subset \mathscr{U}_{x,y}$.

LEMMA 2.5. *Let all hypotheses be as in Lemma 2.4. Then there exists an open compact subgroup B of G such that $D(f) = 0$ for all $D \in I(\overline{\gamma^1_x B^G})$ (respectively, for all $D \in I(\overline{(\gamma^2_{x,y}B)^G})$).*

Proof. Choose B to be an open compact subgroup contained within $(\gamma_x^1)^{-1}V$ (respectively, $(\gamma_{x,y}^2)^{-1}V$).

A major tool is

THEOREM 2.6 (Howe). *Let H be a compact open subgroup of G and X a compact subset of G. Then the map of $I(\overline{X^G})$ into $D(G/H)$ has finite rank.*

Proof. This is conjecture 2 of [11]. A proof for GL (3) is announced on page 379 of that paper.

THEOREM 2.7. *Let H be a compact open subgroup of G.*

(1) *Let $x \in F^*$. Then there exists an open compact subgroup B of G such that the image of $I(\overline{\gamma_x^1 B^G})$ in $D(G/H)$ is contained in the span of the images of D_x^1, D_x^2, and D_x^3 in $D(G/H)$.*

(2) *Let $x, y \in F^* \times F^* - \Delta$. Then there exists an open compact subgroup B of G such that the image of $I(\overline{(\gamma_{x,y}^2 B)^G})$ in $D(G/H)$ is contained in the span of the images of $D_{x,y}^2$ and $D_{x,y}^3$ in $D(G/H)$.*

Proof. Since the proofs of (1) and (2) are altogether similar, only (1) will be proved.

Let $V_0 = \{f \in C(G/H) \mid D_x^1(f) = D_x^2(f) = D_x^3(f) = 0\}$.

Let $V_{00} = \{f \in V_0 \mid D(f) = 0 \text{ for all } D \in I(\gamma_x^1 K^G)\}$.

By Theorem 2.6, V_{00} is of finite codimension in V_0, whence by Lemma 2.5 there is an open compact subgroup B contained in K^G such that $I(\overline{\gamma_x^1 B^G})$ annihilates V_0. This B fills the bill.

COROLLARY 2.8. *Let T be a maximal torus of G and let ω be a G-invariant measure on $T\backslash G$.*

(1) *Let $x \in F^*$. Then there exist locally constant functions $\Lambda_x^{\omega,1}$, $\Lambda_x^{\omega,2}$, $\Lambda_x^{\omega,3}$ on T' such that for each $f \in C(G)$*

$$F_f^\omega = \psi_f^1(x)\Lambda_x^{\omega,1} + \psi_f^2(x, x)\Lambda_x^{\omega,2} + \psi_f^3(x, x)\Lambda_x^{\omega,3}$$

on the intersection of T' and a neighborhood of γ_x^1 which depends on f.

(2) *Let $x, y \in F^* \times F^* - \Delta$. Then there exist locally constant functions $\Lambda_{x,y}^{\omega,2}$ and $\Lambda_{x,y}^{\omega,3}$ on T' such that for each $f \in C(G)$*

$$F_f^\omega = \psi_f^2(x, y)\Lambda_{x,y}^{\omega,2} + \psi_f^3(x, y)\Lambda_{x,y}^{\omega,3}$$

on the intersection of T' and a neighborhood of $\gamma_{x,y}^2$ which depends on f.

Proof. The only assertion that this corollary makes beyond that of Theorem 2.7 is that the $\Lambda_x^{\omega,i}$ and $\Lambda_{x,y}^{\omega,i}$ are locally constant. But this follows from the fact that F_f^{ω} is locally constant together with the existence of functions $f \in C(G)$ satisfying any one of the following three conditions:

(i) $D_x^1(f) = D_x^2(f) = 0$; $D_x^3(f) \neq 0$.

(ii) $D_x^1(f) = 0$; $D_x^2(f) \neq 0$.

(iii) $D_{x,y}^2(f) = 0$; $D_{x,y}^3(f) \neq 0$.

A maximal torus T of G will be said to be split, quadratic, or cubic, depending upon whether the characteristic polynomials of regular elements of T split over F, have an irreducible quadratic factor, or are irreducible. The conjugate classes of quadratic (respectively, cubic) tori of G are in natural one-to-one correspondence with the quadratic (respectively, cubic) field extensions of F.

Let $\{T_\alpha \mid \alpha \in A\}$ be a (finite) set of representatives of the conjugate classes of maximal tori of G. In the rest of this paper it will be assumed that the split element of this collection is the group of diagonal matrices of G and that each quadratic element is a subgroup of the group of matrices of the form (a_{ij}) with $a_{13} = a_{23} = a_{31} = a_{32} = 0$. This assumption implies that the nonregular elements of each T_α are all diagonal matrices. Let A' be the set of $\alpha \in A$ for which T_α is cubic.

For each $\alpha \in A$ choose a G-invariant measure ω_α on $T_\alpha\backslash G$. When used as an index, T_α and ω_α will be systematically replaced by α. For future purposes of comparison with division algebras, it will be assumed that the ω_α have been chosen so that $F_f^\alpha(t) = \int_{Z\backslash G} f(g^{-1}tg)\omega(\bar{g})$ for all $\alpha \in A'$, where ω is a fixed invariant measure on $Z\backslash G$ independent of α.

THEOREM 2.9. *Let $\{\Phi_\alpha \mid \alpha \in A\}$ be a collection of functions Φ_α on T'_α. The following two conditions are equivalent.*

(1) *There exists $f \in C(G)$ such that $\Phi_\alpha = F_f^\alpha$ for all $\alpha \in A$.*

(2) (a) *For each $\alpha \in A$, Φ_α is locally constant, has support which is relatively compact in T_α, and is invariant under conjugation by elements of W_α, and* (b) *There exist functions $\psi_1 \in C(F^*)$ and $\psi_2, \psi_3 \in C(F^* \times F^*)$ such that:*

For each $x \in F^*$

$$\Phi_\alpha = \psi_1(x)\Lambda_x^{\alpha,1} + \psi_2(x, x)\Lambda_x^{\alpha,2} + \psi_3(x, x)\Lambda_{x,x}^{\alpha,3}$$

in a neighborhood of γ_x^1 in T'_α all $\alpha \in A$; and for each $x, y \in F^* \times F^* - \Delta$

$$\Phi_\alpha = \psi_2(x, y)\Lambda_{x,y}^{\alpha,2} + \psi_3(x, y)\Lambda_{x,y}^{\alpha,3}$$

in a neighborhood of $\gamma_{x,y}^2$ in T'_α, all split and quadratic $\alpha \in A$.

Moreover, if these two equivalent conditions are met, then $\psi_1(x) = \psi_f^1(x)$, $\psi_2(x, y) = \psi_f^2(x, y)$, *and* $\psi_3(x, y) = \psi_f^3(x, y)$.

Proof. That (1) implies (2) and the moreover clause is just a restatement of Corollary 2.8.

Assume now that condition (2) is satisfied. Once a function $f \in C(G)$ is produced such that $\psi_f^1 = \psi_1$, $\psi_f^2 = \psi_2$, and $\psi_f^3 = \psi_3$, Lemma 2.3 concludes the proof; for then $\Phi_\alpha - F_f^\alpha$ will be in $C(T')^{W_\alpha}$ for all $\alpha \in A$. In producing such f, five closed subsets of G, X_1, X_2, X_3, X_4, and X_5, will be needed. They are defined as follows.

$$X_1 = Z \qquad X_2 = X_1 \bigcup_{x \in F^*} \mathrm{cl}\,(\gamma_x^2)$$
$$X_3 = X_2 \bigcup_{x \in F^*} \mathrm{cl}\,(\gamma_x^3) \qquad X_4 = X_3 \bigcup_{x, y \in F^* \times F^*} \mathrm{cl}\,(\gamma_{x,y}^2)$$
$$X_5 = X_4 \bigcup_{x, y \in F^* \times F^*} \mathrm{cl}\,(\gamma_{x,y}^3) = \{g \in G \mid g \text{ has multiple eigenvalues}\}\ .$$

The next five maps are all homeomorphisms.

$$\text{(i)} \qquad \begin{aligned} F^* &\longrightarrow X_1 \\ x &\longrightarrow \gamma_x^1 \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} F^* \times G(\gamma_x^2)\backslash G &\longrightarrow X_2 - X_1 \\ x \qquad \bar{g} \quad &\longrightarrow g^{-1}\gamma_x^2 g \end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned} F^* \times G(\gamma_x^3)\backslash G &\longrightarrow X_3 - X_2 \\ x \qquad \bar{g} \quad &\longrightarrow g^{-1}\gamma_x^3 g \end{aligned}$$

$$\text{(iv)} \qquad \begin{aligned} (F^* \times F^* - \Delta) \times G(\gamma_{x,y}^2)\backslash G &\longrightarrow X_4 - X_3 \\ x, y \qquad\qquad \bar{g} \quad &\longrightarrow g^{-1}\gamma_{x,y}^2 g \end{aligned}$$

$$\text{(v)} \qquad \begin{aligned} (F^* \times F^* - \Delta) \times G(\gamma_{x,y}^3)\backslash G &\longrightarrow X_5 - X_4 \\ x, y \qquad\qquad \bar{g} \quad &\longrightarrow g^{-1}\gamma_{x,y}^3 g\ . \end{aligned}$$

Note that the centralizer subgroups $G(\gamma_x^2)$, $G(\gamma_x^3)$, $G(\gamma_{x,y}^2)$, and $G(\gamma_{x,y}^3)$ which appear above do not in fact depend on x and y.

