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A NOTE ON H^1 q-MARTINGALES

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Characterizations of H^1 q-martingales via conjugate transforms are studied. Applications to lacunary Fourier series and to local field analysis are given.

1. Introduction. Characterizations of H^1 space over a local field and H^1 regular martingales by singular integral transforms have been studied in a series of papers ([4], [1], [5], [7], and [2]). That is to say,

$$H^1 = \{f: f,\ T_j f \in L^1,\ j = 1, 2, \cdots, m\}$$

where H^1 is the space of regular functions or martingales with integrable maximal functions and T_j, $j = 1, 2, \cdots, m$, are some sort of nice singular integral transforms. In [4], [1] and [5], multiplier transforms arised from multiplicative characters on local fields are used. In [7], singular integral transforms with matrix operators acting on differences of regular martingales are considered. The dyadic case has been excluded until recently. Two methods of handling the dyadic case are given in [2]. First by noting that the maximal function of a dyadic martingale is equivalent in L^1-norm to the maximal function of its "associated 4-martingale", the space of H^1 dyadic martingales is characterized by 4×4 matrix transforms. Then H^1 space over the dyadic number field is studied via the multiplier transform associated to a multiplicative character of ramification degree 2.

In this note, we shall extend the above concept of higher ramification degree to q-martingales and obtain characterizations for the space H^1. We also show that the conditions are necessary. These results provide an answer to an open problem posed by Gundy and Varopoulos in [6]. Applications to homogeneous and nonhomogeneous multipliers on local fields are also given.

2. Conjugate characterizations. Let q be an integer larger than 1. Let $\{\mathscr{F}_n\}_{n \geq 0}$ be an increasing sequence of σ-fields which are generated by atoms in a probability space such that each atom in $\mathscr{F}_n$ contains exactly q atoms in $\mathscr{F}_{n+1}$ of equal measure. One example having such a structure is the group Z_q^∞. Another is the ring of integers of a local field whose residue class field has q elements. A martingale $f = \{f_n\}$ relative to $\{\mathscr{F}_n\}$ is called a *q-adic martingale* or, simply, a *q-martingale*.

Let $V_q = \{(x_i) \in \boldsymbol{C}^q : \Sigma_i x_i = 0\}$. For a q-martingale f, the q values of the martingale difference $d_n = f_n - f_{n-1}$ on q atoms of $\mathscr{F}_n$ in a fixed atom of $\mathscr{F}_{n-1}$ is called a *local* (*first*) *difference* of f. These local differences can be regarded as vectors in V_q. Similarly, we call the q^k values of $f_n - f_{n-k} (n \geqq k \geqq 1)$ on q^k atoms of $\mathscr{F}_n$ in a fixed atom of $\mathscr{F}_{n-k}$, a *local kth difference* of f.

Associated to each q-martingale $f = \{f_n\}_{n \geqq 0}$ relative to $\{\mathscr{F}_n\}_{n \geqq 0}$, $F = \{f_{nk}\}_{n \geqq 0}$ is a q^k-martingale relative to $\{\mathscr{F}_{nk}\}_{n \geqq 0}$. On the other hand, given a q^k-martingale, we can construct a q-martingale in a unique way by filling in the intermediate levels. The local kth differences of a q-martingale are local differences of its associated q^k-martingale. The H^1 norm of a martingale is taken to be the L^1-norm of its maximal function. We note that the H^1 norms of a q-martingale and of its associated q^k-martingale are equivalent. (See e.g., [2] or [6].)

One type of transforms on q-martingales was introduced in [7] as follows. Given a q-martingale f, applying A to each local difference of f on fixed atoms, we obtain another set of local differences which constitute the difference sequence of a q-martingale Af, say. The transform $f \to Af$ is a sort of singular integral transform on martingales which is bounded on $H^p (0 < p < \infty)$ and BMO. (See [7] and [3].) Moreover it is shown in [7] that if A has no real eigenvector in V_q, then $H^1 = \{f \in L^1 : Af \in L^1\}$.

