PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 1, 1982

THE EXPECTED MEASURE OF THE LEVEL SETS OF A REGULAR STATIONARY GAUSSIAN PROCESS

SALOMON BENZAQUEN AND ENRIQUE M. CABANA

If $X(t_1, t_2, \cdots, t_d)$ is a sufficiently regular, centered, stationary Gaussian process, the (random) level set over a measurable domain $T \subset R^d$

$$A(u) = \{t \in T: X(t) = u\}$$

is a $d-1$-dimensional manifold embedded in R^d. Our main result states that its expected measure is given by

$$E\mu_{d-1}(A(u)) = \lambda(T)E\|\operatorname{grad} X\|\, e^{-u^2/2}/\sqrt{2\pi} \tag{1}$$

where $\mu_{d-1}(A)$ is the $d-1$-dimensional volume of the hypersurface A, λ is the Lebesgue measure on R^d and the variance of X is assumed to be one.

The expression (1) holds even for $d=1$. In that case $\mu_0(A)$ is a counting measure that gives the number of points in A. (μ_1 and μ_2 give respectively length and area.)

1. Preliminary notations and results.

DEFINITION 1. (i) A stationary centered Gaussian process X with parameter $t = (t_1, t_2, \cdots, t_d) \in R^d$ and covariance function

$$\Gamma(t) = EX(s)X(s+t) \tag{2}$$

is said to be regular when it has continuous derivatives $\dot{X} = \operatorname{grad} X = (X_1, \cdots, X_d)$, $X_i(t) = \partial X(t)/\partial t_i$ $(i = 1, \cdots, d)$ and Γ is continuously differentiable up to the sixth order.

(ii) When, in addition, $\Gamma(0) = -\Gamma_{ii}(0) = 1$ $(i = 1, 2, \cdots, d)$ and $\Gamma_{ij}(0) = 0$ $(i \neq j,\ i, j = 1, \cdots, d)$, the process is said to be normalized. (Here and in what follows, the partial derivatives of Γ are denoted by $\Gamma_i = \partial\Gamma/\partial t_i$, $\Gamma_{ij} = \partial^2\Gamma/\partial t_i \partial t_j, \cdots$.)

The strong requirement imposed to the covariance of a *regular* process (which is justified by the use of a theorem by R. J. Adler and A. M. Hasofer, here stated as Lemma 1 (ii)) largely implies the existence of a version of X with continuous derivatives (see for instance [5]). The vector variable

$$\dot{X} = (X_1, X_2, \cdots, X_d)$$

has covariances

$$\operatorname{Var} \dot{X} = -((\Gamma_{ij}(0))) .$$

A change of scale in the process and a linear change of parameter, lead to a normalized process, namely,

$$Y(t) = X\Big(\Big(\Big(\frac{-\Gamma_{ij}(0)}{\Gamma(0)}\Big)\Big)^{-1/2} t\Big)\Big/\sqrt{\Gamma(0)}\ .$$

Even if a process is not normalized, we shall assume in what follows, without loss of generality, that $\Gamma(0) = 1$.

The following lemma states known results. We indicate the corresponding references and omit the proofs.

LEMMA 1. *Let X be a regular process. Then:*

(i) *(Belyaiev [3] Thm. 3.2). Given u and the interval*

$$T = \prod_{i=1}^{d} [a_i, b_i] \subset R^d\ , \tag{3}$$

with probability one there is no point $t \in T$ such that

$$X(t) = u \quad \text{and} \quad \dot{X}(t) = 0\ . \tag{4}$$

(ii) *(Adler [1] (proof of Theorem 2) and Hasofer [2, 7]). The number of points $t \in T$ such that $X(t) = u$ and all but one of the scalar conditions (4) hold, has a finite expectation. (The references give in fact the actual value of the expectation.)*

Given T by (3), let us introduce the notations $T_i^{(-1)} = \{a_i\}$, $T_i^{(0)} = (a_i, b_i)$, $T_i^{(1)} = \{b_i\}$. If $k = (k_1, \cdots, k_d)$ is a multi-index with components $k_i = -1, 0$ or 1 $(i = 1, \cdots, d)$, we abbreviate T^k for $\prod_{i=1}^{d} T_i^{(k_i)}$. The set T^k will be called a face of T of dimension $|k| = \sum_{i=1}^{d}(1 - |k_i|)$. (In particular, the interior T^0 of T is the only d-dimensional face.)