Using now Proposition 1.1 and the above homeomorphisms, the next five maps can be seen to be surjective. This is precisely what is required to produce the sought for $f \in C(G)$.

$$\text{(i)} \qquad \begin{aligned} C(G) &\longrightarrow C(F^*) \\ f \ &\longrightarrow \psi_f^1(x) \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} C(G - X_1) &\longrightarrow C(F^*) \\ f \qquad &\longrightarrow \psi_f^2(x, x) \end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned} C(G - X_2) &\longrightarrow C(F^*) \\ f \qquad &\longrightarrow \psi_f^3(x, x) \end{aligned}$$

$$\text{(iv)} \qquad \begin{aligned} C(G - X_3) &\longrightarrow C(F^* \times F^* - \Delta) \\ f \quad &\longrightarrow \psi_f^2(x, y) \end{aligned}$$

$$\text{(v)} \qquad \begin{aligned} C(G - X_4) &\longrightarrow C(F^* \times F^* - \Delta) \\ f \quad &\longrightarrow \psi_f^3(x, y) \,. \end{aligned}$$

For the application in mind, the germs $\Lambda_x^{\alpha,1}$ must be known explicitly for all $\alpha \in A$ and all $x \in F^*$. Recall that the ω_α have been chosen so that $F_f^\alpha(t) = \int_{Z\backslash G} f(g^{-1}tg)\omega(\bar{g})$ for all $\alpha \in A'$, where ω is an invariant measure on $Z\backslash G$ independent of α.

THEOREM 2.10. (1) *For a split or quadratic torus T_α, $\Lambda_x^{\alpha,1}$ is zero near γ_x^1.*

(2) *For $\alpha \in A'$, $\Lambda_x^{\alpha,1}$ is a nonzero constant $\Lambda(\omega)$ depending on ω but independent of x and α. For $\omega = dz\backslash dg$, where $\int_K dg = 1$ and $\int_{Z(R)} dz = 1$, $\Lambda(\omega) = 3q^{-3}/((1 - 1/q)(1 - 1/q^2))$.*

Proof. (1) For T_α split or quadratic, there are standard formulas, q.v. [8], p. 92-93, which rewrite the integral defining F_f^α. If T_α split, they imply directly that $\Lambda_x^{\alpha,1} = 0$. If T_α is quadratic, they together with the theory of orbital integrals on GL (2, F), which is presented in §4 of [17], imply that $\Lambda_x^{\alpha,1} = 0$.

(2) This has been proved as Lemma 7.4 of [16].

The analogue of Theorem 2.9 for a division algebra is much more trivial. Let G' be the group of nonzero elements in a central division algebra D of finite rank over a nonarchimedean local field F of any characteristic. Let ν and τ be the reduced norm and trace from D to F. Let Z be the center of G', and let ω' be a nonzero invariant measure on the compact group $Z\backslash G'$.

THEOREM 2.11. *The following map is defined and surjective.*

$$\begin{aligned} R\colon C(G') &\longrightarrow C(G')^{\mathrm{Inv}} \\ \varphi \quad &\longrightarrow R\varphi\colon \gamma \longrightarrow \int_{Z\backslash G'} \varphi(g^{-1}\gamma g)\omega'(\bar{g}) \,, \end{aligned}$$

where $C(G')^{\mathrm{Inv}}$ is the set of class functions in $C(G')$.

Proof. The integral converges, since $Z\backslash G'$ is compact. That $R\varphi$ is locally constant and has compact support is due to the fact that conjugation preserves the valuation on D. $R\varphi$ is clearly a class function.

If ω' gives measure 1 to $Z\backslash G'$, then the restriction of R to $C(G')^{\mathrm{Inv}}$ is the identity; thus for any ω', R is the multiple of a projection onto $C(G')^{\mathrm{Inv}}$.

Let now F be of characteristic zero, and assume that D is of rank 3^2 over F. Let $\{T_\alpha | \alpha \in A'\}$ be a set of representatives of the conjugate classes of maximal tori of G', where the indices α are the same as those which index the cubic tori in the previously chosen set of maximal tori of G. As is suggested by the notation, the subgroups T_α of G and T_α of G' will be frequently identified by means of an isomorphism which will be fixed once for all. For $\varphi \in C(G')$, define $F_\varphi^{\omega'_\alpha} = F_\varphi^\alpha$ to be the restriction of $R\varphi$ to T'_α. It depends on ω'.

Let ψ be a nontrivial additive character of F. Let dx and dx' be the Haar measures on $M_3(F)$ and D respectively which are self dual with respect to the characters $\psi \circ \mathrm{Tr}$ and $\psi \circ \tau$. The invariant measures $(dx/|\det x|_F^3)$ on G and $(dx'/|\nu(x')|_F^3)$ on G' are said to be *associated*. Invariant measures ω on $Z\backslash G$ and ω' on $Z\backslash G'$ are said to be *associated* if $\omega = dz\backslash dg$ and $\omega' = dz\backslash dg'$ where dg on G and dg' on G' are associated measures and dz is an invariant measure on the center Z of G and G'.

THEOREM 2.12. *There exists a linear map*

$$\begin{array}{ccc} C(G') & \longrightarrow & C(G) \\ \varphi & \longrightarrow & f \end{array}$$

such that

(1) *For each pair ω, ω' of associated measures on $Z\backslash G$ and $Z\backslash G'$,*

$$F_\varphi^{\omega'_\alpha} = F_f^{\omega_\alpha} \quad \textit{for all} \quad \alpha \in A'$$

(2) $\varphi(z) = f(z)$ *all* $z \in Z$

(3) $D(\gamma, f) = 0$ *if* $\gamma \in G$ *is not an element of a cubic torus.*

Proof. Given φ, f will be produced by Theorem 2.9. The functions ψ_2 and ψ_3 in that theorem are to be taken identically zero. Because $\Lambda_x^{\alpha,1}$ is zero for split and quadratic α, Φ_α can be taken identically zero for such α.

Let ω, ω' be a pair of associated measures on $Z\backslash G$ and $Z\backslash G'$. For $\alpha \in A'$, take $\Phi_\alpha = F_\varphi^{\omega'_\alpha}$. Theorem 2.11 implies that for each $x \in F^*$, $\Phi_\alpha = \varphi(x)\Lambda(\omega')$ in a neighborhood of x in T'_α, where $\Lambda(\omega') = \int_{Z\backslash G'} \omega'(\bar{g})$. Theorem 2.12 will be established upon verification that $\Lambda(\omega) = \Lambda(\omega')$.

Let $\omega' = dz\backslash dg'$, where dg' is the invariant measure on G'

determined by an additive character ψ of F of order zero and where dz is such that $\int_{Z(R)} dz = 1$. An easy computation shows that

$$A(\omega') = \int_{Z\backslash G'} \omega'(\bar{g}) = 3q^{-3}\left(1 - \frac{1}{q^3}\right).$$

If dg is the measure on G associated to dg', then a similar computation shows that

$$\int_K dg = \left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right)\left(1 - \frac{1}{q^3}\right).$$

Comparing these results with Proposition 2.10 now concludes the argument.

3. Local computations. Let F be a nonarchimedean local field of characteristic zero. Let G, G', and all other notation in this section be as in §2. Associated Haar measures dg on G and dg' on G' and a Haar measure dz on Z will be fixed once for all. Denote by ω and ω' the quotient measures $dz\backslash dg$ and $dz\backslash dg'$ on $Z\backslash G$ and $Z\backslash G'$ respectively.