Note that first of all the characterization fails to hold when $q = 2$ since V_2 is a one-dimensional space and so the effect of such a transform A is just multiplying by a constant. Secondly, over local fields, such transforms include only those singular integrals whose kernels are ramified of degree one. We shall now study two related generalizations of these transforms to characterize H^1 q-martingales, both are introduced in [2] for the dyadic case.

The first method is to utilize the one-to-one correspondence between q-martingales and their associated q^k-martingales. Let A be a linear operator on V_{q^k}. For a q-martingale f, let F be its associated q^k-martingale. Apply A to local differences of F to obtain local differences of a q^k-martingale AF and then construct from it the q-martingale Af. The result in [7] together with the equivalence of the corresponding H^1 norms give:

THEOREM 1. *Let A_j, $j = 1, 2, \cdots, m$, be linear operators on V_{q^k} with $k \geqq 1$. Then $H^1 = \{f : f, A_j f \in L^1, j = 1, 2, \cdots, m\}$ if any only if A_j, $j = 1, 2, \cdots, m$, do not have a common real eigenvector in V_{q^k}.*

However, this method does not apply to transforms which correspond to multipliers on local fields that are ramified of degree

$k \geqq 2$. For simplicity, we treat here only the case $k=2$ in detail.

Given a linear operator A on V_q^q, we define the transform $T = T_A$ relative to A as follows. Let $f = \{f_n\}$ be a q-martingale with martingale difference sequence $\{d_n\}$. On each atom in $\mathscr{F}_{n-2}$, the q^2 values of d_n is a vector in V_q^q. Let Ad_n denote the image of d_n under A on the same atom. Those values of Ad_n on various atoms form the martingale difference sequence of a q-martingale $Tf = T_A f$. Multipliers of ramification degree 2 on a local field are such transforms. One detailed example is given in [2]. A local second difference of Tf, $(Tf)_n - (Tf)_{n-2} = Ad_n + Ad_{n-1}$, on an atom in $\mathscr{F}_{n-2}$, dependes on both d_n and d_{n-1}. It has values in two levels, i.e., Ad_n and Ad_{n-1}, mixed together. This is why the first method does not apply here. Note that if we assume $d_n = 0$ for all odd n, then we have $Tf = Af$ where Af is obtained by the first method. Thus one idea for further generalizations is to apply the previous argument to the q^2-martingales, Σd_{2n} and ΣAd_{2n}, which only involve the martingale difference of even levels. Do the same for Σd_{2n-1} and ΣAd_{2n-1}. We could then characterize H^1 q-martingale f by the L^1-boundedness of Σd_{2n}, ΣAd_{2n}, Σd_{2n-1} and ΣAd_{2n-1}.

Moreover, consider such a linear operator on V_2^2 given by the matrix

$$A = \frac{i}{2}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

which corresponds to the multiplier transform T studied in [2; § 3]. We observe that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ is a real eigenvector of A in V_4. The argument in [7] does not apply to A for 4-martingales. However A has no eigenvector in $\boldsymbol{R}^4 \cap V_2^2$. When restricted on V_2^2, the submartingale property involved is still valid as seen in [2]. This suggests that the earlier submartingale results could be sharpened for application to such transforms. The above ideas lead to the following generalization.

For a q-martingale f, let $Ef = \Sigma d_{2n}$ and $Gf = \Sigma d_{2n-1}$. Note that Ef and Gf are q^2-martingales with local differences in V_q^q.

THEOREM 2. *Suppose* $A_j\colon V_q^q \to V_q^q$, $j = 1, 2, \cdots, m$, *do not have a common real eigenvector in* V_q^q *and* $T_j = T_{A_j}$. *Then* Ef, T_jEf, Gf *and* T_jGf, $j = 1, 2, \cdots, m$, *are* L^1*-bounded if and only if* $f \in H^1$.

