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CLASSIFICATION OF THE C^*-ALGEBRAS ASSOCIATED WITH MINIMAL ROTATIONS

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It is shown that the set of eigenvalues of a minimal rotation on a compact group is a complete invariant for the associated C^*-algebra.

1. Introduction. The works of Rieffel [7] and Pimsner, Voiculescu [5] taken together give a complete classification of the C^*-algebras associated with irrational rotations on the one dimensional torus. Our purpose in the present article is to generalize this result to the C^*-algebras which are associated with arbitrary minimal rotations on compact abelian metric groups. Each of these C^*-algebras is simple and it has a unique tracial state. Moreover, we shall see that the weighted shift algebras considered in [1] are contained in the class of minimal rotation C^*-algebras.

Speaking more precisely, we will show that the set of eigenvalues of a minimal rotation is a complete invariant for the C^*-algebra associated with it. In our proof of this fact we will use the results of Pimsner, Voiculescu and Rieffel.

2. Let F be a compact abelian group and let ρ be an arbitrary element in F. We denote by R_ρ the rotation associated with ρ, i.e., $R_\rho x = \rho x$ for each $x \in F$. Then R_ρ is a minimal homeomorphism and it is called a minimal rotation if and only if the cyclic group generated by ρ is dense in F (see [8], p. 121, Example (ii)). Moreover it follows from the duality theory for abelian groups that the group generated by ρ is dense in F if and only if the character χ_ρ of the dual group $\hat{F}$ of F which is associated with ρ is faithful. If R_ρ is minimal, then this shows that R_ρ is uniquely determined by the subgroup $\chi_\rho(\hat{F})$ of the torus $\boldsymbol{T}$. The group $\chi_\rho(\hat{F})$ is countable if and only if F is metrizable. Conversely, we can associate with each subgroup G of $\boldsymbol{T}$ a minimal rotation on the compact group $\hat{G}$. Namely, let $\rho_G\colon G \to \boldsymbol{T}$ be the inclusion mapping of G. Then ρ_G is an element of $\hat{G}$ and R_{ρ_G} is a minimal rotation. In the sequel we shall write R_G instead of R_{ρ_G}.

Now we assume that G is a countable subgroup of $\boldsymbol{T}$. We consider the automorphism $\bar{R}_G$ of the C^*-algebra $C(\hat{G})$ of all continuous functions on $\hat{G}$ with values $\boldsymbol{C}$, which is induced by R_G. Let μ be the normalized Haar measure on $\hat{G}$. We denote by $\mathscr{A}_G$ the C^*-algebra which is generated by the multiplication operators on $L^2(\mu)$ which are associated with continuous functions and by the unitary operator

U_G on $L^2(\mu)$ which is induced by $\bar{R}_G$. For each $\lambda \in G$ we denote by $\pi_G(\lambda)$ the unitary operator on $L^2(\mu)$ which is determined by the character of $\hat{G}$ associated with λ. Then for any $\lambda \in G$ the equality $U_G\pi_G(\lambda) = \lambda\pi_G(\lambda)U_G$ holds. We shall see later that these relations determine the C^*-algebra $\mathscr{A}_G$ already up to isomorphisms. First we note the following.

PROPOSITION 2.1. *For any countable subgroup G of $\boldsymbol{T}$ the C^*-algebra $\mathscr{A}_G$ is simple and $\mathscr{A}_G$ has a unique tracial state.*

Proof. Since R_G is a minimal homeomorphism it follows from [6] that $\mathscr{A}_G$ is simple. Moreover, it can be seen by direct calculations that $\mathscr{A}_G$ has a unique tracial state.