The Weyl integration formulas are as follows. See [8], Lemma 42. For each $\alpha \in A$, let ν_α be a Haar measure on $Z\backslash T_\alpha$. For each $\alpha \in A$ (respectively, A'), let ω_α (resp., ω'_α) be the invariant measure on $T_\alpha\backslash G$ (resp., $T_\alpha\backslash G'$) which is the quotient of ω by ν_α (resp., ω' by ν_α). Then for integrable f on $Z\backslash G$ and integrable φ on $Z\backslash G'$,

$$\int_{Z\backslash G} f(x)\omega(x) = \sum_{\alpha \in A} \frac{1}{|W_\alpha|} \int_{Z\backslash T'_\alpha} \delta(t) \int_{T_\alpha\backslash G} f(g^{-1}tg)\omega_\alpha(\bar{g})\nu_\alpha(t)$$

and

$$\int_{Z\backslash G'} \varphi(x')\omega'(x') = \sum_{\alpha \in A'} \frac{1}{|W_\alpha|} \int_{Z\backslash T'_\alpha} \delta(t) \int_{T_\alpha\backslash G'} \varphi(g^{-1}tg)\omega'_\alpha(\bar{g})\nu_\alpha(t)$$

where in both these formulas, $\delta(t) = |\prod_{i \neq j}(1 - \gamma_i/\gamma_j)|_F$, where γ_1, γ_2, γ_3 are the distinct roots of the reduced characteristic polynomial of t.

Let $\mathscr{C} = \bigcup_{\alpha \in A'} Z\backslash T'_\alpha$. The union is to be regarded as a discrete union. Define the measure μ on $\mathscr{C}$ by

$$\int_{\mathscr{C}} f(c)\mu(c) = \sum_{\alpha \in A'} \frac{1}{|W_\alpha|} \frac{1}{\operatorname{meas}_{\nu_\alpha}(Z\backslash T_\alpha)} \int_{Z\backslash T_\alpha} f(t_\alpha)\delta(t_\alpha)\nu_\alpha(t_\alpha).$$

Let η be a unitary character of Z. Define the space $\mathscr{L}^2(\eta)$ to be the space of complex-valued measurable functions f on $\bigcup_{\alpha \in A'} T'_\alpha$ such that

(i) $f(zt) = \eta(z)f(t)$ for all $z \in Z$ and all $t \in \bigcup_{\alpha \in A'} T'_\alpha$.

(ii) the restriction of f to T'_α is invariant under conjugation by elements of W_α.

(iii) $\int_{\mathscr{C}} f(c)\overline{f(c)}\mu(c) < \infty$.

$\mathscr{L}^2(\eta)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_\eta = \int_{\mathscr{C}} f(c)\overline{g(c)}\mu(c) .$$

Let π' be an admissible irreducible representation of G'. Because $Z\backslash G'$ is compact, π' is finite dimensional. The *character* $\Theta_{\pi'}$ of π' is the function on G' defined by the equation $\Theta_{\pi'}(g) = \operatorname{Trace} \pi'(g)$. $C(G')$ acts on the space of π' via the formula

$$\pi'(\varphi) = \int_{G'} \varphi(g')\pi'(g')dg' .$$

The trace of $\pi'(\varphi)$ equals $\int_{G'} \Theta_{\pi'}(g')\varphi(g')dg'$.

A representation π' of G' is said to be an η-representation for a quasicharacter η of Z in case $\pi'(z) = \eta(z)1$ for all $z \in Z$. For an admissible irreducible η-representation π' of G', $\Theta_{\pi'}(zg) = \eta(z)\Theta_{\pi'}(g)$ for all $z \in Z$ and all $g \in G'$. Denote by $\mathscr{E}(G', \eta)$ the set of equivalence classes of admissible irreducible η-representations of G'. By abuse of notation an η-representation π' will sometimes be identified with its equivalence class in $\mathscr{E}(G', \eta)$.

PROPOSITION 3.1. *Let η be a unitary character of Z.*

(i) *$\{\Theta_{\pi'}\}_{\pi' \in \mathscr{E}(G',\eta)}$ is a complete orthonormal set for $\mathscr{L}^2(\eta)$.*

(ii) *Let $\varphi \in C(G')$ and let $\pi' \in \mathscr{E}(G', \eta)$.*

Then

$$\operatorname{Trace} \pi'(\varphi) = \langle \Theta_{\pi'}, \bar{J}_\varphi\rangle_\eta$$

where

$$J_\varphi(c) = \int_{Z\backslash G'} P\varphi(g^{-1}cg)\omega'(g)$$

and

$$P\varphi(g) = \int_Z \varphi(gz)\eta(z)dz .$$

Proof. The first assertion is a consequence of the Weyl integration formula and the Peter-Weyl theorem. The second follows from the Weyl integration formula.

The analogous theory for G is much more difficult. Most admissible irreducible representations π of G are infinite-dimensional;

thus a character cannot be defined in the same manner as for G'. It is true, though, that for all $f \in C(G)$, $\pi(f)$ has finite rank. It has been shown, [9], p. 189, that there exists a locally constant function Θ_π on $\bigcup_{\alpha \in A} T'^G_\alpha$ which is locally integrable on G and such that

$$\operatorname{Trace} \pi(f) = \int_G \Theta_\pi(g) f(g) dg$$

for all $f \in C(G)$. Θ_π is called the character of π.

PROPOSITION 3.2. *Let η be a unitary character of Z, let π be an admissible irreducible η-representation of G, and let $\varphi \in C(G')$. Let $f \in C(G)$ be the image of φ under the map of Theorem* 2.12. *Then*

$$\operatorname{Trace} \pi(f) = \langle \Theta_\pi, \bar{J}_\varphi \rangle_\eta$$

where J_φ is as in Proposition 3.1.

Proof. This is an immediate consequence of the Weyl integration formula and the fact that

$$J_\varphi(c) = \int_{Z \backslash G} Qf(g^{-1} c g) \omega(g)$$

where

$$Qf(g) = \int_Z f(gz) \eta(z) dz .$$

Let P be a parabolic subgroup of G. Let $P = MN$ be a Levi decomposition of P, with N the unipotent radical of P. Let σ be an admissible representation of M. Lift σ to P by demanding that $\sigma(mn) = \sigma(m)$. Let $\pi = \operatorname{Ind}_P^G(\sigma)$ denote the representation of G by right translations on the space of locally constant functions f on G with values in the space of σ and such that $f(pg) = \Delta_P^{1/2}(p)\sigma(p)f(g)$ for all $p \in P$ and $g \in G$. Because $P \backslash G$ is compact, π is admissible. The importance of induced representations of this type is given by a theorem of Jacquet, [12]; every admissible irreducible representation of G is a subquotient of $\operatorname{Ind}_P^G(\sigma)$ for some parabolic subgroup $P = MN$ and some admissible irreducible supercuspidal representation σ of M. In fact, for most σ, $\operatorname{Ind}_P^G(\sigma)$ is known to be irreducible, [2].

Let P be the subgroup of upper triangular matrices of G, and let M be the subgroup of P consisting of the diagonal matrices. Then $\operatorname{Ind}_P^G(\Delta_P^{-1/2})$ is not irreducible. It has an important quotient representation which will be denoted $\operatorname{Sp}(1)$; this is an admissible irreducible unitary representation whose character is equal to the Steinberg character; its existence is proved in [3]. For what follows it is necessary only to know that the restriction of $\Theta_{\mathrm{Sp}(1)}$ to $\bigcup_{\alpha \in A'} T'_\alpha$ is identically one, the same as the restriction of $\Theta_{\pi'}$ to $\bigcup_{\alpha \in A'} T'_\alpha$ where π' is the trivial one-dimensional representation of G'. For a quasicharacter χ of F^*, $\mathrm{Sp}(\chi)$ will denote the representation $\mathrm{Sp}(1) \otimes \chi$ of G. $\mathrm{Sp}(\chi)$ is a χ^3 representation of G. The representations $\mathrm{Sp}(\chi)$ will be called the *special representations* of G.

Gelfand and Kazhdan have defined in [5] the concept of a nondegenerate representation of G. It is known, see [1], p. 65, that for a Jordan Holder series of $\mathrm{Ind}_P^G(\sigma)$, with σ supercuspidal, exactly one irreducible subquotient is nondegenerate. A complete list of the isomorphism classes of admissible irreducible nondegenerate representations of G has been obtained in [15] by determining explicitly the lattice of subrepresentations of $\mathrm{Ind}_P^G(\sigma)$ for all parabolic subgroups P and supercuspidal representations σ. The result includes

PROPOSITION 3.3. *Let π be an admissible irreducible nondegenerate representation of G. Then exactly one of the following is true.*

(i) *π is supercuspidal.*

(ii) *π is special.*

(iii) *π is isomorphic to a representation* $\mathrm{Ind}_P^G(\sigma)$, *where* $P = \{(a_{ij}) \subset G \mid a_{31} = a_{32} = 0\}$ *and σ is an admissible irreducible but not necessarily supercuspidal representation of*

$$M = \{(a_{ij}) \subset P \mid a_{13} = a_{23} = 0\} .$$

COROLLARY 3.4. *Let π be an admissible irreducible nondegenerate representation of G which is neither supercuspidal nor special. Then the restriction of Θ_π to $\bigcup_{\alpha \in A'} T'^G_\alpha$ is identically zero.*

Proof. The reason is that no element of $\bigcup_{\alpha \in A'} T'^G_\alpha$ is conjugate to an element of P. Formally, one uses Theorem 1 of [9].