The necessity of the condition in Theorem 2 is obvious. The proof of the sufficiency will be given in § 3. Another version of the theorem is the following.

COROLLARY 1. *If the linear operator* A *on* V_q^q *has no real eigenvector in* V_q^q *and* $T = T_A$. *Then* $H^1 = \{f: f,\ Tf,\ Ef,\ TEf \in L^1\}$.

These results can be generalized to $k > 2$. In that case we have operators A_j on $V_q^{q^{k-1}} \subset V_{q^k}$. The analogue of Theorem 2 holds with k operators in the place of E and G, and hence there are $(m+1)k$ transforms (including possibly the identity) involved.

Theorem 2 fails if A_j, $j = 1, 2, \cdots, m$, have a common eigenvector in $\boldsymbol{R}^{q^2} \cap V_q^q$. It is shown in [7] how the eigenvector is used to construct a q^2-martingale F in L^1 but not in H^1 such that $A_j F = \lambda_j F$, $\lambda_j \in \boldsymbol{C}$, $j = 1, 2, \cdots, m$. Let f be the q-martingale corresponding to F. Then $A_j f = \lambda_j f$, $j = 1, 2, \cdots, m$. The local differences of F are by construction all proportional to the eigenvector and thus are elements in V_q^q. This implies that $Ef = f$ and $Gf = 0$. Also, as noted before, $T_j Ef = A_j Ef = A_j f = \lambda_j f$, $j = 1, 2, \cdots, m$. Therefore Ef, $T_j Ef = \lambda_j f$, $Gf = 0$ and $T_j Gf = 0$, $j = 1, 2, \cdots, m$, all belong to L^1, but $f \notin H^1$.

3. Basic lemma.

We present here the basic lemma of which Theorem 2 is a consequence.

LEMMA. *Let* W *be a closed cone*[1] *consisting of elements of the form* $x = (x^1, \cdots, x^m)$ *where* $x^j = (x_1^j, \cdots, x_q^j) \in V_q$ *such that if* $x^j = \eta_j(\lambda_1, \cdots, \lambda_q)$ *for some* $\eta_j \in \boldsymbol{C}$, $j = 1, 2, \cdots, m$, *and* $\lambda_i \in \boldsymbol{R}$, $i = 1, 2, \cdots, q$, *then* $x = (0, 0, \cdots, 0)$. *Then there is a positive* $p < 1$ *such that*

$$|a|^p \leqq \frac{1}{q} \sum_{i=1}^{q} |(a^j + x_i^j)_{j=1}^m|^p \tag{1}$$

for $a = (a^j)_j \in \boldsymbol{C}^m$ *and* $x = (x^j)_j \in W$.

Proof. This follows from essentially the same argument used in [7] which we outline below.

Let α be the maximum of $\sum_i (\mathrm{Re} \sum_j \bar{a}^j x_i^j)^2$ on the compact set

$$K_1 = \{(a, x) \in \boldsymbol{C}^m \times W : |a| = 1 \text{ and } \|x\| = (\sum_{i,j} |x_i^j|^2)^{1/2} = 1\} .$$

Schwartz's inequality gives $\alpha \leqq 1$. Equality would imply $\sum_j \bar{a}^j x_i^j \in \boldsymbol{R}$ and $a^j x_i^k = \lambda a^k x_i^j$ for some $(a, x) \in K_1$ and $\lambda \in \boldsymbol{R}$. Thus we would have $x_i^k = \sum_j \bar{a}^j a^j x_i^k = \lambda a^k \sum_j \bar{a}^j x_i^j$ which contradicts the hypothesis. Hence

[1] By a cone W, we mean $x \in W$ and $t \geqq 0$ imply $tx \in W$.

$\alpha < 1$.

