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STRICT LOCAL INCLUSION RESULTS BETWEEN SPACES OF FOURIER TRANSFORMS

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Let G denote a noncompact Hausdorff locally compact abelian group, Γ its character group, and write $(L^s, l^t)^\wedge$ for the space of Fourier transforms of functions in the amalgam (L^s, l^t). We show that for $1 \leqq p < q \leqq \infty$ the local inclusion $(L^1, l^p)^\wedge \overset{\text{loc}}{\subset} (L^\infty, l^q)^\wedge$ is strict, that is, given any nonvoid open subset Ω of Γ there exists $f \in (L^\infty, l^q)$ such that $\hat{f} - \hat{g}$ does not vanish on Ω for any $g \in (L^1, l^p)$. If in addition G is assumed to be second countable then we show there exists such an f independent of the choice of Ω. Of special interest is the case, included in the above results, where the amalgams (L^1, l^q), (L^∞, l^p) are replaced by $L^p(G)$, $L^q(G)$ respectively.

Throughout G will denote a noncompact Hausdorff locally compact abelian group, with Haar measure λ and character group Γ. $A(\Gamma)$ will denote the space of Fourier transforms of functions integrable over G, and $A_c(\Gamma)$ the subspace formed of functions whose supports are compact. For each compact set $\Xi \subset \Gamma$, write $A_\Xi(\Gamma) = \{h \in A(\Gamma): \operatorname{supp}(h) \subset \Xi\}$. We give each space $A_\Xi(\Gamma)$ its normed topology as $\|\hat{f}\| = \|f\|_1$, $\hat{f} \in A_\Xi(\Gamma)$, and topologise $A_c(\Gamma)$ as the internal inductive limit of the spaces $A_\Xi(\Gamma)$.

For each $s, t \in [1, \infty]$ the amalgam (L^s, l^t) is defined in the following way. Using the structure theorem ([6], (24.30)) we write $G = R^a \times G_0$, where a is a nonnegative integer and G_0 contains a compact open subgroup H. We put $J = Z^a \times G_0/H$, $L = [0, 1)^a \times H$ and write G as the disjoint union $\bigcup_\alpha L_\alpha$ where, for each $\alpha = (n_1, \cdots, n_a, y+H) \in J$, $L_\alpha = (n_1, \cdots, n_a, y) + L$. Given $f \in L^s_{\text{loc}}(G)$ write

$$\|f\|_{s,t} = \left(\sum_\alpha \left(\int_{L_\alpha} |f|^s d\lambda\right)^{t/s}\right)^{1/t},$$

with the usual modification if $\max\{s, t\} = \infty$, and

$$(L^s, l^t) = \{f \in L^s_{\text{loc}}(G): \|f\|_{s,t} < \infty\}.$$

Each amalgam (L^s, l^t) is a Banach space and, provided $s, t < \infty$, its dual space is isometrically isomorphic to $(L^{s'}, l^{t'})$ (where s', t' denote the indices conjugate to s, t respectively); for this and other results on amalgams see [9], § 3.

Here we give three results for amalgams, which will be referred to in the sequel.

THEOREM A. *The translation operators* τ_b *defined by* $\tau_b f\colon x \to f(x-b)$ *are uniformly bounded on* (L^s, l^t) *for all* $s, t \in [1, \infty]$.

THEOREM B. *For any compact set* $\Xi \subset \Gamma$ *there exists* $k \in (L^\infty, l^1)$ *with* $\hat{k} = 1$ *on* Ξ.

THEOREM C. *Let* $f \in (L^{p_1}, l^{p_2})$, $g \in (L^{q_1}, l^{q_2})$ *and suppose that* $1/r_i = 1/p_i + 1/q_i - 1 \geqq 0$ *for* $i = 1, 2$. *Then* $f * g \in (L^{r_1}, l^{r_2})$ *and* $\|f*g\|_{r_1, r_2} \leqq K\|f\|_{p_1,p_2}\|g\|_{q_1,q_2}$, *where* K *is a constant.*

The proof of Theorem 3.3 in [9] applies equally well to give Theorem A, and Theorem B is just [9], Theorem 3.1. Theorem C follows from Theorem A and the results of [1], § 7(i).