Let η be a quasicharacter of F^*. Denote by $\mathscr{E}_2(G, \eta)$ the set of equivalence classes of admissible irreducible η-representations of G which contain either special or supercuspidal representations.

PROPOSITION 3.5. *Let η be a unitary character of Z. Then $\{\Theta_\pi\}_{\pi \in \mathscr{E}_2(G,\eta)}$ is an orthonormal subset of $\mathscr{L}^2(\eta)$.*

Proof. That the set of Θ_π with supercuspidal $\pi \in \mathscr{E}_2(G, \eta)$ is an orthonormal set is precisely Theorem 17 of [8]. That the set of $\Theta_{\mathrm{Sp}(\chi)}$ with χ a quasicharacter of F^* such that $\chi^3 = \eta$ is an orthonormal set follows from Proposition 3.1 and the fact that $\Theta_{\mathrm{Sp}(\chi)}$ gives the same element of $\mathscr{L}^2(\eta)$ as the character of the representation $\chi\circ\nu$ of G'. It remains to be shown that $\Theta_{\mathrm{Sp}(\chi)}$ and Θ_π are orthogonal in $\mathscr{L}^2(\eta)$, where $\mathrm{Sp}\,(\chi)$, $\pi\in\mathscr{E}_2(G,\eta)$ and π is supercuspidal.

More generally, let π_1 and π_2 be nonisomorphic admissible irreducible η-representations of G, with π_2 surercuspidal. It will be shown that $\langle\Theta_{\pi_1},\Theta_{\pi_2}\rangle_\eta=0$. Let v be a unit vector in the (unitary) space of π_2. Define the function f on G by the equation $f(g)=\langle v,\pi_2(g)v\rangle$. Since π_2 is supercuspidal, f has support which is compact in $Z\backslash G$. Define $\pi_1(f)$ to be the endomorphism $\int_{Z\backslash G} f(g)\pi_1(g)\omega(\bar g)$ of the space of π_1. Then $\pi_1(f)$ is zero because π_1 and π_2 are not isomorphic. Hence $\mathrm{Trace}\,\pi_1(f)=0$. But

$$\begin{aligned}\mathrm{Trace}\,\pi_1(f)&=\int_{Z\backslash G}\Theta_{\pi_1}(g)f(g)\omega(\bar g)\\&=\sum_{\alpha\in A}\frac{1}{|W_\alpha|}\int_{Z\backslash T'_\alpha}\delta(t)\Theta_{\pi_1}(t)\int_{T_\alpha\backslash G}f(g^{-1}tg)\omega_\alpha(g)\nu_\alpha(t)\ .\end{aligned}$$

By Lemma 45 of [8], $\int_{T_\alpha\backslash G}f(g^{-1}tg)\omega_\alpha(g)=0$ for all $\alpha\notin A'$. For $\alpha\in A'$,

$$\int_{T_\alpha\backslash G}f(g^{-1}tg)\omega_\alpha(g)=\frac{1}{\mathrm{meas}_{\nu_\alpha}(Z\backslash T_\alpha)}\int_{Z\backslash G}f(g^{-1}tg)\omega(g)\ .$$

It is shown on p. 94 of [8] that there exists a nonzero constant d such that

$$\int_{Z\backslash G}f(g^{-1}tg)\omega(g)=d\cdot\overline{\Theta_{\pi_2}(t)}$$

for all $\alpha\in A'$ and all $t\in T'_\alpha$. Thus

$\mathrm{Trace}\,\pi_1(f)=\langle\Theta_{\pi_1},\Theta_{\pi_2}\rangle_\eta$, and the proof is concluded.

4. Independence of characters—nonramified representations. Let now G be $\mathrm{GL}\,(n,F)$, where F is a nonarchimedean local field with valuation ring R, and let K be $\mathrm{GL}\,(n,R)$. Let q be the module of F.

For a representation (π,V) of G, let V^K be the subspace of elements of V which are fixed by all elements of K. An admissible irreducible representation (π,V) of G is said to be nonramified if V^K is not the zero subspace of V, in which case it has dimension one. For this and other facts about nonramified representations used here see [19] and [12].

The set of isomorphism classes of nonramified representations of G is in one-to-one correspondence with the set C^{*n}/S_n, where S_n is the symmetric group which permutes the factors of C^{*n}. The representation which corresponds to the n-tuple $z = (z_1, \cdots, z_n)$ may be realized as an irreducible subquotient π_z of the G-representation $\mathrm{Ind}_B^G(\chi_z)$, where B is the group of upper triangular matrices, and χ_z is the quasicharacter of B whose value on the matrix (b_{ij}) is $\prod_i z_i^{\mathrm{ord}\, b_{ii}}$.

LEMMA 4.1. *If π_z is a unitary representation of G, then*

(1) $(\bar{z}_1, \cdots, \bar{z}_n)\widetilde{S_\pi}(z_1^{-1}, \cdots, z_n^{-1})$,

and

(2) $q^{-(n-1)/2} \leqq |z_j| \leqq q^{(n-1)/2}$, $j = 1, 2, \cdots, n$.

Proof. (1) From the construction of π_z it may be seen that $\pi_{\bar{z}}$ is isomorphic to the representation complex conjugate to π_z and that $\pi_{z^{-1}}$ is isomorphic to the representation contragredient to π_z. If π_z is unitary, these two must be isomorphic.

(2) See [6], pp. 81–82.

Let $X^u = \{z \in \boldsymbol{C}^{*n}$ such that the two conditions of 4.1 hold$\}$. Let X be the compact Hausdorff space X^u/S_n.

Let H be the algebra of measures on G which are left and right translation invariant by elements of K and have compact support. Once a Haar measure dg on G is fixed, H can be identified with $C(G//K)$, the space of functions on G which are left and right translation invariant by elements of K and have compact support, via the correspondence

$$\begin{aligned} C(G//K) &\longleftrightarrow H \\ f &\longleftrightarrow f(g)dg\,. \end{aligned}$$

In this section, though, this identification will not be made use of.

H acts on the space of every smooth representation (π, V) of G; for $\mu \in H$, $\pi(\mu)$ is defined by the equation $\pi(\mu)v = \int_G \pi(g)v\mu(g)$ for each $v \in V$. This integral is actually a finite sum. The transformation $\pi(\mu)$ maps V into V^K. Hence if (π, V) is admissible, then $\pi(\mu)$ is of finite rank so that the trace of $\pi(\mu)$ is defined. Trace $\pi(\mu)$ will be denoted by $\check{\mu}(\pi)$.

H is a finitely generated commutative $\boldsymbol{C}$-algebra whose structure is completely known; H is isomorphic to $\boldsymbol{C}[Z_1, Z_1^{-1}, \cdots, Z_n, Z_n^{-1}]^{S_n}$, where S_n is the symmetric group on the indeterminants $Z_1, \cdots, Z_n$. The isomorphism is such that $\check{\mu}(\pi_z)$ is equal to the value that the polynomial in $\boldsymbol{C}[Z_1, Z_1^{-1}, \cdots, Z_n, Z_n^{-1}]^{S_n}$ corresponding to μ takes on the point $(Z_1, \cdots, Z_n) = (z_1, \cdots, z_n)$.

Let $\{F_\lambda | \lambda \in \Lambda\}$ be a family of nonarchimedean local fields. Let G_λ be $\mathrm{GL}(n, F_\lambda)$. Similarly R_λ, q_λ, K_λ, H_λ, X_λ will be used to denote the previously defined unsubscripted objects when the local field F is replaced by F_λ. Let e_λ denote the identity element of H_λ.

Let G_A be the product of the groups G_λ restricted with respect to the open compact subgroups K_λ.

The notion of a restricted tensor product has been defined in [13]. Let H_A be $\bigotimes_{e_\lambda} H_\lambda$, the tensor product of the H_λ restricted with respect to the e_λ.

For each $\lambda \in A$, let (π_λ, V_λ) be an admissible irreducible nonramified representation of G_λ, and let v_λ be a nonzero element of $V_\lambda^{K_\lambda}$. The irreducible G_A-representation and H-module $\pi = \bigotimes_{v_\lambda} \pi_\lambda$ may be defined. The isomorphism class of π is independent of the v_λ. The π_λ are determined up to isomorphism by π and are each unitary if π is a unitary G_A-representation. For each $\mu \in H_A$, $\pi(\mu)$ has rank zero or one. Trace $\pi(\mu)$ will be denoted by $\check{\mu}(\pi)$.