REMARK. Suppose that X_1, X_2 are compact metric spaces and S_1, S_2 are minimal homeomorphisms of X_1, X_2 respectively, both having topological discrete spectrum, i.e., the set of eigenfunctions of S_i, which is given by $\{f \in C(X_i)/f \circ S_i = \lambda f \text{ for some } \lambda \in \boldsymbol{T}\}$, is total in $C(X_i)$ for $i = 1, 2$. An element $\lambda \in \boldsymbol{T}$ satisfying $f \circ S_i = \lambda f$ for some $f \in C(X_i)$ is called an eigenvalue for S_i $(i = 1, 2)$. S_1 is called topologically conjugate to S_2 if there exists a homeomorphism Φ from X_1 onto X_2 such that $\Phi \circ S_1 = S_2 \circ \Phi$. By [8], 5.9 S_1 *is topologically conjugate to* S_2 if and only if S_1 and S_2 have the same eigenvalues. Since the C^*-algebras associated with topologically conjugate homeomorphisms are isomorphic and since the eigenvalues of R_G as defined above are exactly the elements of G, this shows that the C^*-algebras $\mathscr{A}_G$ are already the most general ones which may occur within our framework.

We have the following characterization of the minimal rotation algebras.

PROPOSITION 2.2. *Let G be an infinite countable subgroup of $\boldsymbol{T}$. Suppose that π is a homomorphism from G into the unitary group $U(\mathscr{A})$ of a C^*-algebra $\mathscr{A}$, and U is an element in $U(\mathscr{A})$ such that (2.2.1) $U\pi(\lambda)U^* = \lambda\pi(\lambda)$ holds for each $\lambda \in G$.*

Then π extends uniquely to an isomorphism from $\mathscr{A}_G$ onto the C^*-algebra generated by U and $\{\pi(\lambda)\}_{\lambda \in G}$ such that U is the image of U_G. (G will be considered to be identified with the group of characters of $\hat{G}$. Thus we have $G \subseteq C(\hat{G})$.)

Proof. Since $C(\hat{G})$ is the enveloping C^*-algebra of the group G, the homomorphism π extends in a unique manner to a homomorphism $\bar{\pi}$ of the C^*-algebra $C(\hat{G})$. The relations in (2.2.1) show that the homomorphism $\bar{\pi}$ of $C(\hat{G})$ and the unitary representation $n \mapsto U^n$ of $\boldsymbol{Z}$ give rise to a covariant representation of the C^*-dynamical system associated with $\bar{R}_G$. This covariant representation induces a homomorphism π_0 from $\mathscr{A}_G$ into $\mathscr{A}$ (see [4], 7.6.6, 7.7.5). Since $\mathscr{A}_G$ is a simple C^*-algebra and the homomorphism π_0 is nontrivial, π_0 is an isomorphism from $\mathscr{A}_G$ onto the C^*-algebra generated by U and $\{\pi(\lambda)\}_{\lambda \in G}$.

NOTATIONS. (a) For any subgroup G of $\boldsymbol{T}$ we denote by $G^{\uparrow}$ the subgroup $\{t \in \boldsymbol{R} / \exp(2\pi i t) \in G\}$ of $\boldsymbol{R}$. For any subgroup G of $\boldsymbol{R}$ we denote by $G^{\downarrow}$ the subgroup $\{\exp(2\pi i t) / t \in G\}$ of $\boldsymbol{T}$.

(b) Suppose that $\mathscr{A}$ is a C^*-algebra with a tracial state τ. Then τ induces a natural homomorphism $\hat{\tau}$ from the group $K_0(\mathscr{A})$ into $\boldsymbol{R}$ (see [2], §8). We denote by $D_\tau(\mathscr{A})$ the image of $K_0(\mathscr{A})$ with respect to the homomorphism $\hat{\tau}$. If $\mathscr{A}$ admits exactly one tracial state then we will also write $D(\mathscr{A})$ instead of $D_\tau(\mathscr{A})$.

EXAMPLES. (1) Suppose that G is an infinite cyclic subgroup of $\boldsymbol{T}$. Then there exists an irrational number α such that $G^{\uparrow} = \boldsymbol{Z} + \alpha \boldsymbol{Z}$. $\mathscr{A}_G$ is an irrational rotation algebra, as considered in [5], [7]. We will also write $\mathscr{A}_\alpha$ instead of $\mathscr{A}_G$. It follows from [5] and [7] that $D(\mathscr{A}_\alpha) = \boldsymbol{Z} + \alpha \boldsymbol{Z}$ holds.

(2) If G is a finite subgroup of $\boldsymbol{T}$ then G is cyclic. $\mathscr{A}_G$ is seen to be isomorphic to the C^*-algebra M_n of all $n \times n$-matrices, where n is the order of G.

