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RIGHT CHAIN RINGS AND THE GENERALIZED SEMIGROUP OF DIVISIBILITY

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Let R be a ring with unit element and without zero-divisors and let $\tilde{H}(R) = \{\tilde{x} \mid 0 \neq x \in R\}$ where $\tilde{x}$ is the mapping from the set of all nonzero principal right ideals of R into itself defined by $\tilde{x}(aR) = xaR$. $\tilde{H}(R)$ is a partially ordered semigroup that can be considered as a generalization of the group of divisibility of a commutative integral domain. We study those rings R for which $\tilde{H}(R)$ is totally ordered.

1. Introduction. Associated with any commutative integral domain A is the partially ordered group $G(A)$ of nonzero fractional principal ideals of A with $aA \leqq bA$ if and only if aA contains bA. It is well known (see [4], [5], [8]) that $G(A)$, the group of divisibility, reflects certain properties of A, like A being a unique factorization domain, the fact that any two elements in A have a greatest common divisor or A being a valuation ring. This concept of a group of divisibility cannot be extended directly to a not necessarily commutative integral domain R.

In this paper we associate with any ring R with unit element and without zero-divisors a partially ordered semigroup $\tilde{H}(R)$ which is isomorphic to the semigroup $H(A) \subseteqq G(A)$ of nonzero principal ideals aA in A if A is a commutative domain.

After observing some basic facts about $\tilde{H}(R)$ we characterize in §3 those rings R with $\tilde{H}(R)$ totally ordered as right chain rings R with $Ja \subseteqq aR$ for all a in R and $J = J(R)$ the Jacobson radical of R. These rings are localizations of right invariant right chain rings. The main result of §4 is the theorem that a ring with $\tilde{H}(R)$ totally ordered and d.c.c. for prime ideals is right invariant. In a final §5 we show by examples that for every totally ordered group G there exists a ring R with $\tilde{H}(R)$ totally ordered and G (not only the positive cone of G) can be embedded into $\tilde{H}(R)$. The value group $G(A)$ is particularly useful in case A is a commutative valuation ring. The nonzero principal right ideals in a right chain ring R form a semigroup $H(R)$ under ideal multiplication only if R is right invariant. In the general case it is the semigroup $\tilde{H}(R)$ which takes the place of $H(R)$. Mathiak in [6] studies right and left chain domains with the help of a group that could be considered a generalization of $G(A)$. We found that in the case of one-sided conditions a generalization of $H(A)$, which will be a semigroup only, will be more natural.

2. Definition and preliminary results. We consider only rings with unit element and without zero-divisors. We call a ring R *right invariant* if $Ra \subseteqq aR$ (if and only if $RaR = aR$) holds for all elements a in R and R is a *right chain ring* (sometimes called a *right valuation ring*) if for a, b in R either $aR \subseteqq bR$ or $bR \subseteqq aR$ holds. Here $I \subset L$ always means that the set $I \neq L$ is contained in L; $J = J(R)$ is the *Jacobson radical* and $U = U(R)$ the *group of units of* R.

Let $W = \{aR \mid 0 \neq a \text{ in } R\}$ be the set of nonzero principal right ideals of R. Every element $0 \neq x$ in R induces a mapping $\tilde{x}$ on W with $\tilde{x}(aR) = xaR$; and $\widetilde{xy} = \tilde{x}\tilde{y}$ follows. With $\tilde{x} \geqq \tilde{y}$ defined as $xaR \subseteqq yaR$ for all a in R we can consider $\tilde{H}(R) = \{\tilde{x} \mid 0 \neq x \text{ in } R\}$ as a partially ordered semigroup. Further, $\widetilde{x+y} \geqq \inf(\tilde{x}, \tilde{y})$; i.e., $\tilde{z} \leqq \tilde{x}$, $\tilde{z} \leqq \tilde{y}$ implies $\tilde{z} \leqq \widetilde{x+y}$. The mapping '$\sim$' from $R^*(=R\backslash 0)$ to $\tilde{H}(R)$ is called the regular right valuation of R with the value-semigroup $\tilde{H}(R)$. This semigroup satisfies the following conditions:

(1) $\tilde{H}(R)$ is a partially ordered semigroup with unit element $\tilde{1}$.

(2) $\tilde{x} \leqq \tilde{y}$ if and only if there exists a $\tilde{t}$ in $\tilde{H}(R)$ with $\tilde{x}\tilde{t} = \tilde{y}$ and $\tilde{1} \leqq \tilde{t}$.

(3) $\tilde{x}\tilde{y} = \tilde{x}\tilde{z}$ implies $\tilde{y} = \tilde{z}$ for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ in $\tilde{H}(R)$.

This means that the order in $\tilde{H}$ is a right natural order and H is left cancellative.

We draw a few immediate conclusions from these properties:

(i) $\tilde{x} \leqq \tilde{1}$ implies that $\tilde{x}$ is a unit in $\tilde{H}$, i.e., there exists $\tilde{y}$ with $\tilde{x}\tilde{y} = \tilde{y}\tilde{x} = \tilde{1}$.

(ii) $\tilde{1} \leqq \tilde{x}$ implies $\tilde{x}\tilde{a} = \tilde{a}\tilde{x}'$ for some $\tilde{x}'$ in $\tilde{H}$.