From the series expansion of

$$\begin{aligned}|(a^j + x_i^j)_1^m|^p &= (\sum_j |a^j + x_i^j|^2)^{p/2} \\ &= |a|^p \left\{1 + \frac{2\,\mathrm{Re}\sum_j \bar{a}^j x_i^j}{|a|^2} + \frac{\sum_j |x_i^j|^2}{|a|^2}\right\}^{p/2},\end{aligned}$$

we obtain (1) for $\alpha \leqq p \leqq 1$ and $\|x\| < \varepsilon|a|$ for some $\varepsilon > 0$.

We now let β be the maximum of $|a|$ on the compact set

$$K_2 = \left\{(a,\ x) \in \boldsymbol{C}^m \times W: \frac{1}{q} \sum_i |(a^j + x_i^j)_1^m| = 1 \text{ and } \|x\| \geqq \varepsilon|a|\right\}.$$

From Minkowski's inequality, we have $\beta \leqq 1$. Equality holds only if $a^j + x_i^j = \lambda_i a^j$ with $\lambda_i \geqq 0$. Thus we would have $x_i^j = (\lambda_i - 1)a^j$ which contradicts again the hypothesis.

Therefore, if $\|x\| \geqq \varepsilon|a|$, then $|a| \leqq (\beta/q)\sum_i |(a^j + x_i^j)_1^m|$ with $\beta<1$ from which (1) follows if $(\beta/q)^p \leqq 1/q$. This proves the lemma.

We note that the lemma includes the earlier versions of subregularity (or submartingale property) in [1], [4], [5] and [7] as special cases.

We now complete the proof of Theorem 2.

Proof of Theorem 2. Let $W = \{(x_0, x_1, \cdots, x_m): x_0 \in V_q^q, x_j = A_j x_0, j = 1, 2, \cdots, m\}$. Given a q-martingale f, as noted before, we have $T_j Ef = A_j Ef$, $j=1, 2, \cdots, m$. The local differences of Ef, $A_1 Ef, \cdots, A_m Ef$ can be regarded as elements in W. Since A_j's do not have any common real eigenvector in V_q^q, the hypothesis of lemma is satisfied. Thus $|(Ef, A_1 Ef, \cdots, A_m Ef)|^p$ is a submartingale for some $p < 1$.

By an usual martingale majorant argument, we have that $Ef \in H^1$. Similarly, $Gf \in H^1$. Therefore $f \in H^1$.

Another simple application of the lemma is by letting $m = 1$:

COROLLARY 2. *Let W be a subspace of V_q not containing any nonzero real vector. Then for a q-martingale f having local differences in W, there exists a positive $p < 1$ such that $|f|^p$ is a submartingale. Consequently, $\|f\|_{H^1} \leqq C\|f\|_{L^1}$ for some $C > 0$ independent of f.*

The fact that a q-martingale f having local differences in such a W can be regarded as an analytic property of f. The following is a version of the F. and M. Riesz theorem:

COROLLARY 3. *Let W be a subspace of V_q not containing any* *nonzero real vector. If all local differences of the q-martingale corresponding to a finite measure μ are in W, then μ is absolutely continuous, moreover, $\mu^* \in L^1$.*

4. Integrability and H^1. Our basic lemma in § 3 enables us to characterize H^1 as the set of f for which certain transforms are in L^1 without a priori assuming f to be integrable. Special cases were considered before in [1].

THEOREM 3. *Suppose that A_j, $j = 1, 2, \cdots, m$, are linear operators from V_q to V_q such that there is no nonzero $x \in V_q$ for which all A_jx's are multiples of a common real vector in V_q. Let $T_j = T_{A_j}$ be the transform relative to A_j, $j = 1, 2, \cdots, m$. Then for a q-martingale f, $f \in H^1$ if and only if T_jf, $j = 1, 2, \cdots, m$, are L^1-bounded.*

Proof. Let A be the direct product of $A_1, \cdots, A_m$, i.e., $Ax = (A_1x, \cdots, A_mx)$ for $x \in V_q$. Set $W = \{Ax: x \in V_q\} \subset \boldsymbol{C}^{mq}$. W satisfies the condition in the lemma of § 3 by the hypothesis. It follows from the lemma, by taking a_j to be the corresponding values of T_jf, $j = 1, 2, \cdots, m$, that $|(T_1f, \cdots, T_mf)|^p = (\sum_{j=1}^m |T_jf|^2)^{p/2}$ is a submartingale for some $p<1$. Thus by an usual argument, $(T_jf)^* \in L^1$, $j = 1, 2, \cdots, m$.