(3) If G is an infinite torsion subgroup of $\boldsymbol{T}$ then $\mathscr{A}_G$ is a weighted shift algebra as considered by Bunce and Deddens in [1]. Conversely, each weighted shift algebra arises in this manner.

Indeed, suppose that G is an infinite torsion subgroup of $\boldsymbol{T}$ and $\mathscr{H}$ is a separable Hilbert space with an orthonormal basis $\{e_n\}_{n \in \boldsymbol{N}}$. Let S be the corresponding unilateral shift, i.e., S is given by

$$Se_n = e_{n+1}, \qquad n \in \boldsymbol{N}.$$

For each $\lambda \in G$ let D_λ be the diagonal operator which is given by

$$D_\lambda e_n = \lambda^n e_n, \qquad n \in \boldsymbol{N}.$$

Let $\mathscr{K}$ be the closed ideal of compact operators on $\mathscr{H}$ and let ν be the canonical mapping from $\mathscr{B}(\mathscr{H})$ into the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. If $\lambda \in G$ is a primitive kth root of unity then the C^*-algebra generated by S and D_λ is equal to the C^*-algebra generated by all weighted shifts of aeriod k (see [1]). Thus the C^*-algebra $\mathscr{B}$ generated by $\nu(S)$ and $\nu(D_\lambda)$ for all $\lambda \in G$ is a weighted shift algebra in the sense of [1]. Moreover, since the mapping $\lambda \mapsto \nu(D_\lambda)$

is a homomorphism from G into $U(\mathscr{B})$ satisfying the identity $\nu(S)^*\nu(D_\lambda)\nu(S) = \lambda\nu(D_\lambda)$ for each $\lambda \in G$, we infer from 2.2 that $\mathscr{B}$ is isomorphic to $\mathscr{A}_G$. It is clear that each weighted shift algebra arises in this manner.

The examples considered above have in common that the equality $D(\mathscr{A}_G) = G^\dagger$ holds for the corresponding groups. In the next section we will show that this equality is true in the general case also. Finally, let us mention that $\mathscr{A}_G$ is not an AF-algebra of G is an infinite subgroup of $\boldsymbol{T}$. This follows from [3], 6.6. In case G is an infinite torsion group this was shown in [1].

3. The proof of our main result will be done by showing that for each countable subgroup G of $\boldsymbol{T}$ the group $G^\dagger$ is the intersection of certain subgroups of $\boldsymbol{R}$ of the form $D(\mathscr{A})$, where each $\mathscr{A}$ is a suitable C^*-algebra into which $\mathscr{A}_G$ can be embedded. We start with some preliminary results.

LEMMA 3.1. *Suppose that $\mathscr{A}$ is a C^*-algebra with a tracial state ψ and $\mathscr{A}_1 \subseteqq \mathscr{A}_2 \subseteqq \cdots \subseteqq \mathscr{A}$ is a monotonely increasing sequence of C^*-subalgebras of $\mathscr{A}$ such that $\mathscr{A}$ is the norm closure of $\bigcup_{n=1}^\infty \mathscr{A}_n$. For each $n \in \boldsymbol{N}$ let ψ_n be the restriction of ψ to $\mathscr{A}_n$. Then we have $D_\psi(\mathscr{A}) = \bigcup_{n=1}^\infty D_{\psi_n}(\mathscr{A}_n)$.*

Proof. It is clear that the inclusion $D_\psi(\mathscr{A}) \supseteqq \bigcup_{n=1}^\infty D_{\psi_n}(\mathscr{A}_n)$ holds. For each $m \in \boldsymbol{N}$ let $\psi^{(m)}$ be the trace on $M_m(\mathscr{A})$ $(\cong \mathscr{A} \otimes M_m)$ which is induced by ψ. Let λ be an arbitrary element in $D_\psi(\mathscr{A})$. There exists a positive integer m and a projection p in $M_m(\mathscr{A})$ such that $\psi^{(m)}(p) = \lambda$. Since $\bigcup_{n=1}^\infty M_m(\mathscr{A}_n)$ is dense in $M_m(\mathscr{A})$ it follows from [2], A8.1 that there exists a positive integer k and a projection $q \in M_m(\mathscr{A}_k)$ such that q is equivalent to p. From this we obtain $\psi^{(m)}(p) = \psi^{(m)}(q) = \lambda \in D_{\psi_k}(\mathscr{A}_k)$. Thus we have shown that $D_\psi(\mathscr{A}) = \bigcup_{n=1}^\infty D_{\psi_n}(\mathscr{A}_n)$ holds also.