THEOREM 4.2. *Let $\{\pi^\alpha = \bigotimes \pi_\lambda^\alpha \mid \alpha \in A\}$ be a family of pairwise nonisomorphic irreducible unitary representations of G_A as above. (The π_λ^α are all assumed nonramified.) Let $\{c_\alpha \mid \alpha \in A\}$ be a family complex numbers. Suppose that $\sum_{\alpha \in A} c_\alpha \check{\mu}(\pi^\alpha)$ is absolutely convergent to zero for all $\mu \in H_A$. Then $c_\alpha = 0$ for all $\alpha \in A$.*

Proof. Let X_A be the direct product of the spaces X_λ. Each representation π^α will be identified with the point $z^\alpha = (z_\lambda^\alpha)_{\lambda \in A}$ in X_A such that π_λ^α is isomorphic to $\pi_{z^\alpha \lambda}$. Each $\mu \in H_A$ will be identified with the continuous function $\check{\mu}$ on X_A.

H_A separates points of the compact Hausdorff space X_A and contains the constant functions. By Lemma 4.1, H_A contains the complex conjugate of each of its functions. Thus the algebra of functions H_A satisfies the hypotheses of the Stone-Weierstrass theorem; H_A is supnorm dense in $C(X_A)$, the space of continuous complex-valued functions on X_A Theorem 4.2 is now a consequence of

LEMMA 4.3. *Let X be a compact Hausdorff space, and let B be a dense subset of $C(X)$. Let $\{z^\alpha \mid \alpha \in A\}$ be a family of distinct elements of X, and let $\{c_\alpha \mid \alpha \in A\}$ be a family of complex numbers. Suppose that $\sum_{\alpha \in A} c_\alpha f(z^\alpha)$ is absolutely convergent to zero for all $f \in B$. Then $c_\alpha = 0$ for all $\alpha \in A$.*

Proof. Suppose $c_{\alpha_0} \neq 0$. The hypotheses imply that $\sum_{\alpha \in A} |c_\alpha| < \infty$. Choose a finite subset N of A such that $\sum_{\alpha \in A-N} |c_\alpha| < 1/6\,|c_{\alpha_0}|$. Choose $f \in B$ such that

$$|f(z)| \leqq 2 \qquad \text{all} \quad z \in X \tag{1}$$

$$|f(z^{\alpha_0})| \geqq 1 \tag{2}$$

$$|f(z^\alpha)| \leqq \frac{|c_{\alpha_0}|}{3 \cdot |N|(1 + \max\{|c_\alpha| \mid \alpha \in N\})}, \qquad \text{all} \quad \alpha \in N - \{\alpha_0\}. \tag{3}$$

Then

$$\left|\sum_{\alpha\neq\alpha_0} c_\alpha f(z^\alpha)\right| \leqq \sum_{\alpha\in N-\{\alpha_0\}} \frac{|c_{\alpha_0}|}{3|N|} + 2\frac{|c_{\alpha_0}|}{6} \leqq \frac{2}{3}|c_{\alpha_0}| .$$

This contradicts the hypothesis that $\sum_{\alpha\in A} c_\alpha f(z^\alpha) = 0$.

5. Independence of characters—general case. Let G be a locally compact unimodular topological group, and let Z be a closed subgroup of the center of G. Let ω be a Haar measure on $Z\backslash G$, and let ξ be a unitary character of Z.

In this section all representations of G will be understood to be continuous representations of G by bounded operators on a Hilbert space. A representation π of G will be called an ξ-representation of G if $\pi(z) = \xi(z)\cdot 1$ for all $z \in Z$.

Let $\mathscr{L}^1(G, \xi)$ be the Banach $*$-algebra of measurable functions f on G such that $f(zg) = \xi^{-1}(z)f(g)$ for all $z \in Z$ and $g \in G$ and for which $\|f\|_1 = \int_{Z\backslash G} |f(g)|\omega(\bar{g})$ is finite. Multiplication is given by convolution:

$$f_1f_2(g) = \int_{Z\backslash G} f_1(gh^{-1})f_2(h)\omega(\bar{h}) .$$

The involution $*$ is defined by the formula $f^*(g) = \overline{f(g^{-1})}$. For a unitary ξ-representation π of G and function $f \in \mathscr{L}^1(G, \xi)$, define $\pi(f) = \int_{Z\backslash G} f(g)\pi(g)\omega(\bar{g})$.

LEMMA 5.1. *Let B be a dense $*$-closed subalgebra of $\mathscr{L}^1(G, \xi)$. Let π and the elements of the set $\{\pi^\alpha | \alpha \in A\}$ be irreducible unitary ξ-representations of G such that π is not isomorphic to π^α for any $\alpha \in A$. Suppose that $\pi(f)$ and $\pi^\alpha(f)$ for all $\alpha \in A$ are Hilbert-Schmidt operators, for all $f \in B$, and write $\|\ \|$ for the Hilbert-Schmidt norm.*

Let $\{c_\alpha | \alpha \in A\}$ be a family of nonnegative real numbers such that $\sum_{\alpha\in A} c_\alpha \|\pi^\alpha(f)\|^2$ is finite for all $f \in B$.

Then for every $\varepsilon > 0$ there exists $f \in B$ such that

(1) $\pi(f) \neq 0$

and

(2) $\sum_{\alpha\in A} c_\alpha \|\pi^\alpha(f)\|^2 \leqq \varepsilon \|\pi(f)\|^2$.

Proof. This lemma follows trivially from the simple remark on page 496 of [13].

THEOREM 5.2. *Let B be as in Lemma 5.1. Let $\{\pi^\alpha | \alpha \in A\}$ be a family of pairwise nonisomorphic irreducible unitary ξ-representa-*

tions of G. Let $\{c_\alpha \mid \alpha \in A\}$ be a family of complex numbers. Suppose that $\pi^\alpha(f)$ is Hilbert-Schmidt for all $\alpha \in A$ and $f \in B$, and that $\sum_{\alpha \in A} c_\alpha \operatorname{Trace} \pi^\alpha(ff^*)$ is absolutely convergent to zero for all $f \in B$.

Then $c_\alpha = 0$ for all $\alpha \in A$.

Proof. Suppose $c_{\alpha_0} \neq 0$. For all $\alpha \in A$ and $f \in B$, $\operatorname{Tr} \pi^\alpha(ff^*) = \|\pi^\alpha(f)\|^2$. By Lemma 5.1 there exists $f \in B$ such that

$$\sum_{\alpha \in A - \{\alpha_0\}} |c_\alpha| \operatorname{Tr} \pi^\alpha(ff^*) \leqq \frac{1}{2} |c_{\alpha_0}| \operatorname{Tr} \pi^{\alpha_0}(ff^*) \neq 0 .$$

This contradicts the hypothesis that

$$\sum_{\alpha \in A - \{\alpha_0\}} c_\alpha \operatorname{Tr} \pi^\alpha(ff^*) = -c_{\alpha_0} \operatorname{Tr} \pi^{\alpha_0}(ff^*) .$$

6. The Trace formula and the main theorems. Let F be a number field, A the adele ring of F, and A^* the idele group of F. Let G be $\mathrm{GL}(3)$, and let G' be the group of invertible elements in a central division algebra D of rank 3^2 over F. Let ν be the reduced norm from D to F. A^* will be identified with $Z(A)$, the common center of $G(A)$ and $G'(A)$. For finite places v of F, let K_v be the subgroup $G(R_v)$ of $G_v = G(F_v)$, where R_v is the valuation ring of F_v.

Let S be the finite set of places v of F for which D_v is a division algebra. Since the degree of D is odd, S contains no archimedean places. If v does not belong to S, then G_v and G'_v are isomorophic via an isomorphism which will be fixed once for all. For such nonarchimedean v, K_v maps to a subgroup of G'_v to be denoted K'_v. Let G_∞ equal $\prod_{\mathrm{arch}\, v} G_v$, and let G'_∞ equal $\prod_{\mathrm{arch}\, v} G'_v$.

Let Z_∞^+ be the group of ideles a in A^* such that $a_v = 1$ for all nonarchimedean places v of F and for which there exists a positive real number r such that $a_v = r$ for all archimedean places v. Z_∞^+ can be viewed as a subgroup of G_∞ and G'_∞.

Let $dg = \prod_v dg_v$ be a Haar measure on $G(A)$ and let $dg' = \prod_v dg'_v$ be a Haar measure on $G'(A)$ such that for $v \notin S$, dg_v and dg'_v are equal; for almost all finite $v \notin S$, $\int_{K_v} dg_v = 1$; and for all $v \in S$, dg_v and dg'_v are associated as in §2. Let dz be a Haar measure on Z_∞^+, and let $d\bar{g}$ (resp., $d\bar{g}'$) be the measure $dz\backslash dg$ on $Z_\infty^+\backslash G(A)$ (resp., $dz\backslash dg'$ on $Z_\infty^+\backslash G'(A)$).

