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TWISTING TO ALGEBRAICALLY SLICE KNOTS

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It is shown that every knot with zero Arf invariant can be made algebraically slice by a $(-1, -1)$-twist.

Suppose K is a knot in S^3. For each integer k, consider the homeomorphism $H_k\colon D^2 \times I \to D^2 \times I$ defined by $H(re^{i\theta}, t) = (re^{i(\theta+2\pi kt)}, t)$. Orient S^3 by the right-hand rule.

DEFINITION. A knot $K' \subset S^3$ is obtained from K by a (k, l)-*twist* if there exists a smooth embedding $f\colon D^2 \times I \to S^3$ preserving orientation such that:

(i) K intersects $f(D^2 \times \{0\})$ transversely and algebraically l times;

(ii) $K \cap f(D^2 \times I) \subset f((\mathrm{int}\, D^2) \times I)$; and

(iii) $K' = K - (K \cap f(D^2 \times I)) \cup fH_kf^{-1}(K \cap f(D^2 \times I))$.

Example:

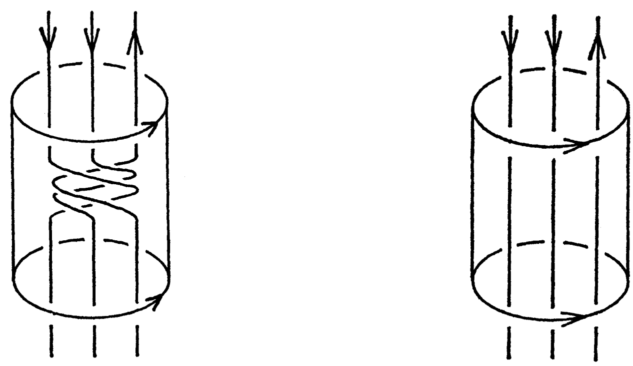

DIAGRAM 1

In [4], Akbulut and Kirby made the following three-tiered conjecture:

CONJECTURE 1. Suppose $K \subset S^3$ is a knot with zero Arf invariant. Then there exists a knot K' obtained from K by a $(-1, -1)$-twist such that

A: K' is algebraically slice

B: K' is slice (or ribbon)

C: K' is the unknot.

Suppose K is a knot with Arf invariant zero. If Conjecture 1B is valid for K, the homology 3-sphere obtained by surgery on K with $+1$ framing must bound an acyclic manifold W with $\pi_1(\partial W) \to$

$\pi_1(W)$ onto. A partial converse also holds. For a given K, if a manifold W as above exists and a certain homotopy CP^2 is genuine, K is concordant to a knot for which Conjecture 1B is true [2]. If Conjecture 1B were known for the (2, 7)-torus knot it would be easy to construct a smooth, closed, simply connected, almost parallelizable 4-manifold with index and second Betti number equal to 16. This is considered unlikely.

Suppose K and K' are knots such that K' is obtained from K by a $(-k, -1)$-twist, It follows from a result of Tristram [5] that if $k \geqq 2$ the k-signatures of K and K' coincide. The k-signatures need not be invariant under a $(-1, -1)$-twist. Akbulut [1] has provided an example of a knot K for which Conjecture 1B is true such that $\sigma_k(K) \neq 0$ for every $k \geqq 2$. In light of this example it is not surprising that Conjecture 1A is true. A proof is supplied below.

THEOREM. *Suppose $K \subset S^3$ is a knot with Arf invariant zero. Then there exists a knot K' such that K' is obtained from K by a $(-1, -1)$-twist and such that K' is algebraically slice.*

REMARK. A. Casson has obtained the following related result: If K is a knot with Arf invariant zero, there exist knots K' and K'' such that K' is concordant to K, K'' is obtained from K' by a $(-1, -1)$-twist, and the Alexander polynomial of K'', $\Delta(t) = 1$. (A knot with $\Delta(t) = 1$ must be algebraically slice.)