To prove (i) we have by (2) an element $\tilde{t}$ with $\tilde{x}\tilde{t} = \tilde{1}$. This implies $\tilde{x}\tilde{t}\tilde{x} = \tilde{x}$ and $\tilde{t}\tilde{x} = \tilde{1}$ using (3). For $\tilde{1} \leqq \tilde{x}$ and $\tilde{a}$ in $\tilde{H}$ we have $\tilde{a} \leqq \tilde{x}\tilde{a}$ and $\tilde{x}\tilde{a} = \tilde{a}\tilde{x}'$ for some $\tilde{x}'$ using (2) again. Let $\tilde{U} = \tilde{U}(R)$ be the subgroup of units of $\tilde{H}(R)$. The following condition is satisfied by $\tilde{H}(R)$:

(4) Let $\tilde{U}'$ be a subgroup of $\tilde{U}$ with $\tilde{U}'\tilde{x} \subseteqq \tilde{x}\tilde{U}$ for all $\tilde{x}$ in $\tilde{H}(R)$. Then $\tilde{U}' = \{\tilde{1}\}$. In particular $\tilde{U} = \{\tilde{1}\}$ for R commutative.

The following is an easy example of a semigroup S satisfying conditions (1)-(3), but not (4).

Let $S = \{(n, a); n, a \in \boldsymbol{Z}; n \geqq 0\}$ considered as a subsemigroup of $G = \boldsymbol{Z} \oplus \boldsymbol{Z}$; $\boldsymbol{Z}$ the integers. We write $(n, a) > (m, b)$ if either $n > m$ or $n = m$ and $a > b$. Conditions (1), (2), (3) hold for S, but $U = \{(0, a); a \in \boldsymbol{Z}\}$ is a subgroup $\neq \{e\}$ of S, violating (4).

Two obvious problems arise: What is the structure of semigroups with (1), (2), (3), (4)? Given a semigroup S satisfying (1), (2), (3), (4) is $S \cong \tilde{H}(R)$ for some R? We are not able to answer these questions in general.

DEFINITION. Let R be a ring. Then

$$\hat{R} = \{r \in R \mid \tilde{r} \geqq 1\} \cup \{0\} = \{r \in R \mid raR \subseteqq aR \text{ for all } a \text{ in } R\} .$$

It is obvious that $\hat{R}$ is a subring of R.

LEMMA 1. (1) *$\hat{R}a \subseteqq a\hat{R}$ for all a in R; in particular $\hat{R}$ is a right invariant subring of R.*

(2) *The mapping $a\hat{R}$ to $\tilde{a}$ for $a \neq 0$ in R defines an isomorphism between the semigroup $C(R)$ of $\hat{R}$-modules $a\hat{R}$ with a in R onto $\tilde{H}(R)$. In $C(R)$ we have $a\hat{R}b\hat{R} = ab\hat{R}$ as operation and $a\hat{R} \leqq b\hat{R}$ if and only if $a\hat{R} \supseteqq b\hat{R}$.*

(3) *$\tilde{H}(R) \simeq R^*/U(\hat{R})$ where $U(\hat{R})$ is the group of units of $\hat{R}$ and $r_1 \equiv r_2$ if and only if $r_1 = r_2u$ with u in $u(\hat{R})$ defines a congruence relation on R^*, the multiplicative semigroup of nonzero elements in R.*

Proof. (1) $\hat{R}a \subseteqq aR$ by definition. If r is in $\hat{R}$ then $ra = ar_1$ and $rab = abr_2 = ar_1b$ for any a, b in R with r_1, r_2 in R. But $r_1b = br_2$ implies r_1 in $\hat{R}$ and $\hat{R}a \subseteqq a\hat{R}$ for $a \neq 0$ in R.

(2) Using (1) it follows that $a\hat{R}b\hat{R} = ab\hat{R}$ for a, b in R. If $\tilde{a} \geqq \tilde{b}$ then $axR \subseteqq bxR$ for all x in R and $a = bs$ and s in $\hat{R}$, hence $a\hat{R} \subseteqq b\hat{R}$ follows. Reversing these arguments yields the converse and $\tilde{H}(R) \simeq \{a\hat{R} \mid 0 \neq a \text{ in } R\}$ as a partially ordered semigroup.

(3) is just a different version of (2). □

REMARK. If R is embeddable into some skew field then $\hat{R} = \bigcap_{0 \neq a \in R} aRa^{-1}$.

If R is a ring such that the product of any two nonzero principal right ideals is again a nonzero principal right ideal we write $H(R)$ for the semigroup of the nonzero principal right ideals of R; $H(R)$ is a partially ordered semigroup with $aR \geqq bR$ if and only if $aR \subseteqq bR$.

If $H(R)$ exists and is isomorphic to $\tilde{H}(R)$ under the mapping that assigns $\tilde{x}$ to xR then R is right invariant. On the other hand $H(R)$ does exist for some rings that are not right invariant; simple rings or not right invariant principal ideal domains are obvious examples.

The following lemma shows that $H(R)$ exists for a local ring R if and only if R is right invariant.