DEFINITION 2. Given a $d - 1$-dimensional manifold $A \subset T$ with continuous normal $\xi(t) = (\xi_1(t), \cdots, \xi_d(t)) \neq 0$, $t \in A$, a point

$$t \in T^k \cap A$$

such that all but one of the $|k|$ conditions

$$\{\xi_i(t) = 0\}_{k_i = 0}$$

hold, will be said to be a *k-critical point*.

COROLLARY 1. *The level set*

$$A(u) = \{t \in T\colon X(t) = u\} \tag{5}$$

of a regular process X is a $d - 1$-manifold with continuous normal $\dot{X} \neq 0$ with probability one. The number $X_k^{(u)}$ of k-critical points *of* $A(u)$ *has a finite expectation for each* $k \in K = \{-1, 0, 1\}^d$.

Proof. The first assertion follows readily from Definition 1 and Lemma 1(i). The second one follows from Lemma 1(ii) applied to the restriction of X to T^k, when $|k| > 1$.

The critical points for $|k| = 1$ are the crossings of the level by the one-dimensional restriction, and its (*finite*) expected number is computed in [5] by a well known formula, namely, (1) with $d = 1$. Finally, for $|k| = 0$, the conclusion is trivial.

LEMMA 2. *Given* A *as in Definition* 2, *if*

$$\psi_i = \psi_i(T) = \max_{t \in T} \# \{x \in R: t + xe_i \in A\} ,$$

$$e_i = (e_{i1}, e_{i2}, \cdots, e_{id}) , \qquad e_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases} ,$$

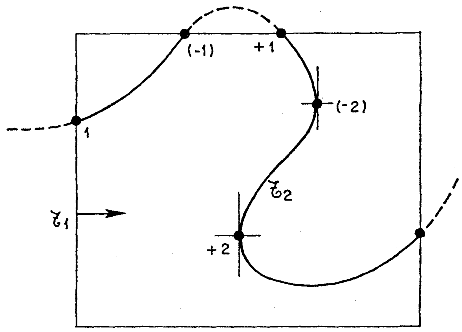
and if $\mathscr{X}_k$ *is the number of k-critical points, then, for each* $i = 1, \cdots, d$,

$$\Psi_i \leqq 2 \sum_{k \in K} \mathscr{X}_k . \tag{6}$$

(*Notice that* Ψ_i *is the maximum number of intersections with* A *of a line, parallel to the* ith *coordinate direction. Our estimate is a very rough one, but sufficient for our purposes.*)

Proof. Let us assume $i = 1$ for ease of description, and proceed to sweep T starting with the closure of a one-dimensional face in the given direction, say $\mathscr{T}_1 = \bar{T}^{(0,1,1,\cdots,1)}$. As a first step, let us translate this face in the second direction, until it describes the closure of the two-dimensional face $\mathscr{T}_2 = \bar{T}^{(0,0,1,1,\cdots 1}$. Then $\mathscr{T}_2$ is translated in the third direction until it descibes $\mathscr{T}_3 = \bar{T}^{(0,0,0,1,\cdots,1)}$ and this procedure is continued until $\mathscr{T}_d = \bar{T}^0 = T$ has been described.

At each step, the maximum number of intersections with A of the lines in the given direction already described, is estimated as follows: At the beginning, we count the intersections of $\mathscr{T}_1$, which are precisely the critical points on the faces that compose $\mathscr{T}_1$, namely $\mathscr{X}_{(1,1,\cdots,1)} + \mathscr{X}_{(0,1,1,\cdots)} + \mathscr{X}_{(-1,1,1,\cdots)}$. When $\mathscr{T}_1$ is translated, each increase of the number of intersections (in the amount of one or two) is produced when the face passes through a critical point. This is a necessary condition, through a critical point may produce a decrease (of one or two intersections) or no change. Therefore $2 \sum_{T_k \subset \mathscr{T}_2} \mathscr{X}_k$ is an upper bound of $\Psi_i(\mathscr{T}_2)$. Now, $\mathscr{T}_2$ is translated, and each increase in the number of intersections must be produced when the face transverses a critical point, hence, since the increase is in the amount of one or two as before, $\Psi_i(\mathscr{T}_3) \leqq 2\sum_{T_k\in\mathscr{T}_3}\mathscr{X}_k$. Going on in the same way, we reach finally the required inequality (6).