Now if $Ax = 0$ for some $x \in V_q$, then $A_jx = 0$ for all j. From the hypothesis, we have $x = 0$. Hence A: $V_q \to V_q^m$ is injective. Let B be a left inverse of A; $BA = I$. Namely, there exist B_j: $V_q \to V_q$, $j = 1, 2, \cdots, m$, such that $I = BA = \sum_{j=1}^m B_jA_j$. Let $S_j = T_{B_j}$, $j = 1, 2, \cdots, m$, be the transform relative to B_j. As noted before such transforms preserve H^1, therefore we have $f = \sum_{j=1}^m S_jT_jf \in H^1$.

Since the converse is obvious, the proof is completed.

For the case that one of the operators is the identity, the result is obtained in [7].

5. Application to lacunary Fourier series. We shall apply the preceeding results to the martingales considered by Gundy and Varopoulos in [6].

Let $f \in L^1(\boldsymbol{T})$ with Fourier series expansion $\sum_k \hat{f}(k) \exp(ik\theta)$. Suppose $\hat{f}(0) = 0$. Define $f_n(\theta) = q^{-n} \sum_{j=0}^{q^n-1} f(\theta + 2\pi jq^{-n}) = \sum_k \hat{f}(kq^n) \exp(ikq^n\theta)$. Let $\mathscr{F}_n$ be the σ-field of all $2\pi q^{-n}$-periodic Borel sets of $\boldsymbol{T}$. Then $\{f_n\}$ forms a backwards q-martingale relative to $\{\mathscr{F}_n\}$. The H^1 space of such martingales is denoted by H_q^1. The difference sequence of the martingale f is given by

$$d_n(\theta) = f_n(\theta) - f_{n+1}(\theta) = \sum_{k; q \nmid k} \hat{f}(kq^n) \exp(ikq^n\theta) .$$

For a fixed θ in an atom S of $\mathscr{F}_{n+1}$, a collection of coset representatives of S contained in an atom of $\mathscr{F}_n$ is $\{\theta_j\}_{j=0}^{q-1}$ where $\theta_j = \theta + 2\pi j q^{-(n+1)}$. Hence if we let x_k be the vector in V_q with components $x_{kj} = \exp(2\pi i j k q^{-1})$, $j = 0, 1, \cdots, q-1$, then the local difference d_n of f at θ in an atom of $\mathscr{F}_{n+1}$ is

$$\sum_{l=1}^{q-1} [\sum_s \hat{f}((sq + l)q^n) \exp(i(sq + l)q^n\theta)]x_l \ .$$

Assume that $B \subset \{1, 2, \cdots, q-1\}$ is such that $m \in B$ implies $q - m \notin B$ and let W be the span of the vectors $\{x_m\}_{m \in B}$. There is no nonzero real vector in W since if $y = \sum_{m \in B} a_m x_m$ is real in W, then $y = \bar{y} = \sum_{m \in B} \bar{a}_m \bar{x}_m = \sum_{m \in B} \bar{a}_m x_{q-m}$ which implies $y = 0$. Let $[k]$ denote the integer between 0 and $q-1$ such that $[k] \equiv k (\mathrm{mod}\, q)$. We observe that $x_k = x_{[k]}$. Moreover, if $\hat{f}(kq^n) = 0$ when $[k] \notin B$, then local differences of f are in W. Therefore Corollary 2 gives the following