NOTATIONS. For any two subgroups G and H of $\boldsymbol{R}$ we denote by $G \otimes H$ the subgroup which is generated by the set $\{xy/x \in G, y \in H\}$. For each subset M of $\boldsymbol{R}$ we denote by $\langle M \rangle$ the subgroup which is generated by M.

LEMMA 3.2. *Suppose that $\mathscr{A}$ is a C^*-algebra with a tracial state ψ, and τ is the unique tracial state on some irrational rotation C^*-algebra $\mathscr{A}_\alpha (\alpha \in \boldsymbol{R}\backslash\boldsymbol{Q})$. Then we have*

$$D_{\psi\otimes\tau}(\mathscr{A} \otimes \mathscr{A}_\alpha) = D_\psi(\mathscr{A}) \otimes D(\mathscr{A}_\alpha) .$$

Proof. It is clear that $D_\psi(\mathscr{A}) \otimes D(\mathscr{A}_\alpha)$ is contained in $D_{\psi\otimes\tau}(\mathscr{A} \otimes \mathscr{A}_\alpha)$. On the other hand it follows from [5] that $\mathscr{A}_\alpha$ can be embedded into an AF-algebra $\mathscr{B}$ whose dimension group is isomorphic to $\boldsymbol{Z} + \alpha\boldsymbol{Z}$. Therefore, if φ is the unique tracial state on $\mathscr{B}$ then $D_\varphi(\mathscr{B}) = D(\mathscr{B}) = \boldsymbol{Z} + \alpha\boldsymbol{Z}$ holds and the restriction of φ to $\mathscr{A}_\alpha$ is equal to τ. Since $D_{\psi\otimes\tau}(\mathscr{A} \otimes \mathscr{A}_\alpha)$ is contained in $D_{\psi\otimes\varphi}(\mathscr{A} \otimes \mathscr{B})$ it suffices to show that $D_{\psi\otimes\varphi}(\mathscr{A} \otimes \mathscr{B})$ is contained in $D_\psi(\mathscr{A}) \otimes D(\mathscr{B})$. Let $\mathscr{B}_1 \subseteqq \mathscr{B}_2 \subseteqq \cdots \subseteqq \mathscr{B}$ be a monotonely increasing sequence of finite dimensional C^*-subalgebras of $\mathscr{B}$ containing the unit of $\mathscr{B}$ such that $\mathscr{B}$ is the norm closure of $\bigcup_{n=1}^\infty \mathscr{B}_n$ and let φ_n be the restriction of φ to $\mathscr{B}_n$ for each $n \in \boldsymbol{N}$. Since $\mathscr{B}_n$ is finite dimensional we have

$$D_{\psi\otimes\varphi_n}(\mathscr{A} \otimes \mathscr{B}_n) = D_\psi(\mathscr{A}) \otimes D_{\varphi_n}(\mathscr{B}_n) .$$

From this and from 3.1 we conclude that $D_\psi(\mathscr{A}) \otimes D(\mathscr{A}_\alpha) = D_{\psi\otimes\tau}(\mathscr{A} \otimes \mathscr{A}_\alpha)$ holds.

For any two countable (infinite) subgroups G_1 and G_2 of $\boldsymbol{T}$ such that $G_1 \subseteqq G_2$ there is a canonical embedding of the C^*-algebra $\mathscr{A}_{G_1}$ into the C^*-algebra $\mathscr{A}_{G_2}$. Namely, let the unitary operator U_{G_i} as well as the homomorphism π_{G_i} of G_i into $U(\mathscr{A}_{G_i})$ be given as in the beginning of §2, for $i = 1, 2$. Then we infer from 2.2 that there is a unique (injective) $*$-homomorphism from $\mathscr{A}_{G_1}$ into $\mathscr{A}_{G_2}$ which maps U_{G_1} onto U_{G_2} and $\pi_{G_1}(\lambda)$ onto $\pi_{G_2}(\lambda)$ for each $\lambda \in G$. This will be used in the sequel without being mentioned explicitely.