COROLLARY 2. *The maximum number* $\Psi_i^{(u)}$ *of intersections of lines in the* ith *direction with the level set* $A(u)$ *of a regular process* X, *has a finite expectation.*

Proof. Use Lemma 2 and Corollary 1.

2. The expected measure of $A(u)$. Given a regular process X, let T and $A(u)$ be defined by (3) and (5). We introduce the cones

$\mathscr{C}_i^\alpha \subset R^d$ defined by

$$\mathscr{C}_i^\alpha = \{\xi = (\xi_1, \cdots, \xi_d): |\xi_j| < \alpha|\xi_i| \text{ for each } j \neq i\}$$

and denote their relative solid angle by

$$\nu(\mathscr{C}_i^\alpha) = \frac{\mu_{d-1}(\mathscr{C}_i^\alpha \cap \{\xi: \|\xi\| = 1\})}{\mu_{d-1}(\{\xi: \|\xi\| = 1\})} .$$

Since $|X_i| > 0$ on $A_i^\alpha(u) = \{t\in A(u): \dot X(t)\in\mathscr{C}_i^\alpha\}$, the portion $A_i^\alpha(u)$ of $A(u)$ can be locally parametrized in the form $t_i = F(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_d)$, where F satisfies $\partial F/\partial t_j = -X_j/X_i$ $(j \neq i)$ because of the Implicit Function Theorem.

Hence the $d-1$-dimensional volume of $A_i^\alpha(u)$ is given by the integral (see for instance [6], p. 334):

$$\mu_{d-1}(A_i^\alpha(u)) = \int_{t\in A_i^\alpha(u)} \frac{\|\dot X(t)\|}{|X_i(t)|} dt_1 \cdots dt_{i-1}dt_{i+1} \cdots dt_d .$$

If $\mathscr{D}_n = \{t\in T;\ 2^n t_i$ is an integer for each $i = 1, \cdots, d\}$, is the set of diadic points in T, then the integral is approximated by the sums.

$$S_i^\alpha(n) = \sum_{\substack{t\in\mathscr{D}_n\\ t+2^{-n}e_i\in T}} 2^{-n(d-1)}\mathbf{1}_{(-\infty,0)}((x(t)-u)(x(t+2^{-n}e_i)-u))\mathbf{1}_{\mathscr{C}_i^\alpha}(\dot X(t))\frac{\|\dot X(t)\|}{|X_i(t)|} .$$

Where 1_C is the indicator function of C, for any set C. More precisely, it is easily seen that for each $\varepsilon > 0$, $\lim_{n\to\infty} S_i^{\alpha-\varepsilon}(n) \leqq \mu_{d-1}(A_i^\alpha(u)) \leqq \lim_{n\to\infty} S_i^\alpha(n)$, and, since

$$1_{\mathscr{C}_i^\alpha}(\dot{X}(t)) \frac{\| \dot{X}(t) \|}{| \dot{X}_i(t) |} \leqq \sqrt{d}\, \alpha$$

and

$$\sum_{\substack{t \in \mathscr{S}_n \\ t+2^{-n}e_i \in T}} 2^{-n(d-1)} 1_{(-\infty,0)}((X(t) - u)(X(t + 2^{-n}e_i) - u) \leqq \Psi_i(u) \frac{\lambda(T)}{b_i - a_i} ,$$

then $S_i^\alpha(n)$ is dominated by the random variable $\sqrt{d}\,\alpha \Psi_i(u)(\lambda(T)/(b_i - a_i))$, whose expectation is finite from Corollary 2.