Let ξ be a unitary character of Z_∞^+. Let $\mathscr{L}^2(G, \xi)$ be the space of measurable functions θ on $G(F)\backslash G(A)$ satisfying

(i) $\theta(zg) = \xi(z)\theta(g)$ for all $z \in Z_\infty^+$ and $g \in G(A)$

and

(ii) $\|\theta\|_2 = \int_{Z_\infty^+ G(F)\backslash G(A)} |\theta(g)|^2 d\bar{g} < \infty .$

$G(A)$ is represented by unitary operators on $\mathscr{L}^2(G, \xi)$ via right translations. Denote this represention by λ. It is an ξ-representation. Make the analogous definitions for the space $\mathscr{L}^2(G', \xi)$ and the $G'(A)$ representation λ'.

Let $(\lambda_d, \mathscr{L}^2_d(G, \xi))$ denote the discrete spectrum of the $G(A)$ representation $(\lambda, \mathscr{L}^2(G, \xi))$. The space of cusp forms in $\mathscr{L}^2(G, \xi)$, denoted $\mathscr{L}^2_0(G, \xi)$, is a $G(A)$ subspace of $\mathscr{L}^2_d(G, \xi)$. The orthogonal complement of $\mathscr{L}^2_0(G, \xi)$ in $\mathscr{L}^2_d(G, \xi)$, to be denoted $\mathscr{L}^2_s(G, \xi)$, is the closed linear span of the characters of $G(A)$ of the form $\chi\circ\det$, where χ is a unitary Hecke character of A^* such that χ^3 restricts to ξ on Z^+_∞.[1] A representation of $G(A)$ is said to be *cuspidal automorphic* if it is a subrepresentation of $\mathscr{L}^2_0(G, \xi)$ for some unitary character ξ of Z^+_∞.

Let $K(G, \xi)$ be the linear span of the functions f on $G(A)$ of the form $f(g) = f_\infty(g_\infty) \prod_{\text{finite } v} f_v(g_v)$, where the functions f_v satisfy the following four conditions.

(i) For finite v, $f_v \in C(G_v)$.

(ii) For almost all finite $v \notin S$, f_v is the characteristic function of K_v.

(iii) f_∞ is an infinitely differentiable function on G_∞ and has support which is compact in $Z^+_\infty\backslash G_\infty$.

(iv) $f_\infty(zg) = \xi^{-1}(z)f_\infty(g)$ for all $z \in Z^+_\infty$ and $g \in G_\infty$.

For each $f \in K(G, \xi)$ and each unitary ξ-representation π of $G(A)$ define the operator $\pi(f)$ to be $\int_{Z^+_\infty\backslash G(A)} f(g)\pi(g)d\bar{g}$. Then $\lambda(f)$ acts on $\mathscr{L}^2(G, \xi)$ via the formula

$$\lambda(f)\theta(h) = \int_{Z^+_\infty\backslash G(A)} f(g)\theta(hg)d\bar{g} \ .$$

Denote by $\lambda_d(f)$ and $\lambda_0(f)$ the restrictions of $\lambda(f)$ to $\mathscr{L}^2_d(G, \xi)$ and $\mathscr{L}^2_0(G, \xi)$ respectively.

Let $\mathscr{L}^2_0(G', \xi)$ be the orthogonal complement in $\mathscr{L}^2(G', \xi)$ of the space $\mathscr{L}^2_s(G', \xi)$ which is the closed linear span of the functions $\chi \circ \nu$, where χ is a unitary Hecke character of A^* such that χ^3 restricts to ξ on Z^+_∞. A representation of $G'(A)$ is said to be *cuspidal automorphic* if it is a subrepresentation of $\mathscr{L}^2_0(G', \xi)$ for some unitary character ξ of Z^+_∞.

Let $K(G', \xi)$ be the linear span of the functions φ on $G'(A)$ of the form $\varphi(g) = \varphi_\infty(g_\infty) \prod_{\text{finite } v} \varphi_v(g_v)$, where the functions φ_v satisfy the same conditions relative to G'_v that the f_v appearing in the definition of $K(G, \xi)$ satisfy relative to G_v. Each $\varphi \in K(G', \xi)$ defines an operator $\lambda'(\varphi)$ on $\mathscr{L}^2(G', \xi)$ by the formula

$$\lambda'(\varphi)\theta(h) = \int_{Z^+_\infty\backslash G'(A)} \varphi(g')\theta(hg')d\bar{g}' \ .$$

[1] Proof of this assertion in preparation.

Denote by $\lambda_0'(\varphi)$ the restriction of $\lambda'(\varphi)$ to $\mathscr{L}_0^2(G', \xi)$. Because $Z_\infty^+ G'(F)\backslash G'(A)$ is compact, it is not difficult to prove that $\lambda'(\varphi)$ is a trace operator for each $\varphi \in K(G', \xi)$ and that $(\lambda', \mathscr{L}^2(G', \xi))$ decomposes discretely with finite multiplicities.

Define a linear map from $K(G', \xi)$ to $K(G, \xi)$ by mapping $\varphi = \varphi_\infty \cdot \prod_{\text{finite } v} \varphi_v \in K(G', \xi)$ to the function $f = f_\infty \cdot \prod_{\text{finite } v} f_v \in K(G, \xi)$, where $f_\infty = \varphi_\infty$; for finite $v \notin S$, $f_v = \varphi_v$; and for $v \in S$, f_v is the image of φ_v under the map of Theorem 2.12. This definition is justified because $K(G', \xi)$ is a restricted tensor product in which the functions φ of the form $\varphi = \varphi_\infty \cdot \prod_{\text{finite } v} \varphi_v$ are the decomposable vectors.

THEOREM 6.1. *Let $\varphi \in K(G', \xi)$. Let $f \in K(G, \xi)$ be the image of φ under the map above. Then $\lambda_d(f)$ is a trace operator, and* $\operatorname{Tr} \lambda_d(f) = \operatorname{Tr} \lambda'(\varphi)$.

Proof. This theorem has been proved by J. Arthur in work yet to appear.

The rest of this thesis is devoted to the deduction of Theorems 1 and 2 from the equality in Theorem 6.1.

THEOREM 6.2. *Let φ and f be as in Theorem* 6.1. *Then*

$$\operatorname{Tr} \lambda_0(f) = \operatorname{Tr} \lambda_0'(\varphi) .$$

Proof. For a unitary Hecke character χ, let π_χ (resp., π'_χ) be a one-dimensional representation of $G(A)$ (resp., $G'(A)$) whose character is $\chi \circ \det$ (resp., $\chi \circ \nu$). The representation λ_s (resp., λ'_s) is isomorphic to the sum of the π_χ (resp., π'_χ) for which χ^3 restricts to ξ on Z_∞^+. It is enough to prove that $\operatorname{Tr} \pi_\chi(f) = \operatorname{Tr} \pi'_\chi(\varphi)$ for all such χ, and that for φ of the form $\varphi = \prod_v \varphi_v$, with $f = \prod_v f_v$. But $\operatorname{Tr} \pi_\chi(f) = \operatorname{prod}_v \operatorname{Tr} \pi_{\chi_v}(f_v)$, and $\operatorname{Tr} \pi'_\chi(\varphi) = \operatorname{prod}_v \operatorname{Tr} \pi'_{\chi_v}(\varphi_v)$. The factors in these products for $v \notin S$ are trivially pairwise equal, and those for $v \in S$ are equal by Propositions 3.1 and 3.2.

It is shown in [4] that every irreducible unitary representation π of $G(A)$ on a Hilbert space is isomorphic to a completed restricted tensor product $\pi_\infty \bigotimes_{\text{finite } v} \pi_v$, where π_∞ (resp., π_v) is an irreducible unitary representation of G_∞ (resp., G_v) whose isomorphism class is determined by π. For almost all finite v, π_v is nonramified. For an irreducible unitary ξ-representation π of $G(A)$ and a function $f = \prod_v f_v \in K(G, \xi)$, the formula $\operatorname{Tr} \pi(f) = \prod_v \operatorname{Tr} \pi_v(f_v)$ is valid, where $\pi_\infty(f_\infty) = \int_{Z_\infty^+\backslash G_\infty} f_\infty(g_\infty)\pi_\infty(g_\infty)d\bar{g}_\infty$ and $\pi_v(f_v) = \int_{G_v} f_v(g_v)\pi_v(g_v)dg_v$ for all finite v. For almost all v, $\pi_v(f_v)$ is the projection onto the one-dimensional subspace of K_v-fixed vectors in the space of π_v, so that $\operatorname{Tr}\pi_v(f_v) = 1$; thus in the product expression for $\operatorname{Tr}\pi(f)$ almost all factors equal one. Similar remarks apply to the group $G'(A)$.