The following lemma is crucial for the proof of our main result.

LEMMA 3.3. *Suppose that $\mathscr{F}$ is a subfield of the real numbers containing the rational field $\boldsymbol{Q}$, and ε is a real number which is transcendent over $\mathscr{F}$. Then for every subgroup G of $\boldsymbol{R}$ which has a basis of the form $\{1, \varepsilon, \alpha_1, \cdots, \alpha_r\}$, where $\alpha_1, \cdots, \alpha_r$ are contained in $\mathscr{F}$, we have $D(\mathscr{A}_{G^\downarrow}) = G$.*

Proof. The proof will be carried out by induction over the number r which occurs in the statement above. If $r = 0$ then $\mathscr{A}_\varepsilon = \mathscr{A}_{G^\downarrow}$ holds and our assertion follows from [5] and [7].

Now we suppose that $r = 1$. Then there exists a $\alpha \in \mathscr{F}$ such that $\{1, \varepsilon, \alpha\}$ is a basis of G. Let $G_1 = \langle\{1, \varepsilon\}\rangle \otimes \langle\{1, \alpha\}\rangle$ and $G_2 = \langle\{1, \varepsilon\}\rangle \otimes \langle\{1, \alpha + \varepsilon\}\rangle$. It follows from 3.2 and from [5] that

$$D(\mathscr{A}_\varepsilon \otimes \mathscr{A}_\alpha) = D(\mathscr{A}_\varepsilon) \otimes D(\mathscr{A}_\alpha) = G_1 ,$$
$$D(\mathscr{A}_\varepsilon \otimes \mathscr{A}_{\alpha+\varepsilon}) = D(\mathscr{A}_\varepsilon) \otimes D(\mathscr{A}_{\alpha+\varepsilon}) = G_2 .$$

By 2.2 $\mathscr{A}_{G^\downarrow}$ can be embedded into $\mathscr{A}_{G_1^\downarrow}$, $\mathscr{A}_{G_2^\downarrow}$, and $\mathscr{A}_\varepsilon$, $\mathscr{A}_\alpha$ can be embedded into $\mathscr{A}_{G^\downarrow}$. Therefore we obtain, using [7], Theorem 1

$$G \subseteqq D(\mathscr{A}_{G^\downarrow}) \subseteqq G_1 \cap G_2 .$$

We claim that $G_1 \cap G_2 = G$ holds. Indeed, if this were not true then we could find an element a in $(G_1 \cap G_2)\backslash G$. Hence there exist integers x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 such that

$$\begin{aligned} &x_0 + x_1\varepsilon + x_2\alpha + x_3\alpha\varepsilon = a \\ &y_0 + y_1\varepsilon + y_2\alpha + y_3\alpha\varepsilon + y_3\varepsilon^2 = a \end{aligned}$$

and $x_3 \neq 0$, $y_3 \neq 0$. This implies that the elements 1, ε, α, $\alpha\varepsilon$, ε^2 are linearly dependent over $\boldsymbol{Q}$. However, since ε is transcendent over the field $\mathscr{F}$ and since α is an irrational element in $\mathscr{F}$ this is a contradiction. Thus the equalities $G = G_1 \cap G_2$ and $D(\mathscr{A}_{G^\downarrow}) = G$ hold.