THEOREM 4. *Let B be any subset of $\{1, 2, \cdots, q-1\}$ such that m and $q - m$ are not both in B. Suppose*

$$\hat{f}(kq^n) = 0 \ \textit{for} \ [k] \notin B \ \textit{and} \ n \geqq 0 \ .$$

Then

$$\| f \|_{H_q^1} \leqq C \| f \|_{L^1} \ \textit{with some} \ C > 0 \ .$$

We show now how the result of Gundy and Varopoulos ([6]; Theorem 2 and Theorem 4) follows from this. For a subset A of integers, we write $f_A(\theta) = \sum_{n \in A} \hat{f}(n) \exp(in\theta)$. Assume that q is odd. We choose, e.g.,

$$B_1 = \left\{1, 2, \cdots, \frac{q-1}{2}\right\} \quad \text{and} \quad B_2 = \left\{\frac{q+1}{2}, \cdots, q-1\right\}$$

and set

$$A_i = \{kq^n : [k] \in B_i, n \geqq 0\}, \ i = 1, 2 \ .$$

Note that the nonzero integers Z^* is a disjoint union of A_1 and A_2. Thus it follows from Theorem 4 that

COROLLARY 4. *Suppose q is odd and A_1, A_2 are given as above. Then $f \in H_q^1$ if and only if f_{A_1} and $f_{A_2} \in L^1$.*

(This is Theorem 2 of [6].)

When q is even, the problem is more delicate as has been seen in [6]. We use the idea in § 2 and consider the q^2-martingales Ef and Gf associated with the (backwards) q-martingale $\{f_n\}$ given above.

Recall that Ef is the q^2-martingale such that $Ef = \sum d_{2n}$ where $\{d_n\}$ is the martingale difference sequence of $\{f_n\}$. Also $d_{2n}(\theta) = \sum_{k;q\nmid k} \hat{f}(kq^{2n}) \exp(ikq^{2n}\theta)$. Hence the local difference of Ef at θ is given by

$$\sum_{1\leqq <lq^2, q\nmid l} \sum_s \hat{f}((sq^2 + l)q^{2n}) \exp(i(sq^2 + l)q^{2n}\theta)x$$

where $x_l = \langle \exp(2\pi ijlq^{-2})\rangle_{j=0}^{q^2-1}$.

Now let, for instance

$$B_1 = \left\{l: 1 \leqq l < \frac{q^2}{2},\ q\nmid l\right\} \quad \text{and} \quad B_2 = \left\{l: \frac{q^2}{2} < l \leqq q^2 - 1,\ q\nmid l\right\} .$$

Set $A_i = \{(sq^2 + l)q^{2n}: s \in \boldsymbol{Z},\ l \in B_i\}$, $i = 1, 2$. Note that $Ef = f_{A_1 \cup A_2}$ and $(Ef)_{A_i} = f_{A_i}$, $i = 1, 2$. Therefore it follows from Theorem 4 that

$$\|f_{A_i}\|_{H^1_{q^2}} \leqq C_i \|f_{A_i}\|_{L^1},\ i = 1, 2 .$$

The equivalence of the norms $\|\cdot\|_{H^1_q}$ and $\|\cdot\|_{H^1_{q^2}}$ gives

$$\|f_{A_i}\|_{H^1_q} \leqq C_i \|f_{A_i}\|_{L^1},\ i = 1, 2 .$$

Similarly, let $A_{i+2} = \{(sq^2 + l)q^{2n+1}: s \in \boldsymbol{Z},\ l \in B_i\}$, $i = 1, 2$, (by considering Gf), we have

$$\|f_{A_i}\|_{H^1_q} \leqq C_i \|f_{A_i}\|_{L^1},\ i = 3, 4 .$$

Since A_i, $i = 1, 2, 3, 4$, form a partition of the nonzero integers, we have

COROLLARY 5. *Let q be a positive even integer and A_i, $i=1, 2, 3, 4$ be defined as above. Then $f \in H^1_q$ if and only if $f_{A_i} \in L^1$, $i=1, 2, 3, 4$.*

For $q = 2^k$, this is Theorem 4(a) of Gundy-Varopoulos [6]. If q is even but not a power of 2, this improves their results by reducing the number of sets in the partition from six to four. Thus it provides an answer to the open problem posed by Gundy and Varopoulos [6], namely, at most three Fourier multipliers are needed to characterize H^1_q.