Applying the Dominated Convergence Theorem, and noticing that $\mu_{d-1}(A_i^\alpha(u))$ is increasing in α, it follows

$$\lim_{n\to\infty} ES_i^{\alpha-\varepsilon}(n) \leqq E\mu_{d-1}(A_i^\alpha(u)) \leqq \lim_{n\to\infty} ES_i^\alpha(n) . \tag{7}$$

In order to compute $\lim_{n\to\infty} ES_i^\alpha(n)$, we write

$$ES_i^\alpha(n) = \sum_{t \in \mathscr{S}_n} 2^{-nd} E\Big(1_{\mathscr{C}_i^\alpha}(\dot{X}(t)) \frac{2^n \| \dot{X}(t) \|}{| \dot{X}_i(t) |} P\{(X(t) - u)(X(t + 2^{-n}e_i) - u) < 0/\dot{X}(t)\}\Big)$$

and the stationarity of X leads to

$$\lim_{n\to\infty} ES_i^\alpha(n) = \lambda(T) \lim_{n\to\infty} 2^n E\Big(1_{\mathscr{C}_i^\alpha}(\dot{X}(0)) \frac{\| \dot{X}(0) \|}{| \dot{X}_i(0) |} P\{(X(0) - u)(X(2^{-n}e_i) - u) < 0/\dot{X}(0)\}\Big) .$$

Let us abbreviate $\Gamma_\cdot(\delta) = (\Gamma_1(\delta), \cdots, \Gamma_d(\delta))^{tr}$. The conditional distribution of $(X(0), X(2^{-n}e_i))$ given $\dot{X}(0)$ is Gaussian, with expectation $(0, -\Gamma_\cdot(2^{-n}e_i)((-\Gamma_{hj}(0)))^{-1}\dot{X}(0))$ and variance

$$\begin{pmatrix} 1 & \Gamma(\delta) \\ \Gamma(\delta) & 1 - \Gamma_\cdot^{tr}(2^{-n}e_i)((-\Gamma_{hj}(0)))^{-1}\Gamma_\cdot(2^{-n}e_i) \end{pmatrix} ,$$

and the Taylor expansions of Γ, Γ_i are

$$\Gamma(t) = 1 + \frac{1}{2} \sum_{ij=1}^d \Gamma_{ij}(0) t_i t_j + \frac{1}{24} \sum_{ijkl=1}^d \Gamma_{ijkl} t_i t_j t_k t_l + \cdots$$

$$\Gamma_i(t) = \sum_{j=1}^d \Gamma_{ij}(0) t_j + \frac{1}{6} \sum_{jkl=1}^d \Gamma_{ijkl} t_j t_k t_l + \cdots ,$$

therefore, it is easily seen that the conditional distribution of

$$\left(X(0),\ Z_n = \frac{X(2^{-n}e_i) - X(0)}{2^{-n}\dot{x}_i}\right)$$

has conditional expectation of $(0, 1 + 0(2^{-n}))$ and conditional variance

$$\begin{pmatrix} 1 & 0(2^{-n}) \\ 0(2^{-n}) & 0(2^{-2n}) \end{pmatrix} .$$

Then we have

$$\begin{aligned}
&\lim_{n\to\infty} -\frac{2^n \|\dot{x}\|}{|\dot{x}_i|} P\{(X(0) - u)(X(2^{-n}e_i) - u) < 0/\dot{X}(0) = \dot{x}\} \\
&= \lim_{n\to\infty} \frac{2^n \|\dot{x}\|}{|\dot{x}_i|} P\{(X(0) - u)(2^{-n}\dot{x}_i Z_n + X(0) - u) < 0/\dot{X}(0) = \dot{x}\} \\
&= \frac{\|\dot{x}\|}{|\dot{x}_i|} \cdot |\dot{x}_i| \frac{1}{\sqrt{2\pi}} e^{-u^2/2} = \|\dot{x}\| \varphi(u) ,
\end{aligned}$$

thus

$$\lim_{n\to\infty} ES_i^\alpha(n) = \lambda(T)\varphi(u)E(1_{\mathscr{C}_i^\alpha}(\dot{X}) \|\dot{X}\|) .$$

This limit is a continuous function of α, hence

$$E\mu_{d-1}(A_i^\alpha(u)) = \lambda(T)\varphi(u)E(1_{\mathscr{C}_i^\alpha}(\dot{X}) \|\dot{X}\|) ;$$

furthermore, the inclusions

$$\bigcup_{i=1}^{d} A_i^1(u) \subset A(u) \subset \bigcup_{i=1}^{d} A_i^\alpha(u) \qquad (\alpha > 1)$$

and the fact that $A_1^1(u), A_2^1(u), \cdots, A_d^1(u)$ are disjoint, imply

$$\sum_{i=1}^{d} E\mu_{d-1}(A_i^1(u)) \leqq E\mu_{d-1}(A(u)) \leqq \sum_{i=1}^{d} E\mu_{d-1}(A_i^\alpha(u)) .$$

We use again the continuity in α to obtain the result stated as follows.