Next we assume that our assertion is true for all groups of the kind described above, whose rank is not greater than $r + 2$ for some $r \geqq 1$. Let $\alpha_1, \cdots, \alpha_{r+1}$ be elements in $\mathscr{F}$ such that $\{1, \varepsilon, \alpha_1, \cdots, \alpha_r, \alpha_{r+1}\}$ is linearly independent over $\boldsymbol{Q}$ and let G be the group which is generated by this set. We consider subgroups of $\boldsymbol{R}$ of the following forms. For any integer p, $1 \leqq p \leqq r + 1$ we set

$$\begin{aligned} H^{(p,p)} &= \langle\{1, \varepsilon\} \cup \{\alpha_i/i \neq p\}\rangle , \\ K^{(p)} &= \langle\{1, \alpha_p\varepsilon^{-1}\}\rangle ; \end{aligned}$$

and for any two integers p, q with $p \neq q$, $1 \leqq p$, $q \leqq r + 1$ we set

$$H^{(p,q)} = \langle\{1, \varepsilon\} \cup \{\alpha_i/i \neq p, q\} \cup \{\alpha_q + \alpha_p\}\rangle .$$

Finally we define for each p, q with $1 \leqq p$, $q \leqq r + 1$

$$G^{(p,q)} = H^{(p,q)} \otimes K^{(p)} .$$

From our assumption, from 3.2 and from [5] we obtain for $1 \leqq p$, $q \leqq r + 1$

$$D(\mathscr{A}_{H^{(p,q)}\downarrow} \otimes \mathscr{A}_{\alpha_p\varepsilon^{-1}}) = D(\mathscr{A}_{H^{(p,q)}\downarrow}) \otimes D(\mathscr{A}_{\alpha_p\varepsilon^{-1}}) = G^{(p,q)} .$$

Since $\mathscr{A}_{G^\downarrow}$ can be embedded into $\mathscr{A}_{G^{(p,q)}\downarrow}$ and $\mathscr{A}_\varepsilon, \mathscr{A}_{\alpha_1}, \cdots, \mathscr{A}_{\alpha_{r+1}}$ can be embedded into $\mathscr{A}_{G^\downarrow}$ we obtain ([7], Theorem 1)

$$G \subseteqq D(\mathscr{A}_{G^\downarrow}) \subseteqq G^{(p,q)}$$

for any p, q with $1 \leqq p$, $q \leqq r + 1$. We claim that $G = \bigcap_{1 \leqq p,q \leqq r+1} G^{(p,q)}$ holds. Suppose that this were not true and suppose that a is an element in $\bigcap_{1 \leqq p,q \leqq r+1} G^{(p,q)}\backslash G$. For technical reasons we distinguish the following two different cases. (Observe that $a \in G^{(p,p)} = \langle\{1, \varepsilon\} \cup \{\alpha_i/1 \leqq i \leqq r + 1\} \cup \{\alpha_p\varepsilon^{-1}\} \cup \{\alpha_i\alpha_p\varepsilon^{-1}/i \neq p\}\rangle$ for $1 \leqq p \leqq r + 1$).

(I) There exists an integer p, $1 \leqq p \leqq r+1$ and some other integers x_0, x, $\bar{x}_0$, x_i, $\bar{x}_i$ $(1 \leqq i \leqq r+1, i \neq p)$ such that

$$x_0 + x\varepsilon + \sum_{i=1}^{r+1} x_i\alpha_i + \bar{x}_0\alpha_p\varepsilon^{-1} + \sum_{i\neq p} \bar{x}_i\alpha_i\alpha_p\varepsilon^{-1} = a$$

and $\bar{x}_k \neq 0$ for some k. Since a is contained in $G^{(p,k)}$ also, there exist integers y_0, y, $\bar{y}_0$, y_i, $\bar{y}_i$ $(1 \leqq i \leqq r+1, i \neq p)$ such that

$$y_0 + y\varepsilon + \sum_{i=1}^{r+1} y_i\alpha_i + \bar{y}_0\alpha_p\varepsilon^{-1} + \sum_{i\neq p} \bar{y}_i\alpha_i\alpha_p\varepsilon^{-1} + \bar{y}_k\alpha_p^2\varepsilon^{-1} = a\ .$$

By subtracting one of the last two relations from the other one we obtain that the set $\{1, \varepsilon_1, \alpha_1, \cdots, \alpha_{r+1}, \alpha_1\alpha_p\varepsilon^{-1}, \cdots, \alpha_{r+1}\alpha_p\varepsilon^{-1}\}$ is not linearly independent over $\boldsymbol{Q}$. Since $\{1, \alpha_1, \cdots, \alpha_{r+1}\}$ is a linearly independent subset of $\mathscr{F}$ and ε is transcendent over $\mathscr{F}$ this is a contradiction.