We remark here that the partition into A_i, $i = 1, 2, 3, 4$, is merely one of many choices in order that Corollary 5 is valid.

6. Application to local field analysis. We apply the main results to multiplier transforms on local fields. Preliminaries on Fourier analysis on local fields are found in [8].

Let K be a local field with residue class field isomorphic to $GF(q)$. $H^1(K)$ is the space of regular functions on $K \times \boldsymbol{Z}$ with integrable maximal functions. Let m be a nonconstant bounded function on $K^* = K\backslash\{0\}$ such that for some integer $k \geqq 1$, $m(x+y) = m(x)$ whenever $|y| \leqq q^{-k}|x|$. If k is the smallest among such integers, we shall say that m is *ramified of degree* k. If m is homogeneous of degree 0, then the mapping $f \rightarrow (m\hat{f})^{\vee}$ is a singular integral transform and a transform on q-martingales (i.e., regular functions on $K \times \boldsymbol{Z}$) of the type studied in connection with Theorem 2. (We note that previous results also hold for σ-finite measure spaces.) If m is not homogeneous, then this mapping is of more general type which can be realized as applying a different operator $A^{(n)}$ on the atoms in each "level" n (i.e., σ-field $\mathscr{F}_n$).

We begin with multipliers with ramification degree one:

THEOREM 5. *Let m_j, $j = 1, 2, \cdots, n$, be bounded functions on K, ramified of degree 1 and such that $\sum_{j=1}^{n} |m_j(x) - m_j(-x)| \geqq c > 0$, for $x \neq 0$. Then*

$$H^1(K) = \{f \in L^1 \colon m_j\hat{f} \in \widehat{L^1}(K),\ j = 1, 2, \cdots, n\} .$$

Proof. We regard a vector in $\boldsymbol{C}^q$ as a function on the finite group (field) $GF(q)$. Let W be the set of all vectors $(x_0, \cdots, x_n) \in V_q^{n+1}$ satisfying $\hat{x}_j(g) = \alpha_j(g)\hat{x}_0(g)$, $j = 1, 2, \cdots, n$, for some function α_j on $GF(q)$ with $|\alpha_j(g)| \leqq \|m_j\|_\infty$ and $\sum_{j=1}^{n} |\alpha_j(g) - \alpha_j(-g)| \geqq c > 0$ for $g \neq 0$. W satisfies the condition of the lemma in § 3. For every sphere in K, the local differences of f and $(m_j\hat{f})^{\vee}$, $j = 1, 2, \cdots, n$, constitute an element in W. Thus, by the lemma,

$$(|f|^2 + \textstyle\sum_j |(m_j\hat{f})^{\vee}|^2)^{p/2}$$

is a submartingale (i.e., subregular) for some $p < 1$. Therefore Theorem 5 follows as usual.

In the homogeneous case, this is proved in [7] and the condition is shown there to be necessary also.

A similar result without a priori assuming $f \in L^1$ is obtained by the method of Theorem 3. We state here one version for the homogeneous case:

THEOREM 6. *Suppose m_j, $j = 1, 2, \cdots, n$, are homogeneous of degree 0 and ramified of degree 1. Then*

$$H^1(K) = \{f \colon m_j\hat{f} \in \widehat{L^1}(K),\ j = 1, 2, \cdots, n\}$$

if and only if $\langle m_j(x)\rangle_{j=1}^n$ *is not a constant multiple of* $\langle m_j(-x)\rangle_{j=1}^n$ *for any* $x \neq 0$.