THEOREM. (i) *The expected measure of the level set of a regular process X corresponding to a measurable set T and a level u, is*

(8) $$E\mu_{d-1}(A(u)) = \lambda(T)\varphi(u)E\|\dot{X}\| ,$$

where $\dot{X}$ is Gaussian, $E\dot{X} = 0$, $\operatorname{Var}\dot{X} = ((-\Gamma_{ij}(0)))$, and $\varphi(u) = 1/\sqrt{2\pi}\, e^{-u^2/2}$.

(ii) *When X is normalized,* (8) *reduces to*

$$E\mu_{d-1}(A(u)) = \lambda(T)e^{-u^2/2}/B\left(\frac{1}{2}, \frac{d}{2}\right)$$

where B is Euler Beta function.

The proof of (8) for an interval T is contained in the preceding context; since the expectation is an additive function of T, the same result holds for measurable T.

When X is normalized, a straightforward calculation gives the final result.

3. Comparisons with previous results. For $d = 1$, (8) reduces to the formula

$$\text{expected number of crossings of } u = \lambda(T)\sqrt{-\Gamma''(0)}e^{-u^2/2}/\pi ,$$

given in [5].

In the case $d = 2$, Benzaquen [4] proved that if π_i is the projection $\pi_i(t_1, \cdots, t_d) = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_d)$ and $\mu_{d-1}^{(i)}(A(u)) = \mu_{d-1}(\pi_i(a(u)))$, where the points are taken with its corresponding multiplicity, then

$$E\mu_{d-1}^{(i)}(A(u)) \leqq \lambda(T)\sqrt{-\Gamma_{ii}(0)}\, e^{-u^2/2}/\pi . \tag{9}$$

It is not hard to prove the equality in (9) with our assumptions of regularity, and to extend the same formula for $d > 2$.

Clearly, the inequalities

$$\mu_{d-1}^{(i)}(A(u)) \leqq \mu_{d-1}(A(u)) \leqq \sum_{j=1}^{d} \mu_{d-1}^{(j)}(A(u))$$

hold, and the compatibility of (8) and (9) require

$$\sqrt{2/\pi}\sqrt{-\Gamma_{ii}(0)} \leqq E\|\dot{X}\| \leqq \sqrt{2/\pi}\sum_{j=1}^{d}\sqrt{-\Gamma_{jj}(0)}$$

and, in the normalized case,

$$\sqrt{2/\pi} \leqq \sqrt{2\pi}/B\left(\frac{1}{2}, \frac{d}{2}\right) \leqq d\sqrt{2/\pi} .$$

These inequalities are trivially verified by a direct calculation of expectations in $|\dot{X}_i| \leqq \|\dot{X}\| \leqq \sum_{j=1}^{d}|\dot{X}_j|$.

References

1. R. J. Adler, *Excursions above a fixed level by n-dimensional random fields*, J. App. Prob., **13** (1976), 276–289.
2. R. J. Adler and A. M. Hasofer, *Level crossings for random fields*, Ann. Prob., **4** (1976), 1–12.

3. Yu. K. Belyaiev, *Point processes and first passage problems,* Proc. Sixth Berkeley Symposium Math. Stat. Prob., **3** (1972), 1–17.
4. S. Benzaquen, *Cruces de un proceso estocástico con una barrera fija,* u. M. Sc. in Math. Thesis, Universidad Simón Bolívar, 1979.
5. H. Cramer and M. R. Leadbetter, *Stationary and Related Stochastic Processes,* J. Wiley and Sons, 1967.
6. C. H. Edwards, *Advanced Calculus of Several Variables,* Academic Press, N. Y., 1973.
7. A. M. Hasofer, *Upcrossings of random fields,* Suppl. Adv. Prob., **10** (1978), 14–21.

Received September 18, 1980 and in revised form June 2, 1981.

UNIVERSITY SIMON BOLIVAR
APARTADO 80659
CARACAS, VENEZUELA