(II) Now we assume that the condition in (I) is not satisfied. Since a is contained in $G^{(r+1,r+1)}$ and $G^{(r,r)}$ there exist integers $x_0, x, \bar{x}_0, x_1, \cdots, x_{r+1}$ and $y_0, y, \bar{y}_0, y_1, \cdots, y_{r+1}$ such that $\bar{x}_0 \neq 0$, $\bar{y}_0 \neq 0$ and

$$x_0 + x\varepsilon + x_1\alpha_1 + \cdots + x_{r+1}\alpha_{r+1} + \bar{x}_0\alpha_{r+1}\varepsilon^{-1} = a\ ,$$
$$y_0 + y\varepsilon + y_1\alpha_1 + \cdots + y_{r+1}\alpha_{r+1} + \bar{y}_0\alpha_r\varepsilon^{-1} = a\ .$$

By subtracting one of these relations from the other one we obtain that the set $\{1, \varepsilon, \alpha_1, \cdots, \alpha_{r+1}, \alpha_r\varepsilon^{-1}, \alpha_{r+1}\varepsilon^{-1}\}$ is not linearly independent. Thus we have reached a contradiction. It follows from (I) and (II) that our claim is true, i.e., $G = \bigcap_{1\leqq p,q\leqq r+1} G^{(p,q)} = D(\mathscr{A}_{G\downarrow})$ holds.

COROLLARY 3.4. *For each finitely generated torsion free subgroup G of $\boldsymbol{T}$ we have $D(\mathscr{A}_G) = G^{\dagger}$.*

Proof. Let $\mathscr{F}$ be the field which is generated by the rational numbers and by the elements of $G^{\dagger}$. Since $\mathscr{F}$ is countable there exists a real number ε which is transcendent over $\mathscr{F}$. For each $n \in \boldsymbol{N}$ let H_n be the group which is generated by $G^{\dagger}$ and n/ε. Since G is torsion free it is clear that H_n is of the kind of groups we have considered in 3.3. Therefore we obtain from 3.3 that

$$D(\mathscr{A}_{H_n^{\downarrow}}) = H_n\ .$$

Since $G^{\dagger} \subseteqq D(\mathscr{A}_G) \subseteqq H_n$ holds for each $n \in \boldsymbol{N}$ (again we use [7], Theorem 1) and $G^{\dagger} = \bigcap_{n=1}^{\infty} H_n$, our assertion follows from this.

By using 3.4 we can now prove a more general version of 3.4.

PROPOSITION 3.5. *For each finitely generated subgroup G of $\boldsymbol{T}$ we have $D(\mathscr{A}_G) = G^\dagger$.*

Proof. The case of finite groups has already been considered in §1. Therefore we may assume that G is not finite. Since G is finitely generated the group $G^\dagger$ is generated by a finite set $\{\alpha_0, \alpha_1, \cdots, \alpha_r\}$ of real numbers which is linearly independent over $\boldsymbol{Q}$. Moreover we may assume that there exists a positive integer n such that $\alpha_0 = 1/n$. For $1 \leqq k \leqq r$ let H_k be the group which is generated by the set $(\{1, \alpha_1, \cdots, \alpha_r\} \backslash \{\alpha_k\}) \cup \{\alpha_k + 1/n\}$. Moreover we set $H_0 = \langle\{1, \alpha_1, \cdots, \alpha_r\}\rangle$. We claim that $\mathscr{A}_G$ can be embedded into the C^*-algebra $\mathscr{A}_{H_k^\downarrow} \otimes M_n$ for each k. Let us fix some k, $0 \leqq k \leqq r$. Let the operator $U_{H_k^\downarrow} \in U(\mathscr{A}_{H_k^\downarrow})$ as well as the homomorphism $\pi_{H_k^\downarrow}$ from $H_k^\downarrow$ into $U(\mathscr{A}_{H_k^\downarrow})$ be given as in the beginning of §2. Suppose that $\{e_1, \cdots, e_n\}$ is the canonical basis in $\boldsymbol{C}^n$ and that V, W are the matrices in M_n which are determined by

$$Ve_l = \exp(2l\pi i/n)e_1 , \qquad 1 \leqq l \leqq n ,$$
$$We_l = e_{1+l} , \qquad 1 \leqq l < n ; \qquad We_n = e_l .$$