Proof of the Theorem. Let F be a Seifert surface for K and $a_1, b_1, \cdots, a_n, b_n$ a system of canonical curves for F. Since $\mathrm{Arf}(K) = 0$, we may assume that the diagonal entry of the Seifert matrix arising from a_i is even for each $i = 1, \cdots, n$ (see [3]).

For each $i = 1, \cdots, n$ be c_i and d_i be the cores of the handles pictured in Diagram 2:

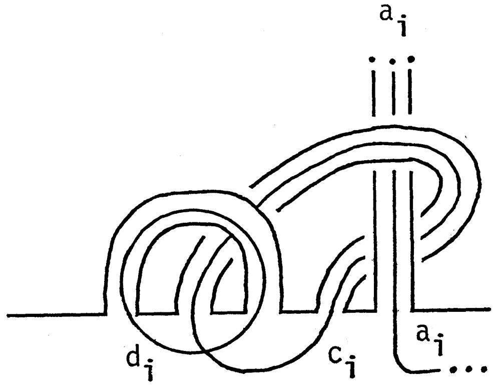

DIAGRAM 2

Let $\bar{F}$ be the union of F with these $2n$ additional handles. Plainly $\partial\bar{F}$ is isotopic to ∂F. (By abuse of notation we write $\partial\bar{F}=K$.) The matrix describing the restriction of the Seifert form to the generators $a_1, \cdots, a_n, c_1, \cdots, c_n$ is $\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}$ where $A = (\alpha_{ij})$ corresponds to the a_i. (Recall that the value of the Seifert form on the pair of curves (x, y) of $\bar{F}$ is defined to be $lk(x, i^*y)$ where i is a normal vector field to $\bar{F}$.) For each $i = 1, \cdots, n$ define classes $[a'_i] \in H_1(\bar{F})$ by $[a'_i] = [a_i] + \sum_{j=1}^{i}(1 - \alpha_{ij})[c_j] + (\alpha_{ii}/2)[c_i]$. The class $[a'_i]$ can be realized by a connected sum of a_i with push-offs of the c_j. Thus there exist disjointly embedded curves $a'_1, \cdots, a'_n$ which represent the $[a'_i]$ and are disjoint from the c_j. A simple calculation shows that the values of the Seifert form on $a'_1, \cdots, a'_n, c_1, \cdots, c_n$ are described by the matrix $\begin{bmatrix} B & I \\ I & O \end{bmatrix}$ where the diagonal entries of B are $+2$ and the off-diagonal entries $+1$.

Choose additional curves $b'_1, \cdots, b'_n, d'_1, \cdots, d'_n$ on $\bar{F}$ so that $\{a'_1, b'_1, \cdots, a'_n, b'_n, c_1, d'_1, \cdots, c_n, d'_n\}$ form a system of canonical curves and regard $\bar{F}$ as a disk with bands with these curves as cores. Let $l \subseteq \bar{F}$ be an arc whose intersection with $\partial\bar{F}$ is one of its endpoints and which misses each of the canonical curves (see Diagram 3). Consider the 2-disk $D \subset S^3$ pictured in Diagram 3 where the strands in the box $\boxed{\begin{matrix} l_i \\ n_i \end{matrix}}$ run parallel to an arc $l_i \subset S^3 - \bar{F}$ and perform $n_i (\in Z)$ full twists about l_i. The (transverse) intersection of D and $\bar{F}$ consists of the co-cores of the bands with cores $a'_1, \cdots, a'_n$ together with l. We will show that for certain choices of (l_i, n_i), the knot K' obtained from K by a $(-1, -1)$-twist along D is algebraically slice.