LEMMA 2. *Assume $H(R)$ exists and let $0 \neq a$ be in R. Then $RaR = bR$ for some b and if $a = bc$ then c is not contained in $J(R)$.*

Proof. It only remains to show that c is not in $J(R)$. We have $b = \sum r_i a s_i$ for some r_i, s_i in R; $b = \sum r_i b c s_i = \sum b r_i' c s_i = b \sum r_i' c s_i$ where

$r_i b = br'_i$ for some r'_i in R. But this is impossible for c in $J(R)$. $\square$

COROLLARY. *If R is local then $H(R)$ exists if and only if R is right invariant.*

3. $\tilde{H}(R)$ totally ordered.

If A is a commutative integral domain its group of divisibility $G(A)$ is totally ordered only if A is a valuation ring. We will discuss the corresponding question for $\tilde{H}(R)$ and characterize the rings with $\tilde{H}(R)$ totally ordered. If x and y are nonzero elements in R then $\tilde{x} \leqq \tilde{y}$ or $\tilde{y} < \tilde{x}$ and $xR \supseteqq yR$ or $yR \supseteqq xR$ follows. Therefore, R is a right chain ring if $\tilde{H}(R)$ is totally ordered. Examples (see §5) show that for R a right chain ring $\tilde{H}(R)$ is not necessarily totally ordered.

THEOREM 1. *For an integral domain R the following conditions are equivalent:*

(1) *$\tilde{H}(R)$ is totally ordered.*

(2) *R is a right chain ring such that r in R, not in $\hat{R}$ implies r^{-1} in $\hat{R}$.*

(3) *$R = R'_P$, the localization of a right invariant right chain ring R' at a prime ideal P of R'.*

(4) *R is a right chain ring such that $Ja \subseteqq aR$ for all a in R.*

(5) *R is a right chain ring and if $Ra \not\subseteqq aR$ then $Ja \subseteqq aJ$ for any a in R.*

(6) *The submodules of the right $\hat{R}$-module R are totally ordered.*

Proof. (1) $\Rightarrow$ (2) We observed that R is a right chain ring if $\tilde{H}(R)$ is totally ordered. For an element r, not in $\hat{R}$, we have $\tilde{r} < \tilde{1}$, hence $raR \supseteqq aR$ for all $a \in R$ and r in $U(R)$, r^{-1} in $\hat{R}$ follows. (2) $\Rightarrow$ (3) It follows from (2) that $\hat{R}$ is a right chain ring and from Lemma 1 that $\hat{R}$ is right invariant. The set $S = \hat{R} \cap U(R)$ is multiplicatively closed and $P = \hat{R} \backslash S$ is a prime ideal in $\hat{R}$. Finally, $R = \hat{R}_P = \hat{R}S^{-1}$ is the localization of $\hat{R}$ at P.

To prove that (3) implies (1) we need a few lemmas.

Let R be a right invariant right chain ring. We write $\tilde{T} = \{\tilde{t} \in \tilde{H}(R) \mid t \in T\}$ for a subset $T \subseteqq R^*$ and we say $\tilde{T}$ ($\neq \varnothing$) is *R-convex* if for $tR \subseteqq sR \subseteqq R$, t in T, the element $\tilde{s}$ is contained in $\tilde{T}$. One can check the following two statements.

LEMMA 3. *There is a one-to-one correspondence between the set of R-convex subsets of $\tilde{H}(R)$ and the right ideals $\neq R$ given by*

$$\tilde{S} \longrightarrow \tilde{S}' = \{x \in R \mid \tilde{x} \notin \tilde{S}\} \cup \{0\}$$
$$I \longrightarrow I' = \{\tilde{x} \in \tilde{H}(R) \mid xR \supset I\}$$

where $\tilde{S}$ is R-convex and I is a right ideal $\neq R$.

LEMMA 4. *The R-convex subset $\tilde{S}$ is a subsemigroup of $\tilde{H}(R)$ if and only if $\tilde{S}' = P$ is a completely prime ideal of R.*

We consider the situation as described in the last lemma. Then $S = \{x \in R \mid \tilde{x} \in \tilde{S}\}$ is a multiplicatively closed saturated (i.e., ab in S implies a, b in S) right Ore system in R. The corresponding prime ideal is $P = R \backslash S$ and $R_P = RS^{-1}$ is the corresponding localization. Set $N = N(S) = \{r \in R \mid ra = as_a,\ s_a \text{ in } S \text{ for all } a \neq 0 \text{ in } R\}$. N is an R-convex subsemigroup of S maximal with the property that $a^{-1}Na \subseteqq N$ for all nonzero a in R. To see this, one observes that with n in N, $nR \subseteqq mR \subseteqq R$, we have $n = mr$ for some r and $na = as_a = am'r'$ for m', r' in R with $ma = am'$, $ra = ar'$. Therefore $m'r' = s_a$ is in S and m' in S, and m in N. Further, n in N and $na = as_a$ implies s_a in N.

To N there corresponds a prime ideal $Q = R \backslash N$ with $P \subseteqq Q \subseteqq J$. We want to describe $\tilde{H}(R_P)$ and we will get the result by considering two special cases:

(i) $N(S) = S$, i.e., $Q = P$ (Lemma 5) and

(ii) $N(S) = U(R)$, i.e., $Q = J$ (Lemma 6).