We now consider multipliers which are ramified of degree $k \geqq 2$: Let, for $n \in \boldsymbol{Z}$,

$$e(x) = \begin{cases} 1 \text{ if } |x| = q^{2n} \\ 0 \text{ if } |x| = q^{2n+1} \end{cases} \text{ and } \quad g(x) = 1 - e(x) .$$

If Ef and Gf are the q^2-martingales associated to a q-martingale (i.e., a regular function) f, then $\widehat{Ef} = e\hat{f}$ and $\widehat{Gf} = g\hat{f}$. Therefore we have

COROLLARY 6. *Suppose* m *is homogeneous of degree* 0 *and ramified of degree* 2. *Then*

$$H^1(K) = \{f\colon e\hat{f},\, me\hat{f},\, g\hat{f},\, mg\hat{f} \in \widehat{L^1}(K)\}$$
$$(= \{f \in L^1(K)\colon m\hat{f},\, e\hat{f},\, me\hat{f} \in \widehat{L^1}(K)\}) ,$$

if and only if $m(x) \neq m(-x)$ *for every nonzero* $x \in K$.

Proof. We note that the linear operator A on V_q^q corresponding to the transform $f \to (m\hat{f})^\vee$ is a convolution on a finite group whose Fourier transform is the restriction of the multiplier m. If $m(x) \neq m(-x)$ for all nonzero $x \in K$, then A does not have real eigenvectors (c.f. [7; § 4]). Therefore the result follows from Theorem 2 and the discussion after it.

More generally, we let $e_j^k(x) = 1$ if $|x| = kn + j$ and $e_j^k(x) = 0$ otherwise, for $k \geqq 1$, $n \in \boldsymbol{Z}$, $j = 1, 2, \cdots, k$. We have

COROLLARY 7. *Suppose* m *is homogeneous of degree* 0 *and ramified of degree* k. *Then*

$$H^1(K) = \{f\colon e_j^k\hat{f},\, me_j^k\hat{f} \in \widehat{L^1}(K),\ j = 1, 2, \cdots, k\}$$

if and only if $m(x) \neq m(-x)$ *for all* $x \neq 0$.

We remark that various versions of the results similar to Corollary 6 and Corollary 7 that involve several multipliers m_j's being ramified of degree k and/or nonhomogeneous also follow from Theorem 2 and Theorem 3. We omit the details here.

In characterizing $H^1(K)$, if q is odd, it suffices to take an m with ramification degree 1 (c.f. [1], [7]). If q is even and thus a power of 2, then K is an extension of a dyadic field or a dyadic series field. For the former case, a multiplier m with ramification degree 2 will do. However, in an extension K of a dyadic series field, $x = -x$ for all $x \in K$. Thus no finite set of multipliers of this type characterizes $H^1(K)$.

REFERENCES

1. J.-A. Chao, *H^p spaces of conjugate systems on local fields*, Studia Math., **49** (1974), 267–287.
2. ———, *Conjugate characterizations of H^1 dyadic martingales*, Math. Ann., **240** (1979), 63–67.
3. ———, *The spaces H^p and BMO on regular martingales*, preprint.
4. J.-A. Chao and M. H. Taibleson, *A subregularity inequality of conjugate systems on local fields*, Studia Math., **46** (1973), 249–257.
5. ———, *Generalized conjugate systems on local fields*, Studia Math., **64** (1979), 213–225.
6. R. F. Gundy and N. Th. Varopoulos, *A martingale that occurs in harmonic analysis*, Ark. Mat., **14** (1976), 179–187.
7. S. Janson, *Characterizations of H^1 by singular integral transforms on martingales and $\boldsymbol{R}^n$*, Math., Scand., **41** (1977), 140–152.
8. M. H. Taibleson, *Fourier analysis on local fields*, Princeton, 1975.

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