For each $f \in (L^1, l^\infty)$ we define the Fourier transform $\hat{f}$ as the continuous linear functional on $A_c(\Gamma)$ given by

$$\hat{f}(h) = f(\hat{h}_\vee),\ h \in A_c(\Gamma)\ ,$$

where $\hat{h}_\vee$ is the reflection of the inverse Fourier transform of h and $f(g) = \int_G fg d\lambda$ (for a similar definition of the Fourier transform see Bertrandias and Dupuis ([2], § 4(a))). That $\hat{f}$ is linear is clear, and continuity can be shown as follows. Since $A_c(\Gamma)$ is the inductive limit of the spaces $A_\Xi(\Gamma)$ we need only prove that $\hat{f}$ is continuous on each $A_\Xi(\Gamma)$. First note that if $h \in A_\Xi(\Gamma)$ then $\hat{h}_\vee = k * \hat{h}_\vee \in (L^\infty, l^1)$ by Theorem C, where $k \in (L^\infty, l^1)$ is chosen as in Theorem B with $\hat{k} = 1$ on $-\Xi$. Hence

$$\begin{aligned} |\hat{f}(h)| = |f(\hat{h}_\vee)| = \left|\int_G f\hat{h}_\vee d\lambda\right| &\leqq \sum_\alpha \int_{L_\alpha} |f\hat{h}_\vee| d\lambda \\ &\leqq \sum_\alpha \left(\int_{L_\alpha} |f| d\lambda \max_{L_\alpha} |\hat{h}_\vee|\right) \\ &\leqq \left(\max_\alpha \int_{L_\alpha} |f| d\lambda\right) \sum_\alpha \max_{L_\alpha} |\hat{h}_\vee| \\ &= \|f\|_{1,\infty}\|\hat{h}_\vee\|_{\infty,1} \\ &\leqq K\|f\|_{1,\infty}\|k\|_{\infty,1}\|\hat{h}_\vee\|_1 \\ &= K\|f\|_{1,\infty}\|k\|_{\infty,1}\|h\|\ , \end{aligned}$$

so that $\hat{f}$ is bounded, hence continuous, on $A_\Xi(\Gamma)$.

It follows from [3], Theorem 3.1 (and the remarks at the beginning of [3], § 3) that this definition of the Fourier transform agrees with that given by Gaudry in [5], § 1 (who defines the Fourier transform as a suitable quasimeasure) for functions in $L^p(G)$, and in particular with that usually taken when $1 < p \leqq 2$. It will also be convenient to think of the Fourier transform of $f \in L^1(G)$ as a linear functional on $A_c(\Gamma)$. We write $(L^s, l^t)^\wedge = \{\hat{f}: f \in (L^s, l^t)\}$.

Let Ω be a nonvoid open subset of Γ. We say that $\hat{f}$ vanishes on Ω if $\hat{f}(h) = 0$ for all $h \in A_c(\Gamma)$ with $\operatorname{supp}(h) \subset \Omega$. Suppose that $1 \leqq p \leqq q \leqq \infty$, $g \in (L^1, l^p)$, and take Ω to be any nonvoid relatively compact open subset of Γ. By Theorem B we have the existence of $k \in (L^\infty, l^1)$ with $\hat{k} = 1$ on Ω. Writing $f = k*g$ it is seen that $f \in (L^\infty, l^q)$ and $\hat{f} - \hat{g}$ vanishes on Ω. Thus we have that $(L^1, l^p)^\wedge \overset{\text{loc}}{\subset} (L^\infty, l^2)^\wedge$, where the inclusion is to be interpreted as holding locally. This has already been observed for the Lebesgue spaces when G is the real line; see [8], Chapter VI, § 4.12.

Our main result (Theorem 1), which we prove with the aid of an extension to amalgams of a multiplier theorem of Hörmander, implies that this local inclusion is strict whenever $p < q$. In the case $q \in (1, 2]$, with the amalgams (L^1, l^p), (L^∞, l^q) replaced by $L^p(G)$, $L^q(G)$ respectively, this result has been given previously by Fournier ([4], Theorem 1) with Ω only required to be a set with positive Haar measure. Fournier's proof is based on the construction of certain positive definite functions on Γ.

THEOREM 1. *Let $1 < q \leqq \infty$ and suppose Ω is a nonvoid open subset of Γ. Then there exists $f \in (L^\infty, l^q)$ such that for any $p \in [1, q)$ and $g \in (L^1, l^p)$, $\hat{f} - \hat{g}$ does not vanish on Ω.*

Proof. We first show that if $h \in L^1(G)$ with $h \geqq 0$, $h \neq 0$, then there exists nonnegative $f \in (L^\infty, l^q)$ such that $h*f \notin (L^1, l^p)$ for any $p \in [1, q)$. This is easy to see if $q = \infty$, since in this case $f = 1$ satisfies the stated conditions.