LEMMA 5. *Let R be a right invariant right chain ring, P a prime ideal in R, $S = R \backslash P$. Assume $N(S) = N = S$. Then R_P is again right invariant and $\tilde{H}(R_P) \simeq \tilde{H}(R)/\tilde{N} \risingdotseq \bar{H}$.*

Proof. That R_P is again right invariant follows from the fact that every principal right ideal in R_P has the form aR_P with a in R and that $sa = as_a$ for all a in R, s_a in S if s is in $S = N$. Hence $rs^{-1}aR_P = raR_P = ar'R_P$ with $ra = ar'$, r, a in R. If one defines $\tilde{r}_1 \equiv \tilde{r}_2$, r_1, r_2 nonzero elements in R, if and only if $r_1 = r_2 n$ or $r_1 n = r_2$ for some n in N, then "$\equiv$" is a congruence relation defined on $\tilde{H}$, and we write $H = \tilde{H}(R) \backslash \tilde{N}$ for the factor semigroup modulo this congruence. Further, $\bar{r}_1 > \bar{r}_2$ in $\bar{H}$ if and only if $r_1 > r_2$ in $\tilde{H}(R)$ and $\tilde{r}_1 \not\equiv \tilde{r}_2$. It follows that $\bar{H} \simeq \tilde{H}(R_P)$ as totally ordered semigroups.

LEMMA 6. *Let R be a right invariant, right chain ring, P a prime ideal in R, $S = R \backslash P$. Assume $N(S) = U(R)$. Then R_P is not right invariant if $P \subset J$ and $\tilde{H}(R_P) \simeq \tilde{H}(R)\tilde{S}^{-1}$.*

Proof. $\tilde{H}(R)$ contains the subsemigroup $\tilde{S}$. We will prove that under the above assumption $\tilde{H}(R)$ can be embedded into the semigroup $\tilde{H}(R)\tilde{S}^{-1} = \{\tilde{r}\tilde{s}^{-1} \mid r \in R^*, s \in S\}$ of fractions for $\tilde{H}(R)$.

The semigroup $\tilde{H}(R)$ is totally ordered and $\alpha\beta = \alpha\gamma$ for α, β, γ in $\tilde{H}(R)$ implies $\beta = \gamma$. Since the other cancellation law does not hold in general, $\tilde{H}(R)$ itself may not be embeddable into a group. But for every $\tilde{r}$ in $\tilde{H}(R)$ and $\tilde{s}$ in $\tilde{S}$ there exists an element $\tilde{a}$ in $\tilde{H}(R)$ with $\tilde{r}\tilde{a} = \tilde{s}$ or $\tilde{r} = \tilde{s}\tilde{a}$ and $\tilde{H}(R)\tilde{S}^{-1}$ exists ([3], Prop. 5.1; page 21) if we can show that $\tilde{r}_1\tilde{s} = \tilde{r}_2\tilde{s}$ implies $\tilde{r}_1 = \tilde{r}_2$ for $\tilde{r}_1$, $\tilde{r}_2$ in $\tilde{H}(R)$, $\tilde{s}$ in $\tilde{S}$.

We can assume $r_1 = r_2c$ for some c in R and we are done if we can show that c is in N. But, $\tilde{r}_1\tilde{s} = \tilde{r}_2\tilde{s}$ implies $r_2cs = r_2s\varepsilon$ for some ε in $U(R)$. Therefore $cs = s\varepsilon$ and c is an element of S. Let a be in R. If a is in S then $ca = ac'$ with c' in S. If a is not in S then $a = sa_1$ for some a_1 in R and $ca = csa_1 = s\varepsilon a_1 = sa_1\varepsilon' = a\varepsilon'$ with ε' in $U(R)$. Hence, c is in $N = U(R)$ and $K = \tilde{H}(R)\tilde{S}^{-1} = \{\tilde{r}\tilde{s}^{-1} \mid r \in R^*, s \in S\}$ exists.

This semigroup is totally ordered if we define $\tilde{r}_1\tilde{s}_1^{-1} \geqq \tilde{r}_2\tilde{s}_2^{-1}$ if and only if for all $\tilde{s}$, $\tilde{s}'$, with $\tilde{s}_1\tilde{s} = \tilde{s}_2\tilde{s}'$ we get $\tilde{r}_1\tilde{s} \geqq \tilde{r}_2\tilde{s}'$.

This last condition is equivalent to $\tilde{r}_1 \geqq \tilde{r}_2\tilde{s}$ if $s_1 = s_2s$ and $\tilde{r}_1\tilde{s} \geqq \tilde{r}_2$ if $s_1s = s_2$ where s is some element in S. For the necessary computations it is the easiest to write any finite number of elements in K in the form $\tilde{r}_i\tilde{s}^{-1}$, $i = 1, \cdots, n$.

It is a bit tedious to check that K is a totally ordered semigroup with unit element such that

(i) $\alpha \geqq \beta$ in K implies that there exists γ in K with $\alpha = \beta\gamma$

(ii) $\gamma\alpha = \gamma\beta$ implies $\alpha = \beta$ where α, β, γ are in K.