A portion of the knot K' and a portion of a genus $3n$ Seifert

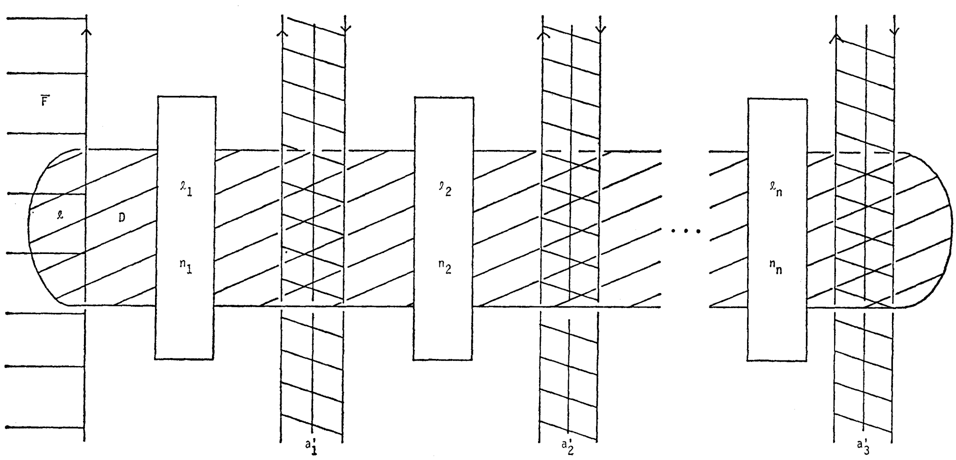

DIAGRAM 3

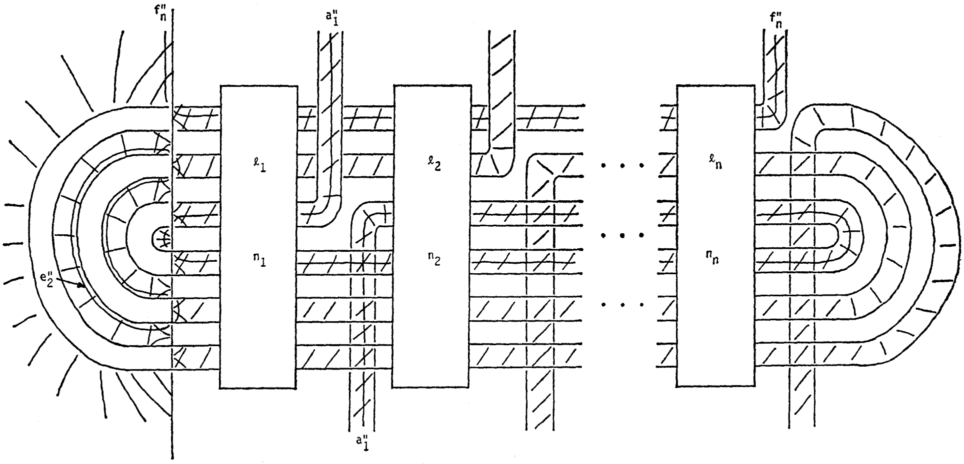

DIAGRAM 4

surface G for K' are pictured in Diagram 4. (G is orientable because the n_i are integral.) The part of (G, K') absent from Diagram 4 agrees identically with the part of $(\bar{F}, K)$ not in Diagram 3. Thus the curves b'_i, c_i, d'_i, $i = 1, \cdots, n$ represent generators of $H_1(G)$. Define b''_1, c''_i, d''_i resp. to be the same curves regarded as curves on G.

Additional curves a''_i, e''_i, $f''_i i = 1, \cdots, n$ completing a symplactic basis are shown in Diagrams 4 and 5. The curve a''_i is obtained as the twist of a'_i along D except near l where a''_i follows a sheet of G through two left half-twists. The curves $e''_1, \cdots, e''_n$ generate the homology of the part of G near l and are shown in Diagram 5. The curve f''_i is dual (in $H_1(G)$) to e''_i. It is obtained as the union of an arc which runs once along l_j, $1 \leqq j \leqq i$ (and is shown in Diagrams 4 and 5) with an arc outside Diagram 4 which misses each of the other canonical curves.