For $q < \infty$ fix $p \in [1, q)$ and assume that $h*f \in (L^1, l^p)$ for all $f \in (L^\infty, l^q)$. Consider the map T, defined on (L^∞, l^q) by $Tf = h*f$. By assumption T maps (L^∞, l^q) into (L^1, l^p). T is obviously linear and commutes with translations. Furthermore the Closed Graph Theorem shows that T is continuous. Now the proof of Hörmander's theorem ([7], Theorem 1.1), which holds for all noncompact locally compact abelian groups, can be modified to show that $T = 0$. This is clearly impossible if h is nonzero, so there exists $f \in (L^\infty, l^q)$ such that $h*f \notin (L^1, l^p)$. Since $h \geqq 0$ the same is true if f is replaced by $|f|$. Now let (p_n), $p_n \geqq 1$, be any strictly increasing sequence of numbers converging to q. Choose a corresponding sequence (f_n) of nonnegative functions in (L^∞, l^q) such that for each $n \in \{1, 2, \cdots\}$, $h*f_n \notin (L^1, l^{p_n})$. We assert that

$$f = \sum_{n=1}^{\infty} n^{-2} \|f_n\|_{\infty,q}^{-1} f_n$$

is a suitable choice of f; indeed, if there exists $p \in [1, q)$ such that $h*f \in (L^1, l^p)$ then, choosing n_0 such that $p_{n_0} \in [p, q)$, we would have (recall that for each n, $h*f_n \geqq 0$)

$$h*f_{n_0} \in (L^1, l^p) \subset (L^1, l^{p_{n_0}}) ,$$

contradicting the choice of f_{n_0}.

Now choose $\gamma \in \Omega$ and nonzero $h \in A_c(\Gamma)$ such that $\operatorname{supp}(h) \subset -\gamma + \Omega$ and $\hat{h} \geqq 0$ (this is possible using [6], (31.34) and the fact that $-\gamma + \Omega$ is a neighbourhood of 0). From the first part of the proof there exists $f_0 \in (L^\infty, l^q)$ such that $\hat{h}*f_0 \notin (L^1, l^p)$ for any $p \in [1, q)$. Then $f = \gamma f_0$ satisfies the conditions of the theorem, for if there exists $p \in [1, q)$ and $g \in (L^1, l^p)$ such that $\hat{f} - \hat{g}$ vanishes on Ω then, since $(\bar{\gamma} f - \bar{\gamma} g)^\wedge$ vanishes on $-\gamma + \Omega$, we would have

$$\hat{h}*(\bar{\gamma} f - \bar{\gamma} g)(x) = (\bar{\gamma} f - \bar{\gamma} g)^\wedge(xh) = 0$$

for all $x \in G$ (where $(xh)(\gamma) = \gamma(x)h(\gamma)$). But this gives $\hat{h}*f_0 = \hat{h}*\bar{\gamma} g \in (L^1, l^p)$, a contradiction of our choice of f_0. □

In the case where G is second countable f in Theorem 1 can be chosen independently of the nonvoid open set Ω.

THEOREM 2. *Let G be a second countable noncompact locally compact abelian group. If $1 \leqq p < q \leqq \infty$ then there exists $f \in (L^\infty, l^q)$ such that, for any nonvoid open set $\Omega \subset \Gamma$, there is no $g \in (L^1, l^p)$ for which $\hat{f} - \hat{g}$ vanishes on Ω.*

Proof. Since G is second countable so is Γ (see [6], (24.14)). Suppose to the contrary that no such f exists when g is restricted to lie in $L^p(G)$. We consider $p > 1$ and make use of Baire's category theorem to derive a contradiction.

For each pair of positive integers m, n define $T_m(\Omega_n) = \{f \in (L^\infty, l^q)$: $\hat{f} - \hat{g}$ vanishes on Ω_n for some $g \in L^p(G)$, $\|g\|_p \leqq m\}$, where $\{\Omega_n: n = 1, 2, \cdots\}$ is a base for the topology of Γ with each Ω_n nonvoid. Our assumption in the previous paragraph just says that $\bigcup_{m,n=1}^\infty T_m(\Omega_n) = (L^\infty, l^q)$. We shall show that for each $m, n \in \{1, 2, \cdots\}$, $T_m(\Omega_n)$ is closed.