Further, it follows from these conditions that all elements $\gamma \leqq \tilde{1}$ in K have an inverse in K.

It remains to show that $K \simeq \tilde{H}(R_P)$ as ordered semigroups where the isomorphism is given by $\tilde{r}\tilde{s}^{-1} \leftrightarrow \widetilde{rs^{-1}}$. (Here $\tilde{r}$, $\tilde{s}$ are elements in $\tilde{H}(R)$, $\widetilde{rs^{-1}}$ is an element in $\tilde{H}(R_P)$.) We shall show here that the given correspondence is one-to-one and omit the rest.

Let $\widetilde{r_1s^{-1}} = \widetilde{r_2s^{-1}}$ i.e., $r_1s^{-1}aR_P = r_2s^{-1}aR_P$ for all a in R_P; in particular $r_1s^{-1}sbR_P = r_2s^{-1}sbR_P$ for all b in R and $r_1bR_P = r_2bR_P$, $r_1b = r_2bs'$ or $r_1bs' = r_2b$ for some s' in S follows. Comparing r_1 and r_2 yields $r_1 = r_2c$ or $r_2 = r_1c$ for some c in N and $\tilde{r}_1 = \tilde{r}_2$ in $\tilde{H}(R)$. If conversely $\tilde{r}_1\tilde{s}^{-1} = \tilde{r}_2\tilde{s}^{-1}$ in K we get $\tilde{r}_1 = \tilde{r}_2$ in $\tilde{H}(R)$ and therefore $r_1s^{-1}aR_P = r_2s^{-1}aR_P$ for all a in R: If a is in S this is obvious, otherwise $a = sb$ and $r_1bR = r_2bR$ implies $r_1s^{-1}aR_P = r_2s^{-1}aR_P$ in that case. Finally let s be in $S \backslash U(R)$. Then there exists a in R with $sa = ag$ and g not in S since s is not in N. This shows that $s^{-1}aR_P \supset aR_P$ and R_P is not right invariant.

If we combine Lemma 5 and Lemma 6 we get the following result:

THEOREM 2. *Let R be a right invariant right chain ring, P a* *prime ideal in* R, $S = R\backslash P$; $N = \{x \in R \mid xa = as_a,\ s_a$ *in* S *for all* $a \in R\}$. *Then:*

(1) $\tilde{H}(R_P) \simeq \bar{H}\bar{S}^{-1}$ *is a totally ordered semigroup with* $\bar{H} = \tilde{H}(R_Q) \simeq \tilde{H}(R)/\tilde{N}$ *and* $\bar{S} \simeq \tilde{S}/\tilde{N}$; $Q = R\backslash N$ *is a prime ideal and* R_Q *is right invariant.*

(2) R_P *is right invariant if and only if* $N = S$.

With Theorem 2 the equivalence of (1), (2), (3) in Theorem 1 is proven.

We prove the equivalence of (1) and (4). If $\tilde{H}(R)$ is totally ordered and j in $J(R)$, then $\tilde{j} \leqq \tilde{1}$ is impossible, since this implies $jR = R$, j a unit. Hence $jaR \subseteq aR$ for all a in R. Conversely if R is a right chain ring with $Ja \subseteq aR$ for all a in R we must show that for any nonzero elements x, y in R either $\tilde{x} \leqq \tilde{y}$ or $\tilde{y} \leqq \tilde{x}$. If we assume on the contrary that there exist a, b in R with $xaR \subset yaR$ and $ybR \subset xbR$ we obtain $xa = yav_1$, $yb = xbv_2$ and say $a = bs$ for v_1, v_2, s in J (the case $b = as$ is similar). Then $ya = ybs = xbv_2s = xbsv_2' = xav_2' = yav_1v_2'$ and $ya = 0$ where $v_2s = sv_2'$ for some v_2' in R, using (4).

The implication $(5) \Rightarrow (4)$ is obvious. To prove $(4) \Rightarrow (5)$ assume there is an a *in* R with $Ra \nsubseteq aR$ and $Ja \nsubseteq aJ$, but $Ja \subseteq aR$. Then there exist elements u in $U(R)$, n in J with $uaR \supset aR$ and $uan = a$; and elements n' in J, u' in $U(R)$ with $n'a$ in aR, but not in aJ, hence $n'au' = a$. This leads to $un'au'n = a$ and with $Ja \subseteq aR$ to $a = 0$, a contradiction. The equivalence of (1) and (6) follows from Lemma 1(2) and with this Theorem 1 is proved completely.

DEFINITION. A right chain ring R that satisfies the equivalent conditions of Theorem 1 is called *semi-invariant*.

Since $\tilde{H}(R)$ is not known even if R is right invariant unless R is also right noetherian or satisfies some other extra condition (see [1]) we cannot describe the structure of $\tilde{H}(R)$ for a semi-invariant ring R. It follows from Theorem 2 that this semigroup is a group of fractions of a semigroup $H = \tilde{H}(R')$ where R' is a right invariant right chain ring with respect to a subsemigroup T of H which satisfies

(1) If t is in T, h in H and e the unit element in H with $e \leqq h \leqq t$, then h is in T.