THEOREM 6.3. *For each finite* $v \notin S$ *let* π_v^0 *be an irreducible unitary representation of* G_v. *Let* π_∞^0 *be an irreducible unitary* ξ-*representation of* G_∞. *For each* $v \in S$, *let* $\varphi_v \in C(G'_v)$, *and let* $f_v \in C(G_v)$ *be the image of* φ_v *under the map of Theorem* 2.12. *Then*

$$\sum_{\pi} \operatorname*{prod}_{v \in S} \operatorname{Tr}\pi_v(f_v) = \sum_{\pi'} \operatorname*{prod}_{v \in S} \operatorname{Tr}\pi'(\varphi_v)$$

where the sum is taken over those representations π (resp., π') *in a decomposition of* $\mathscr{L}_0^2(G, \xi)$ (resp., $\mathscr{L}_0^2(G', \xi)$) *into a Hilbert direct sum of irreducible representations for which* π_v (resp., π'_v) *is isomorphic to* π_v^0 *for* $v = \infty$ *and for all finite* $v \notin S$.

REMARK. By the strong form of the "multiplicity one" theorem for GL (3), for which see [14] and [21], the sum on the left contains at most one nonzero term. At this stage the sum on the right is known only to converge absolutely, though it will be shown later that it, too, contains at most one nonzero term.

Proof. It may be assumed that π_v^0 is nonramified for almost all finite v, for otherwise the sums in the theorem are empty. Let V be the finite whose elements are the symbol ∞ and the finite places v of F for which either $v \in S$ or $v \notin S$ and π_v^0 is not nonramified. For each finite $v \in V$, let $\varphi_v^0 \in C(G'_v)$. For $v = \infty$, let φ_v^0 be a function on G_∞ which is infinitely differentiable, has support compact in $Z_\infty^+\backslash G'_\infty$, and satisfies the condition $\varphi_\infty^0(zg) = \xi^{-1}(z)\varphi_\infty^0(g)$ for all $z \in Z_\infty^+$ and $g \in G'_\infty$. For $v \in S$, let $f_v^0 \in C(G_v)$ be the image of φ_v^0 under the map of Theorem 2.12. For $v \in V - S$, let $f_v^0 = \varphi_v^0$ as a function on $G_v = G'_v$.

Let $K(G', \xi, \varphi^0)$ be the subspace of $K(G', \xi)$ spanned by the functions of the form $\varphi = \prod_{v \in V} \varphi_v^0 \prod_{v \notin V} \varphi_v$, where for all $v \notin V$, $\varphi_v \in H'_v = C(G'_v//K'_v)$, and for almost all $v \notin V$, φ_v is the characteristic function of K'_v. Let $K(G, \xi, \varphi^0)$ be the image of $K(G', \xi, \varphi^0)$ in $K(G, \xi)$.

If π' is an irreducible unitary ξ-representation of $G'(A)$ for which there exists a finite place $v \notin V$ such that π'_v is not nonramified, then $\pi'(\varphi)$ is the zero map for all $\varphi \in K(G', \xi, \varphi^0)$. A similar remark applies to $G(A)$. Thus

$$(*) \qquad \sum_{\pi} \operatorname{Tr}\pi(f) = \sum_{\pi'} \operatorname{Tr}\pi'(\varphi)$$

for all $\varphi \in K(G', \xi, \varphi^0)$ where the sum is taken over the π (resp., π') in a decomposition of $\mathscr{L}_0^2(G, \xi)$ (resp., $\mathscr{L}_0^2(G', \xi)$) into a direct sum of irreducible representations for which π_v (resp., π'_v) is nonramified for all finite $v \notin V$.

Theorem 4.2 can now be applied. The set Λ of that theorem will be the set of finite places v of F such that $v \notin V$. The indexing set A of that theorem will be the set of isomorphism classes of representations $\pi^{\Lambda} = \bigotimes_{v \in \Lambda} \pi_v$, where for each $v \in \Lambda$, π_v is an irreducible unitary nonramified representation of $G_v = G'_v$. The constant $C_{\pi^{\Lambda}}$ will equal

$$\sum_{\pi} \operatorname*{prod}_{v \in V} \operatorname{Tr} \pi(f_v^0) - \sum_{\pi'} \operatorname*{prod}_{v \in V} \operatorname{Tr} \pi'(\varphi_v^0)$$

where the sums are taken over those π and π' as before for which $\bigotimes_{v \in \Lambda} \pi_v$ and $\bigotimes_{v \in \Lambda} \pi'_v$ are isomorphic to π^{Λ}. The fact that the representations π_v occurring in the theorem at hand are continuous representations on a Hilbert space but that the representations occurring in Theorem 4.2 are admissible causes no problem. The spaces here are the completions of the spaces of Theorem 4.2, and an element of $C(G_v)$ has the same trace on either. So one deduces that

$$(^{**}) \qquad \sum_{\pi} \operatorname*{prod}_{v \in V} \operatorname{Tr} \pi(f_v^0) = \sum_{\pi'} \operatorname*{prod}_{v \in V} \operatorname{Tr} \pi'(\varphi_v^0)$$

where the sums are now over the π and π' such that π_v and π'_v are isomorphic to π_v^0 for all finite $v \notin V$.

The proof is concluded by applying Theorem 5.2 to the groups G_v for $v \in V - S$ and the equation (**) in a manner entirely analogous to the just completed application of Theorem 4.2 to the group G_{Λ} and the equation (*).

Part of Theorem 2 can now be proved and will be stated as

COROLLARY 6.4. *Let $\pi' = \bigotimes_v \pi'_v$ be an irreducible subrepresentation of λ'_0. Then there exists a unique irreducible subrepresentation $\pi = \bigotimes_v \pi_v$ of λ_0 such that $\pi_v \simeq \pi'_v$ for almost all $v \notin S$. Moreover, $\pi_v \simeq \pi'_v$ for all $v \notin S$, and π_v is special or supercuspidal for all $v \in S$.*

Proof. The uniqueness comes from the "strong multiplicity one" theorem for GL (3).

If π did not exist with $\pi_v \simeq \pi'_v$ for all $v \notin S$, then the left hand side of the equality of Theorem 6.3 would be zero. That would contradict the conclusion reached by applying Theorem 5.2 to the group $\prod_{v \in S} G'_v$.

It has been proved in [21] that for every irreducible subrepresentation $\pi = \bigotimes_v \pi_v$ of λ_0, π_v is nondegenerate for all v. Thus, if there existed $v \in S$ for which π_v were not special or supercuspidal, then by Proposition 3.2 and Corollary 3.4, the left hand side of the equality of Theorem 6.3 would still be zero. That would lead to the same contradiction as before.

Most of Theorem 2 is an immediate consequence of the combination of Corollary 6.4 and Lemma 6.5 below. The only assertion of Theorem 2 left unproved is that if $\pi \sim \pi'$, where π and π' are irreducible subrepresentations of λ_0 and λ_0' respectively, then $\pi_v \sim \pi_v'$ for all $v \in S$. That assertion is equivalent to the fact that the constants a_{π_v} which appear in the statement of Lemma 6.5 are all equal to one. That in turn is a consequence of Theorem 1, whose proof has yet to be discussed.

LEMMA 6.5. *Let $\pi = \bigotimes_v \pi_v$ be an irreducible subrepresentation of λ_0 such that π_v is special or supercuspidal for all $v \in S$. Then there exists a unique irreducible subrepresentation $\pi' = \bigotimes_v \pi_v'$ of λ_0' such that $\pi_v' \simeq \pi_v$ for all $v \notin S$. Moreover, for each $v \in S$ there is a constant, $a_{\pi_v} = \pm 1$ such that $\Theta_{\pi_v} = a_{\pi_v}\Theta_{\pi_v'}$ on $\bigcup_{\alpha \in A_v'} T_\alpha'$, and* $\operatorname{prod}_{v \in S} a_{\pi_v} = 1$.

Proof. Let η be the central character of π. For each $v \in S$, let π_v^0 be an admissible irreducible η_v-representation of G_v' such that $\langle \Theta_{\pi_v}, \Theta_{\pi_v^0} \rangle_{\eta_v} = a_{\pi_v} \neq 0$. The existence of π_v^0 is assured by Propositions 3.1 and 3.5. Because Θ_{π_v} and $\Theta_{\pi_v^0}$ are both unit vectors in $\mathscr{L}^2(\eta_v)$, $|a_{\pi_v}| \leqq 1$. For each $v \in S$, let $\varphi_v^0 \in C(G_v')$ be such that $\operatorname{Tr} \pi_v^0(\varphi_v^0) = 1$ and $\operatorname{Tr} \pi_v'(\varphi_v^0) = 0$ for all admissible irreducible η_v-representations π_v' of G_v' for which $\pi_v' \neq \pi_v^0$.

Theorem 6.3 yields the equation

$$\operatorname*{prod}_{v \in S} a_{\pi_v} = \sum_{\pi'} 1$$

where the sum is taken over those representations π' in a decomposition of $\mathscr{L}_0^2(G', \xi)$ into a direct sum of irreducible representations for which $\pi_v' \simeq \pi_v$ for all $v \notin S$ and $\pi_v' \simeq \pi_v^0$ for all $v \in S$. It is immediate that there is exactly one term on the right hand side, that $\operatorname{prod}_{v \in S} a_{\pi_v} = 1$, that $|a_{\pi_v}| = 1$ for all $v \in S$, and that $\Theta_{\pi_v} = a_{\pi_v}\Theta_{\pi_v^0}$ on $\bigcup_{\alpha \in A_v'} T_\alpha'$ for all $v \in S$.