We set $U_0 = U_{H_k^\downarrow} \otimes W$ and we denote by $\mathscr{G}$ the set of all elements of the form $\pi_{H_k^\downarrow}(\lambda) \otimes V^l$, $\lambda \in H_k^\downarrow$, $l \in \boldsymbol{N}$. It is seen that $\mathscr{G}$ is a subgroup of $U(\mathscr{A}_{H_k^\downarrow} \otimes M_n)$. Moreover, for each $\lambda \in G$ there exists a unique element V_λ in $\mathscr{G}$ such that

$$U_0 V_\lambda U_0^* = \lambda V_\lambda$$

and the application $\lambda \mapsto V_\lambda$ is a homomorphism from G into $\mathscr{G}$. Therefore we infer from 2.2 that our claim is true. From this and from 3.4 we obtain

$$D(\mathscr{A}_G) \subseteqq D(\mathscr{A}_{H_k^\downarrow} \otimes M_n) = 1/nH_k$$

for $0 \leqq k \leqq r$. One can check that $G^\dagger = \bigcap_{k=0}^{n} 1/nH_k$ holds. This shows that the inclusion $D(\mathscr{A}_G) \subseteqq G^\dagger$ is satisfied. By using [7] Theorem 1 once more it is seen that the inclusion $G^\dagger \subseteqq D(\mathscr{A}_G)$ is also valid.

THEOREM 3.6. *For each countable subgroup G of $\boldsymbol{T}$ we have $D(\mathscr{A}_G) = G^\dagger$.*

Proof. First we assume that G is not a torsion subgroup of $\boldsymbol{T}$. Since G is countable there exists a monotonely increasing sequence $G_1 \subseteqq G_2 \subseteqq \cdots \subseteqq G$ of finitely generated infinite subgroups of G such that $\bigcup_{n=1}^\infty G_n = G$. For each $n \in \boldsymbol{N}$ let $\varphi_n: \mathscr{A}_{G_n} \to \mathscr{A}_G$ be the canonical embedding of $\mathscr{A}_{G_n}$ into $\mathscr{A}_G$. Then $\mathscr{A}_G$ is the inductive limit of the sequence $\{\varphi_n(\mathscr{A}_{G_n})\}_{n\in N}$. Therefore, since $G^\dagger = \bigcup_{n=1}^\infty G_n^\dagger$ holds, our assertion follows from 3.5 and 3.1 in this case.

Now let G be an infinite torsion subgroup of G. Suppose that δ is an element in T which generates an infinite cyclic group. For each $n \in N$ let G_n be the group generated by G and δ^n. Since G_n is not a torsion group we have $D(\mathscr{A}_{G_n}) = G_n^\dagger$. Since $\mathscr{A}_G$ can be embedded into $\mathscr{A}_{G_n}$ for each $n \in N$ we obtain

$$D(\mathscr{A}_G) \subseteqq \bigcap_{n=1}^{\infty} D(\mathscr{A}_{G_n}) = \bigcap_{n=1}^{\infty} G_n^\dagger = G^\dagger .$$

On the other hand, since the restriction of the unique tracial state of $\mathscr{A}_G$ to $C(\hat{G})$ coincides with the normalized Haar measure on $C(\hat{G})$ and since G is a torsion group the inclusion $D(\mathscr{A}_G) \supseteqq G^\dagger$ holds too.

COROLLARY 3.7. *For any two countable subgroups G_1, G_2 of T the associated minimal rotation C^*-algebras $\mathscr{A}_{G_1}$, $\mathscr{A}_{G_2}$ are isomorphic if and only if G_1 is equal to G_2.*

Proof. Since the group $D(\mathscr{A}_{G_i})$ is an isomorphism invariant for the C^*-algebra $\mathscr{A}_{G_i}$ $(i = 1, 2)$ the corollary is an immediate consequence of 3.6.

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