(2) For every $e \neq t$ in T there exist h and k in H with $th = hk$ and k not in T.

(3) $h_1t = h_2t$ for t in T, h_1, h_2 in H implies $h_1 = h_2$.

One sees that $\tilde{H}(R)$, R semi-invariant, not a division ring, is not a group, but we will show that for every totally ordered group G

there exists a semi-invariant ring R such that G can be embedded into $\tilde{H}(R)$.

4. Semi-invariant right chain rings with d.c.c. for prime ideals. Investigating the condition $\tilde{H}(R)$ totally ordered, we were led to semi-invariant right chain rings. The valuation semigroup can then be described using Theorem 2. In many cases we actually have $H(R) \cong \tilde{H}(R)$. The reason for this is the result we will prove in this section: Semi-invariant right chain rings with d.c.c. for prime ideals are right invariant. We recall that an ideal P in R is called completely prime if ab in P implies a or b in P and P is called prime if aRb in P implies a or b in P where a, b are elements in R. It follows from a result of Thierrin ([10]) that a prime ideal P is completely prime if a^2 in P implies a in P.

LEMMA 7. *Every prime ideal P in the semi-invariant ring R is completely prime.*

Proof. Assume a^2 in P and a not in P. Then there exists t_1 in R with at_1a not in P and t_2 in R with $at_2(at_1a)$ not in P. We can assume $R \neq P$ and a in J. Hence $a(t_2at_1)a = a^2r$ for some r in R using (4) of Theorem 1. This contradiction proves the lemma. □

The next result shows how to produce certain prime ideals.

LEMMA 8. *Let z be an element in R, a semi-invariant ring. Then $D = \bigcap z^nR$ is a prime ideal.*

Proof. We can assume that z is in J. Then D is a right ideal and we will first show that a^2 in D implies a in D for a in R. Assume a is not in D, then a is in J and $aj = z^n$ for some natural number n and j in J. But then $ajaj = a^2j'j = z^{2n}$ is not in D contradicting a^2 in D. It remains to prove that D is a left ideal. Let x be in D and $x = z^nq_n$, q_n in J follows. For r in R we get $rxrx = rxrz^nq_n = z^nvq_n$ for some v in R. This shows that $(rx)^2$ is in D and hence rx in D. □

The next theorem will be proved in three steps, Lemmas 9–11.

THEOREM 3. *A semi-invariant right chain ring with* d.c.c. *for ideals is right invariant.*

Let a be an element in the semi-invariant right chain ring R. By (5) Theorem 1 we have either $Ra \subseteq aR$ or $Ja \subseteq aJ$. In the first case we are done and in the second we define a mapping ϕ from the set of prime ideals $P \neq R$ into itself by defining P^{ϕ} as the smallest prime ideal with $Pa \subseteqq aP^{\phi}$. We will show that either $J^{\phi} = J$ which implies $Ra \subseteqq aR$ or $J^{\phi} \subset J$ and $\{J^{\phi^n}\}$ is a strictly decreasing chain of prime ideals of R.

LEMMA 9. *Let $J = J^{\phi}$ and $J = mR$, then $Ra \subseteqq aR$.*

Proof. We have $ma = am^k v$ for some unit v in R, some integer k, some generator m of J, since as a right ideal $J^{\phi} = J$ using Lemma 8.

If $Ra \not\subseteqq aR$ there exists a unit u in R and an element q in J with $ua = aq$. Since q is in J and $u^{k+1}a = aq^{k+1}$ we obtain $q^{k+1}R \subset m^k R$ and we can assume $qR \subset m^k R$ and $q = m^k vt$ with t in J. With $us = m$, $ma = am^k v$, $mat = am^k vt = aq = ua$ we obtain $sat = a$, s, t in J and $a = 0$ follows.

LEMMA 10. *Let R be semi-invariant, J not finitely generated as a right ideal and $0 \neq a$ an element in R with $Ja \subseteqq aJ^{\phi}$, $J^{\phi} = J$. Then $Ra \subseteqq aR$.*

Proof. Assume $j \neq 0$ in J. We want to find r, s in J with $ra = as$ and $sR \supseteqq jR$. Let $P = \cap j^n R$. By Lemma 8, P is a prime ideal and $P \subset J$. Since $J^{\phi} = J$ there exist elements r_1, s_1 in J with s_1 not in P such that $r_1 a = as_1$. Either $s_1 R \supseteqq jR$ and we are done or there exists an n with $jR \supset \cdots \supset j^{n-1}R \supset s_1 R \supseteqq j^n R$. Hence $s_1 q = j^n$ for some q in R. We choose an element z in J with $r_1 = z^m v$ with v in J and some $m > n$. This is possible, since J is not finitely generated: Let $r_1 R \subset xR \neq R$. We obtain $r_1 = xy$ for x, y in J. Choose z_1 in J with $z_1 R \supset xR$ and $z_1 R \supset yR$ and $r_1 = z_1^2 u_1$ follows with u_1 in J. Repeating this process yields an element z with $r_1 = z^m v$, z, v in J, $m > n$. Consider $za = az'$, z, z' in J. We claim $z'R \supseteqq jR$. Otherwise $jw = z'$ for some w in J. But $r_1 a = z^m va = az'^m v' = as_1$ for some element v' in J with $va = av'$.