All that remains to be proved is the assertion that a_{π_v} is real. Clearly $\Theta_{\tilde{\pi}_v} = a_{\pi_v}\Theta_{\tilde{\pi}_v^0}$ and $\Theta_{\bar{\pi}} = \bar{a}_{\pi_v}\Theta_{\bar{\pi}_v^0}$, where $\sim$ means contragredient and—means complex conjugate. But π_v and π_v^0 are unitary, so that $\bar{\pi}_v \simeq \tilde{\pi}_v$ and $\bar{\pi}_v^0 \simeq \tilde{\pi}_v^0$.

In the deduction of Theorem 1 from the above global results, the following existence lemma will be made use of.

LEMMA 6.6. *Let η be a unitary Hecke character of A^* such that*

η restricts to ξ on Z_∞^+. Let V be a finite set of finite places of F. For each $v \in V$, let π_v^0 be an admissible irreducible η_v-representation of G'_v which is supercuspidal if $v \notin S$. Then there exists an irreducible subrepresentation π' of λ' with central character η and such that $\pi'_v \simeq \pi_v^0$ for all $v \in V$.

Proof. For a unitary Hecke character χ of A^*, let λ'_χ be the representation of $G'(A)$ by right translations on the space $\mathscr{L}^2(G', \chi)$ of complex-valued measurable functions θ on $G(F)\backslash G'(A)$ satisfying (i) $\theta(zg) = \chi(z)\theta(g)$ for all $z \in Z(A)$ and $g \in G'(A)$ and (ii) $\|\theta\|_2 = \int_{Z(A)G'(F)\backslash G'(A)} |\theta(g)|^2 d\bar{g} < \infty$.

Because $Z_\infty^+ Z(F)\backslash Z(A)$ is compact, $\lambda' \simeq \bigoplus_\chi \lambda'_\chi$, where the sum is over all unitary Hecke characters χ such that χ restricts to ξ on Z_∞^+. The representation π' demanded in the theorem will be found as a subrepresentation of λ'_η.

For each $v \in V$, let φ_v be a matrix entry of π_v^0; that is, φ_v is a function on G'_v defined by the equation $\varphi_v(g) = \langle gw_v, \tilde{w}_v \rangle$ where w_v is a vector in the space of π_v^0 and $\tilde{w}_v$ is a vector in the space of the represention contragredient to π_v^0. Assume that $\varphi_v(1) \neq 0$ for each $v \in V$. Define the function $\varphi = \bigotimes_{v \in V} \varphi_v$ on $G'_V = \prod_{v \in V} G'_v$. The support of φ is compact in $Z_V\backslash G'_V$.

Let φ' be a continuous complex-valued function on the restricted product $G'_{V^c} = \prod'_{v \notin V} G'_v$ satisfying the three properties

(i) $\varphi'(zg) = \eta(z)\varphi'(g)$ for all $z \in Z_{V^c} = Z(A) \cap G'_{V^c}$ and all $g \in G'_{V^c}$.

(ii) The support of φ' is compact in $Z_{V^c}\backslash G'_{V^c}$.

(iii) $\varphi'(1) = 1$.

Define the function Φ on $G'(A)$ by the formula $\Phi(g) = \sum_{\gamma \in Z(F)\backslash G'(F)} \varphi \otimes \varphi'(\gamma g)$. The sum converges; in fact, because $Z(F)\backslash G'(F)$ is discrete in $Z(A)\backslash G'(A)$, only finitely many γ enter nontrivially into the sum for any g in any fixed set which is compact mod $Z(A)$. Hence $\Phi \in \mathscr{L}^2(G', \eta)$.

Notice next that for a fixed neighborhood X of 1 in G'_V which is compact mod Z_V, φ' can be taken, by shrinking its support if necessary, so that only the term $\gamma = 1$ gives a nonzero contribution to the sum defining $\Phi(g)$ for any $g \in X$. Let $\tilde{\varphi}$ be the function on G'_V defined by $\tilde{\varphi}(g) = \varphi(g^{-1})$. By the preceding remark applied to $X = \operatorname{supp} \tilde{\varphi}$, choose φ' so that only the term $\gamma = 1$ enters nontrivially into the sum for $\tilde{\varphi}(g)\Phi(g) = \sum_{\gamma \in Z(F)\backslash G'(F)} \tilde{\varphi}(g)\varphi \otimes \varphi'(\gamma g)$ for any $g \in G'_V$.

The function $\tilde{\varphi}$ acts on the space of λ'_η via the formula

$$\lambda'_\eta(\tilde{\varphi}) = \int_{Z_V\backslash G'_V} \tilde{\varphi}(g)\lambda'_\eta(g)d\bar{g} \,.$$

Moreover, $\lambda'_\eta(\tilde{\varphi})\Phi \neq 0$, In fact, $\lambda'_\eta(\tilde{\varphi})\Phi(1) \neq 0$, as is seen from the calculation

$$\lambda'_\eta(\tilde{\varphi})\Phi(1) = \int_{Z_V \backslash G'_V} \tilde{\varphi}(g)\Phi(g)d\bar{g} = \int_{Z_V \backslash G'_V} \tilde{\varphi}(g)\varphi(g)d\bar{g} \neq 0 .$$

This concludes the proof. For on the space of an irreducible η-representation $\pi' = \bigotimes_v \pi'_v$ of $G'(A)$ for which there exists $v \in V$ such that $\pi'_v \neq \pi^0_v$, the operator

$$\bigotimes_{v \in V} 1 \otimes \int_{Z_V \backslash G'_V} \tilde{\varphi}(g)\pi'_V(g)d\bar{g} ,$$

where $\pi'_V = \bigotimes_{v \in V} \pi'_v$, is the zero transformation.

Theorem 1 can now be proved.

Let E be a nonarchimedean local field of characteristic zero, and let H be a central division algebra of rank 3^2 over E. Let σ' be an admissible representation of H^*. A special or supercuspidal representation σ of GL (3, E) such that $\sigma \sim \sigma'$ is sought. Of course, σ will have the same central character as σ'. After noting that $\sim$ is compatible with twists by quasicharacters of E^* and that σ' is the twist by such a quasicharacter of a unitary representation, it may and will be assumed that σ' unitary. That σ is unique up to isomorphism follows from Proposition 3.5.

If σ' is one-dimensional and hence of the form $\chi \circ \nu$ for some character χ of E^*, then σ can be taken isomorphic to Sp (χ).

Suppose the dimension of σ' is greater than one. Let F be a number field with a place v_0 such that $F_{v_0} \simeq E$, let D be a central division algebra over F such that $D(F_{v_0}) \simeq H$, let $G = \mathrm{GL}(3)$, and let $G' = D^*$. Identify $G'(F_{v_0})$ with H^*.

Let S be the set of places of F at which D does not split. Let η be a unitary Hecke character such that η_{v_0} extends the central character of σ' and η_v is a cube for all $v \in S - \{v_0\}$. Let ξ be the restriction of η to Z^+_∞. Apply Lemma 6.6 to the case in which $V = S$, $\pi^0_{v_0} \simeq \sigma'$, and π^0_v is a one-dimensional η_v-representation for all $v \in S - \{v_0\}$. The conclusion is that there exists an irreducible subrepresentation $\pi' = \bigotimes_v \pi'_v$ of $\mathscr{L}^2(G', \xi)$ such that $\pi'_{v_0} \simeq \sigma'$ and $\pi'_v \simeq \pi^0_v$ for all $v \in S - \{v_0\}$. Since the dimension of π'_{v_0} is greater than one, π' is actually a subrepresentation of $\mathscr{L}^2_0(G', \xi)$.

Let π be the subrepresentation of $\mathscr{L}^2_0(G, \xi)$ such that $\pi \sim \pi'$. With the notation of Lemma 6.5, $\Theta_{\pi_v} = a_{\pi_v}\Theta_{\pi'_v}$ for all $v \in S$. Proposition 3.5 together with the fact that π'_v is one-dimensional implies that π_v is special for all $v \in S - \{v_0\}$; thus $a_{\pi_v} = 1$ for all such v. The relation $\mathrm{prod}_{v \in S}\, a_{\pi_v} = 1$ establishes that $a_{\pi_{v_0}} = 1$; that is, $\sigma = \pi_{v_0} \sim \sigma'$.

Proposition 3.5 and the completeness assertion of Proposition 3.1 together imply that for every admissible irreducible special or supercuspidal representation σ of GL (3, E) there exists an irreducible admissible representation σ' of H^* such that $\sigma \sim \sigma'$. Herewith Theorem 1 is proved.

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