Let (f_s) be a sequence of functions in $T_m(\Omega_n)$ converging in (L^∞, l^q) to f, say. Now for each $s \in \{1, 2, \cdots\}$ there exists $g_s \in L^p(G)$ such that $\|g_s\|_p \leqq m$ and $\hat{f}_s - \hat{g}_s$ vanishes on Ω_n. Using the theorem of Alaoglu we can deduce the existence of $g \in L^p(G)$ with $\|g\|_p \leqq m$, a weak*-cluster point of the sequence (g_s).

Now let $\varepsilon > 0$ and $h \in A_c(\Gamma)$ with $\operatorname{supp}(h) \subset \Omega_n$ be given. Choose s such that $|g(\hat{h}_\vee) - g_s(\hat{h}_\vee)| < \varepsilon/2$ and $|f(\hat{h}_\vee) - f_s(\hat{h}_\vee)| < \varepsilon/2$ (note that $\hat{h}_\vee \in (L^\infty, l^1)$). Then

$$\begin{aligned} |(\hat{f} - \hat{g})(h)| &\leqq |f(\hat{h}_\vee) - f_s(\hat{h}_\vee)| + |f_s(\hat{h}_\vee) - g_s(\hat{h}_\vee)| \\ &\quad + |g_s(\hat{h}_\vee) - g(\hat{h}_\vee)| \\ &< \varepsilon + |(\hat{f}_s - \hat{g}_s)(h)| . \end{aligned}$$

But $\hat{f}_s - \hat{g}_s$ vanishes on Ω_n and thus $|(\hat{f} - \hat{g})(h)| < \varepsilon$. Since $\varepsilon > 0$ and $h \in A_c(\Gamma)$ with $\operatorname{supp}(h) \subset \Omega_n$ were chosen arbitrarily we deduce that $\hat{f} - \hat{g}$ vanishes on Ω_n, so that $f \in T_m(\Omega_n)$. Hence $T_m(\Omega_n)$ is closed.

Now (L^∞, l^q) is a complete metric space and thus we can apply Baire's category theorem which gives us the existence of positive integers m_0, n_0 such that $T_{m_0}(\Omega_{n_0})$ has nonvoid interior. This means we can find $\delta > 0$ and $f_0 \in T_{m_0}(\Omega_{n_0})$ such that

$$V = \{f \in (L^\infty, l^q) \colon \|f - f_0\|_{\infty,q} < \delta\} \subset T_{m_0}(\Omega_{n_0}) .$$

Let $k \in (L^\infty, l^q)$ and choose nonzero α such that $\|\alpha k\|_{\infty,q} < \delta$. Then f_0, $\alpha k + f_0$ belong to V and so there exist $g_0, g_1 \in L^p(G)$ with $\hat{f}_0 - \hat{g}_0$ and $(\alpha k + f_0)\hat{} - \hat{g}_1$ vanishing on Ω_{n_0}. The linearity of the Fourier transform entails that

$$\hat{k} - \alpha^{-1}(g_1 - g_0)\hat{} = \alpha^{-1}((\alpha k + f_0 - g_1)\hat{} - (f_0 - g_0)\hat{})$$

vanishes on Ω_{n_0}. But $\alpha^{-1}(g_1 - g_0) \in L^p(G)$; since $k \in (L^\infty, l^q)$ was chosen arbitrarily we have a contradiction of Theorem 1.

Hence our initial assumption was false and, for $p_0 \in (p, q)$ with p, q as in the statement of the theorem, we have the existence of $f \in (L^\infty, l^q)$ such that, for any nonvoid open set $\Omega \subset \Gamma$, there is no $g \in L^{p_0}(G)$ for which $\hat{f} - \hat{g}$ vanishes on Ω. Then the same is true with $L^{p_0}(G)$ replaced by (L^1, l^p). For suppose to the contrary that $\Omega \subset \Gamma$ and $g \in (L^1, l^p)$ exist such that $\hat{f} - \hat{g}$ vanishes on Ω; without loss of generality we may assume that Ω is relatively compact. Then, choosing $k \in (L^{p_0}, l^1)$ with $\hat{k} = 1$ on Ω, we have that

$$\hat{f} - (k*g)\hat{} = (\hat{f} - \hat{g}) + (\hat{g} - (k*g)\hat{})$$

vanishes on Ω and $k*g \in L^{p_0}(G)$, contradicting our choice of f. This completes the proof of the theorem. □

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