Hence $s_1 = z'^m v' = (jw)^m v' = j^m bv'$ for some element b in R. This implies $j^n = s_1 q = j^m bv'q$, a contradiction, since $m > n$. We conclude that we have found an element $r = z$, $s = z'$ with $sR \supseteqq jR$ and $ra = as$ for the given element j in J.

If $Ra \not\subseteqq aR$ there exist a unit u in R and an element t in J with $ua = at$. By the above argument we have s, r in J with $ra = as$ and $sR \supset tR$. Hence, $sv = t$ for some v in J and $rav = asv = at = ua$. We obtain $a = u^{-1}rav = ak$, k in J and $a = 0$, a contradiction.

REMARK. Under the hypothesis of Lemma 10 we have proved that $J^\phi = J$ is even the smallest two-sided ideal I satisfying $Ja \subseteqq aI$.

LEMMA 11. *Let R be semi-invariant, a in R with $Ja \subseteqq aJ^\phi$ and $J^\phi \subset J$. Then $J^{\phi^{n+1}} \subset J^\phi$ for all n.*

Proof. We will write $J^{(n)}$ instead of J^{ϕ^n}. Then $J^{(n+1)} \subseteqq J^{(n)}$ and we assume n minimal with $J^{(n)} = J^{(n+1)}$. Let r be in $J^{(n-1)} \backslash J^{(n)}$, $ra = as$ with s in $J^{(n)}$. Then there exists a q in $J^{(n)}$ with $qa = aq'$ and $q'R \supset s^kR$ for some k, since otherwise $J^{(n+1)} = J^{(n)} \subseteqq \bigcap s^iR \subset J^{(n)}$. After replacing r by r^k if $k > 1$ we can assume that there is an r in $J^{(n-1)} \backslash J^{(n)}$ with $ra = as$ and an element q in $J^{(n)}$ with $qa = aq'$ and $q'R \supset sR$. Hence $q't = s$ for some t in J and $rv = q$ for some v in $J^{(n)}$. This yields $ra = as = aq't = qat = rvat = rav't$ with v' in J and the contradiction $ra = 0$ proves the lemma. □

5. Examples, problems and comments.

We begin with an example of a semi-invariant right chain ring R such that $\tilde{H}(R)$ contains G where G is a given totally ordered group.

EXAMPLE 1. For very totally ordered group G there exists a semi-invariant right chain ring R such that $\tilde{H}(R)$ contains G.

Let $K = \bigoplus_{i \in \boldsymbol{Z}} G_i$ where $G_i \simeq G$ for all $i \in \boldsymbol{Z}$. K is an ordered group with the lexicographic ordering. Next, let $L = \{t^n k \mid n \in \boldsymbol{Z}, k \in K\}$ with $t^n k_1 \cdot t^m k_2 = t^{n+m}(k_1^{(m)} k_2)$ be the ordered group where $k = (g_i)$ and $k^{(m)} = (g_i')$ with $g_i' = g_{i+m}$. Further $t^n k_1 > t^m k_2$ if and only if $n > m$ or $n = m$ and $k_1 > k_2$ in K.

Let $H = \{t^n k \in L \mid t^n k \geqq e, k = (g_i)$ with $n \geqq 0$ and $g_i = 1_{G_i}$ for $i > 0\}$. Then H is a totally ordered semigroup with unit element and both cancellation laws. Further, H is naturally ordered in the sense that $h_1 \geqq h_2$ for h_i in H holds if and only if there exists an element $h \geqq e$ in H with $h_1 = h_2 h$. Therefore it is possible to construct the generalized power series ring.

$$R' = \{\alpha = \sum x_h a_h \mid h \in H,\ a_h \in \boldsymbol{R} \text{ and } T(\alpha) = \{h \mid a_h \neq 0\} \text{ well ordered in } H\}\,.$$

R' is a right invariant right chain ring with $\tilde{H}(R') \simeq H$ ([7]).

To the subsemigroup $M = \{t^0(g_i) \mid g_i = 1_{G_i}$ for $i \neq 0\}$ there corresponds an R'-convex subsemigroup in $\tilde{H}(R')$ and a prime ideal P in R'. We put $R'_P = R$. Since for h in M we have $ht = th'$ with h' not in M unless $h = 1$, we conclude that $\tilde{H}(R) \simeq HM^{-1} = H \cup M^{-1}$. It follows that G can be embedded into $\tilde{H}(R)$ where R is a semi-invariant right chain ring. We observe that the right ideal $x_t R$ is not a left ideal and Rx_t is not a right ideal. On the other hand we know ([2]) that for every a in a semi-invariant right and left chain ring either aR or Ra is a two-sided ideal.