Suppose that choices of (l_i, n_i) have been made so that the entries of the Seifert matrix of G are defined. We shall later modify these choices. Let $\Lambda \subset H_1(G)$ be the subgroup generated by the classes of $a''_1, \cdots, a''_n, c''_1 \# e''_1, \cdots, c''_n \# e''_n$. (The connected sums are taken along G.) Let $\Gamma \subset H_1(G)$ be the subgroup generated by the elements of Λ together with the classes of $f''_1 \# (-d''_1), \cdots, f''_n \# (-d''_n)$. It follows from the calculations above and the nature of the linking in Diagram 4 that the intersection form of $H_1(G)$ vanishes on Γ and the Seifert form vanishes on Λ.

Let $C_1, \cdots, C_n$ be disjoint curves in $S^3 - G$ satisfying

(i) $lk(C_i, x) = -lk(f''_i \# (-d''_i), x)$ for each generator x of Λ, and

(ii) $lk(C_i, C_j) = -lk(f''_i \# (-d''_i), f''_j \# (-d''_j))$ for each $j \neq i$.

Replace l_i by $l_i \# C_i i = 1, \cdots, n$. The entries of the Seifert matrix on Γ are now zeros except possibly for the diagonal entries corresponding to the $f''_i \# (-d''_i)$. Clearly, these can be made zero by

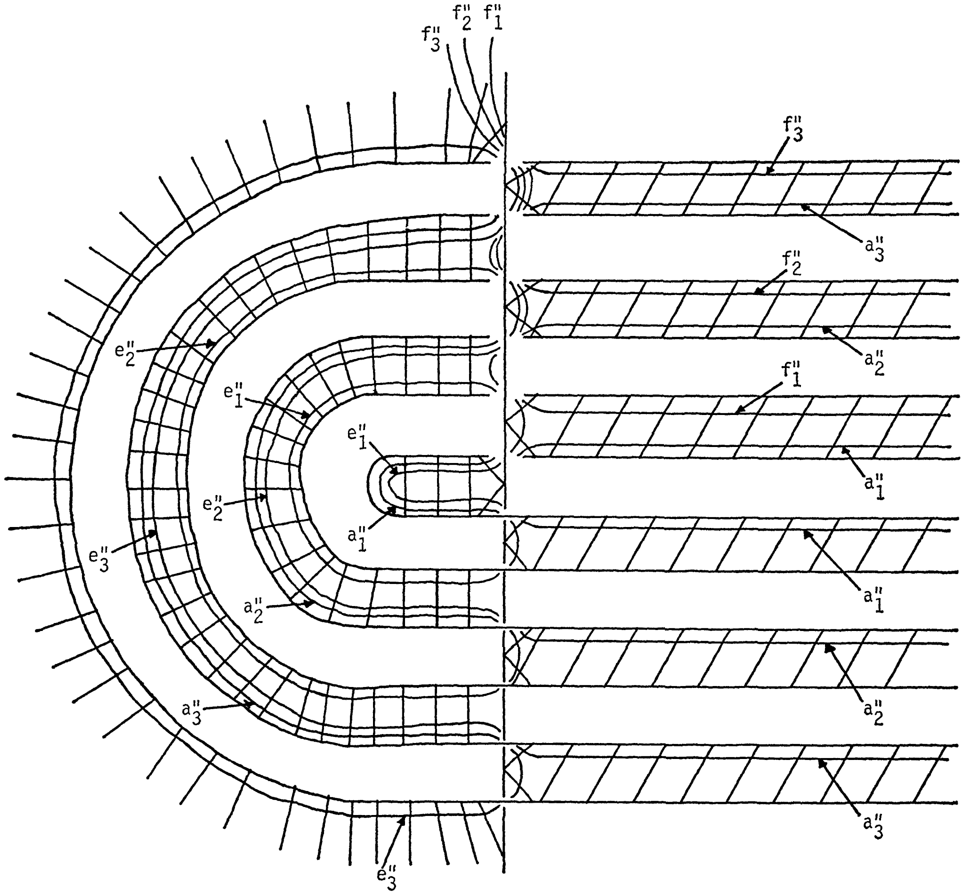

DIAGRAM 5

rechoosing the n_i. This completes the proof.

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