EXAMPLE 2. In our next example we construct a right chain ring R such that $\tilde{H}(R)$ is not totally ordered, but that the subgroup $\tilde{U}(R) = \{\tilde{u} \mid u \text{ in } U(R)\}$ of $\tilde{H}(R)$ is totally ordered with respect to the order as defined in $\tilde{H}(R)$. This condition

$$(U)\ \tilde{U}(R) \text{ is totally ordered}$$

is therefore weaker than the condition $\tilde{H}(R)$ totally ordered and implies among other things that for a right chain ring R with (U), a in R, there exists a unit ε in $U(R)$ with $a\varepsilon$ in $\hat{R}$ (see Lemma 12 (ii) below). The basic idea of this construction has been used in [9], [2] and [6]: Let R_1 be a right and left chain ring, $D = Q(R_1)$ the division ring of quotients of R_1, H a totally ordered semigroup with unit element that satisfies both cancellation laws. Further, let $h_1 \geqq h_2$ hold for elements h_1, h_2 in H if and only if $h_1 = h_2 h$ for some h in H. Finally, let τ be a mapping from H into the semigroup $M(D)$ of monomorphism from D to D with $\tau(h_1h_2) = \tau(h_1)\tau(h_2)$. One then can form the generalized power series ring $D\{\{H\}\} = \{\sum x_h d_h = \alpha \mid h$ in H, d_h in D, $T(\alpha) = \{h \mid d_h \neq 0\}$ well ordered in $H\}$ where multiplication is defined by $x_{h_1}x_{h_2} = x_{h_1h_2}$ and $dx_h = x_h d^{\tau(h)}$. The subring R of $D\{\{H\}\}$ consisting of those elements α with d_e in R_1 is a right chain ring where e is the unit element in H. It does not seem to be easy to determine $\tilde{H}(R)$ in general.

To consider a special case let $F = Q(x, y)$, the field of rational functions in the two indeterminates x and y over the field Q of rational numbers. Then F contains $R_1 = Q[x, y]_{(x)}$, a chain ring one obtains by localizing the polynomial ring $Q[x, y]$ at the prime ideal (x). We form the skew power series ring $F[[t, \tau]]$, where τ is the automorphism of F exchanging x and y. Finally, R consists of all those power series $\sum t^i f_i(x, y)$ with $f_0(x, y)$ in R_1. The principal right ideals of R are of the form $t^n x^m R$ with $n = 0, 1, 2, \cdots$ and m in $\boldsymbol{Z}$, but $m \geqq 0$ if $n = 0$. The semigroup $\tilde{H}(R) = \{\widetilde{t^n x^m y^k} \mid n = 0, 1, 2, \cdots; m, k \text{ in } \boldsymbol{Z} \text{ and } m \geqq 0 \text{ if } n = 0\}$. It is $\widetilde{t^{n_1}x^{m_1}y^{k_1}} > \widetilde{t^{n_2}x^{m_2}y^{k_2}}$ if $n_1 > n_2$ or $n_1 = n_2$ and $m_1 > m_2$ with $k_1 \geqq k_2$ or $n_1 = n_2$, and $m_1 = m_2$ and $k_1 > k_2$. Finally, we have $\tilde{U}(R) = \{\tilde{y}^k, k \in \boldsymbol{Z}\} \cong \boldsymbol{Z}$ as ordered groups. Therefore, $\tilde{H}(R)$ satisfies condition (U), but is not totally ordered: $\tilde{x}\tilde{y}^{-1}$ and $\tilde{1}$ for example cannot be compared.

We conclude this paper with some observation for right chain rings that satisfy condition (U).

LEMMA 12. *Let R be a ring satisfying condition U.*

(i) *Let a, b in R with $aR = bR$. Then either $\tilde{a} \leqq \tilde{b}$ or $\tilde{b} < \tilde{a}$.*

(ii) *For any a in R, R local, exists x in $\hat{R}$ with $aR = xR$.*

(iii) *Let R be a local ring and $aR \supset bR$. Then there exists for every x with $xR = aR$ a y in R with $\tilde{x} < \tilde{y}$ and $yR = bR$. Similarly for every y in R with $yR = bR$ exists x with $xR = aR$ and $\tilde{x} < \tilde{y}$.*

Proof. (i) is obvious, using condition (U). Statement (ii) is correct if a is a unit. We can therefore assume a in J, a not in $\hat{R}$. Hence $1 + a$ is in $U(R)\backslash\hat{R}$ and $(1 + a)(1 + x) = (1 + x)(1 + a) = 1$ for some x in R. But $1 + x$ and x are in $\hat{R}$ and $a(1 + x) = (1 + x)a = -x$ is in $\hat{R}$. Since $aR = xR$, (ii) follows.

To prove (iii) assume $b = xp$. Using (ii) there exists a unit u in R with pu in $\hat{R}$ and $bu = xpu$ implies $\tilde{x} < \widetilde{bu}$. If $y = ap$ the second part of (iii) is correct for p in $\hat{R}$. Otherwise we obtain with (ii): $(1 + p)^{-1}p$ is in $\hat{R}$, $y = a(1 + p)(1 + p)^{-1}p$ and $x = a(1 + p)$. □

PROBLEMS.

(1) Describe all rings R for which $\tilde{H}(R)$ satisfies (U). (This class of rings contains all right invariant, in particular all commutative rings.)

(2) Which conditions characterize the semigroups S with $S \cong \tilde{H}(R)$, R a ring or additionally: R a right chain ring.

(3) Find the class of rings R with $\tilde{H}(R)$ lattice ordered.

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